Proving irrationality: an alternative approach

The usual proof that $\sqrt{2}$ is irrational, and its generalization to other square roots, cube roots, etc., of integers involves a lot of messing about with divisibility conditions. I here outline an alternative approach which some may find more appealing.

Let's start, as always, with $\sqrt{2}$, and assume (to obtain a contradiction) that $\sqrt{2}$ **is** rational; put $\sqrt{2} = a/b$ with a and b positive integers. The key to this proof is that if r and s are integers then

$$r + s\sqrt{2} = r + s\frac{a}{b} = \frac{rb + sa}{b} \tag{1}$$

is also a rational with denominator b. This puts a severe restriction on what sort of numbers can be expressed in this form. If we can find integers r and s with $0 < r + s\sqrt{2} < 1/b$ then we would have a contradiction since the number $r + s\sqrt{2}$ can't be a rational with denominator b.

To find such r and s we look at numbers of the form $(\sqrt{2} - 1)^n$. We calculate

$$\begin{aligned}
(\sqrt{2} - 1)^1 &= \sqrt{2} - 1 \\
(\sqrt{2} - 1)^2 &= 3 - 2\sqrt{2} \\
(\sqrt{2} - 1)^3 &= 5\sqrt{2} - 7 \\
(\sqrt{2} - 1)^4 &= 17 - 12\sqrt{2} \\
(\sqrt{2} - 1)^5 &= 29\sqrt{2} - 41
\end{aligned}$$

and so on. It seems as if we can write $(\sqrt{2} - 1)^n = r_n + s_n\sqrt{2}$ for integers r_n and s_n for each positive integer n. This can be easily proved by induction (exercise!) or directly, by expanding $(\sqrt{2} - 1)^n$ by the binomial theorem. But why are we doing this? Well $\sqrt{2} - 1 = 0 \cdot 4142\cdots$, in particular $0 < \sqrt{2} - 1 < 1$. It follows that for n large enough we have $0 < (\sqrt{2} - 1)^n < 1/b$. Hence

$$0 < r_n + s_n\sqrt{2} = r_n + s_n\frac{a}{b} < \frac{1}{b}$$

and so $0 < r_n b + s_n a < 1$ which is impossible since $r_n b + s_n a$ is an integer. Again we conclude that $\sqrt{2}$ is irrational.

We can play the same game with other square roots. Suppose m is a positive integer, but not a perfect square. Again suppose that $\sqrt{m} = a/b$ with a and b positive integers. Again if r and s are integers then $r+s\sqrt{m}$ is a rational with denominator b. This time, since m isn't a perfect square we consider powers of $\sqrt{m} - t$ where t is the natural number with $t < \sqrt{m} < t+1$. (For instance if $m = 77$ we would let $t = 8$.) Again $(\sqrt{m}-t)^n = r_n+s_n\sqrt{m}$ with r_n and s_n integers, and if n is large enough we have $0 < (\sqrt{m} - t)^n < 1/b$ and so $0 < r_n b + s_n a < 1$ giving the contradiction that shows that $\sqrt{m}$ cannot be rational.

Now let's look at cube roots. Take first $\sqrt[3]{2}$ and suppose that it is rational, say $\sqrt[3]{2} = a/b$ with a, b natural numbers. Noting that $0 < \sqrt[3]{2} - 1 < 1$ we may decide to consider powers of this number. We calculate

$$(\sqrt[3]{2} - 1)^1 = \sqrt[3]{2} - 1$$

$$
\begin{aligned}
(\sqrt[3]{2}-1)^2 &= \sqrt[3]{4}-2\sqrt[3]{2}+1\\
(\sqrt[3]{2}-1)^3 &= -3\sqrt[3]{4}+3\sqrt[3]{2}+1\\
(\sqrt[3]{2}-1)^4 &= 6\sqrt[3]{4}-2\sqrt[3]{2}-7\\
(\sqrt[3]{2}-1)^5 &= -8\sqrt[3]{4}-5\sqrt[3]{2}+19
\end{aligned}
$$

and so on. I hope that you can convince yourself that $(\sqrt[3]{2}-1)^n = r_n + s_n\sqrt[3]{2} + t_n\sqrt[3]{4}$ where r_n, s_n and t_n are integers. Hence

$$(\sqrt[3]{2}-1)^n = \frac{r_n b^2 + s_n ab + t_n a^2}{b^2} = \frac{c_n}{b^2}$$

where c_n is an integer. But since $\sqrt[3]{2}-1 = 0\cdot 2599\cdots$ then for n large enough we have $0 < (\sqrt[3]{2}-1)^n = c_n/b < 1/b$ which is impossible as c_n is an integer. This contradiction means that $\sqrt[3]{2}$ is irrational. Now this argument can easily be extended to numbers of the form $\sqrt[k]{m}$ provided that m isn't a k-th power of an integer already. If this isn't the case then $r < \sqrt[k]{m} < r+1$ for some integer r, and we consider powers of $(\sqrt[k]{m}-r)$.

This approach works for other types of irrationals as well, not just k-th roots of integers. For example let $\xi = 2\cos 2\pi/9 (= 2\cos 40^\circ)$. Putting $\theta = 2\pi/9$ into the identity $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ we get $-\frac{1}{2} = \frac{1}{2}(\xi^3 - 3\xi)$ and so $\xi^3 - 3\xi + 1 = 0$, or $\xi^3 = 3\xi - 1$. Now $\xi = 1\cdot 5320\cdots$ and so we may be tempted to consider powers of $(\xi - 1)$ since this lies in the interval $(0,1)$. Now

$$
\begin{aligned}
(\xi-1)^1 &= \xi - 1\\
(\xi-1)^2 &= \xi^2 - 2\xi + 1\\
(\xi-1)^3 &= \xi^3 - 3\xi^2 + 3\xi - 1 = -3\xi^2 + 6\xi - 2\\
(\xi-1)^4 &= (\xi-1)(-3\xi^2+6\xi-2) = -3\xi^3 + 9\xi^2 - 8\xi + 2 = 9\xi^2 - 17\xi + 5\\
(\xi-1)^5 &= (\xi-1)(9\xi^2 - 17\xi + 5) = 9\xi^3 - 26\xi^2 + 22\xi - 5 = -26\xi^2 + 49\xi - 14
\end{aligned}
$$

and so on. I hope that you can prove by induction that $(\xi-1)^n = r_n + s_n\xi + t_n\xi^2$ for some integers r_n, s_n and t_n. If $\xi = a/b$ is rational, then $(\xi-1)^n = c_n/b^2$ with c_n an integer and again this is impossible for large enough n. Now one can extend this argument again to show that if $\xi^k + u_1\xi^{k-1} + \cdots + u_{n-1}\xi + u_n = 0$ with the u_js integers, then if ξ isn't an integer, then ξ must be irrational.

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