Some Problems and Investigations

These are some fairly open-ended mathematical problems/investigations based on material that you will meet during your first year. Most will repay a little computational effort using a pocket calculator or `MAPLE`. If you are feeling more ambitious you may want to formulate some of your observations as theorems and then prove them.

1. Let

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots.$$

Show that $xf'(x) = \sum_{n=1}^{\infty} nx^n$, and so find a closed expression for this sum. Similarly find closed expressions for $\sum_{n=1}^{\infty} n^2x^n$, $\sum_{n=1}^{\infty} n^3x^n$, and so on. (If your differentiation is a bit unreliable try `MAPLE`!) What can you say about the formulae you get?

2. Let $G = S_8$. Pick a pair of elements σ and τ from G, and compute $\sigma\tau$ and $\tau\sigma$. Repeat this for a few other choices of σ and τ. In general you should find that $\sigma\tau$ and $\tau\sigma$ aren't always equal, but that there is some relation between them. What is this relation, and can you explain why it holds?

3. There are two ways of completely bracketing together three variables in a product and five ways of bracketing together four variables: $a(bc)$ and $(ab)c$; $a(b(cd))$, $a((bc)d)$, $(ab)(cd)$, $(a(bc))d$ and $a(b(cd))$ respectively. How many ways of bracketing together five variables? six variables? ten? twenty? and so on.

4. Pick square matrices A and B of the same size (2 by 2s are fine). Work out AB and BA and find their characteristic equations. Repeat a few times. In general $AB \neq BA$ but what about their characteristic equations?

5. In this investigation consider a permutation of size n as a listing of the numbers 1 to n in some order, e.g., when $n = 6$ typical permutations might be 423165 or 621345. In a permutation an *ascent* is a pair of adjacent numbers with the second bigger than the first and a *descent* is a pair of adjacent numbers with the first bigger than the second. (Pretty obvious jargon really!) For instance 423165 has two ascents (23, 16) and three descents (42, 31, 65).

With these definition let's count the number of permutations of size n with a given number of descents. Let $D_{n,k}$ be the number of permutations of size n with k descents. For instance $D_{3,0} = 1$, $D_{3,1} = 4$ and $D_{3,2} = 1$. Find $D_{n,k}$ for all n up to 5 and with all appropriate k. If you are very patient, are good at training `MAPLE` to do such things, or can find a cunning way to compute $D_{n,k}$ without listing all permutations, then go a bit further. What can you say about these numbers $D_{n,k}$?

A harder problem along the same lines is the enumeration of *up-down* or *zig-zag* permutations. These are permutations which begin with an ascent, then a descent then an ascent etc. Let Z_n be the number of up-down permutations of size n. For instance $Z_4 = 5$ as the up-down permutations of size 4 are 1324, 1423, 2314, 2413 and 3412. Compute the first few values of Z_n. (There is a pattern but it's extremely subtle!)

6. Consider the sum

$$G_n = \sum_{r=0}^{n-1} e^{2\pi i r^2/n} = \sum_{r=0}^{n-1} \cos\frac{2\pi r^2}{n} + i\sum_{r=0}^{n-1} \sin\frac{2\pi r^2}{n}$$

for positive integers n. Calculate G_1, G_2, G_3, G_4, G_6 and G_{12} exactly, and calculate G_n numerically for n at least up to 20. What pattern do you observe?

7. Express

$$\sum_{r=0}^{\infty} \frac{x^{3r}}{(3r)!} = 1 + \frac{x^3}{3!} + \frac{x^3}{6!} + \frac{x^9}{9!} + \cdots$$

in closed form. What about

$$\sum_{r=0}^{\infty} \frac{x^{3r+1}}{(3r+1)!} \qquad \text{and} \qquad \sum_{r=0}^{\infty} \frac{x^{3r+2}}{(3r+2)!}?$$

Can you generalize?

8. Let $f : [\,0, 1,] \to \mathbf{R}$ be a continuous function with the property that $f(0) = f(1)$. Show that there exists x with $f(x) = f(x + \frac{1}{2})$. (Hint: apply the Intermediate Value Theorem to g defined by $g(x) = f(x + \frac{1}{2}) - f(x)$ on $[\,0, \frac{1}{2}\,]$.)

Must there exist x with $f(x) = f(x + \frac{1}{3})$? $f(x) = f(x + \frac{2}{3})$? $f(x) = f(x + \frac{1}{4})$? etc.

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