Thematic studies: Number theory

Solutions to questions 1, 2, 3(iii), 9(iv), 10(i), and 11 should be submitted for assessment by noon on Thursday 19 March 2009

1. Prove that there are infinitely many prime numbers p satisfying $p \equiv 5 \pmod{6}$. [15 marks]

2. Let $n \in \mathbf{N}$ with $n \geq 2$. Consider the numbers $n! + 2, n! + 3, \ldots, n! + n$. Prove that none of them is prime, and deduce that there are arbitrarily long finite stretches of consecutive non-prime numbers in $\mathbf{N}$. [15 marks]

3. In each case, using the Euclidean algorithm, find $g = \gcd(a, b)$ and integers r and s such that $g = ra + sb$:

   (i) $a = 105$, $b = 72$;

   (ii) $a = 667$, $b = 413$;

   (iii) $a = 2009$, $b = 1000$. [15 marks]

4. The Fibonacci numbers (as seen in *The Da Vinci Code*) F_1, F_2, $F_3, \ldots$ are defined recursively by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$. Write down F_n for all n up to 15.

   Prove, by induction, that $\gcd(F_{n+1}, F_n) = 1$ for all n.

5. Find the prime factorizations of each of the following numbers:

   (i) 105; (ii) 720; (iii) 1001; (iv) 2009; (v) 14820.

6. Prove that the only number p such that p, $p+2$ and $p+4$ are all prime is $p = 3$.

   Find a number p and an integer $d > 2$ such that p, $p + d$ and $p + 2d$ are all prime.

7. (Harder) Prove that if $n \in \mathbf{N}$ and $n \geq 2$ then $n^4 + 4^n$ is not a prime number.

8. Prove that if a is odd then $a^2 \equiv 1 \pmod{8}$.

   Also prove that if b is coprime to 6 then $b^2 \equiv 1 \pmod{24}$.

9. Solve the following linear congruences:

   (i) $9x \equiv 2 \pmod{19}$;

   (ii) $79x \equiv 199 \pmod{281}$;

   (iii) $696x \equiv 1 \pmod{1001}$;

   (iv) $666x \equiv 355 \pmod{2009}$. [15 marks]

10. (i) Let n and k be integers with $n \geq 2$ and $k \geq 2$. Prove that $(n-1) \mid (n^k - 1)$. Hence prove that if $n^k - 1$ is prime then $n = 2$ and k is prime. [25 marks]

    (ii) Primes of the form $2^p - 1$ are called Mersenne primes. The largest known prime is the Mersenne prime $2^{43112609} - 1$. Determine, explaining your method, how many decimal digits it has. (A 10-digit pocket calculator should suffice for this!) Also determine its first and last digits.

11. From Fermat's little theorem deduce that when p is prime,

$$n^p \equiv n \pmod{p}$$

    for all integers n. [15 marks]

12. For each of the following numbers n, determine whether n can be written as the sum of two squares of integers, and if so write n in that form.

    (i) $n = 146$; (ii) $n = 234$, (iii) $n = 721$, (iv) $n = 1728$, (v) $n = 34045$.

13. (Representations $n = a^2 + b^2 = c^2 + d^2$ are *essentially different* if $(a, b) \neq (\pm c, \pm d)$ and $(a, b) \neq (\pm d, \pm c)$).

    (i) Find two essentially different representations of $n = 5917 = 61 \times 97$ as a sum of two squares

    (ii) Find four essentially different representations of $n = 53655 = 5 \times 29 \times 37$ as a sum of two squares.

    (iii) Find four essentially different representations of $n = 55981 = 17 \times 37 \times 89$ as a sum of two squares.

14. In each case find **positive** integers c and d such that $n = c^2 + d^2$.

    (i) $n = 289 = 17^2$; (ii) $n = 1681 = 41^2$; (iii) $n = 11881 = 109^2$.

    Prove that if p is a prime number with $p \equiv 1 \pmod{4}$ then $p^2 = c^2 + d^2$ for some integers c and d with $c > 0$ and $d > 0$.

15. Suppose that p is a prime, and the congruence $x^2 \equiv -2 \pmod{p}$ has a solution. Using the pigeonhole principle prove that there are $a, b \in \mathbf{Z}$ with $p = a^2 + 2b^2$.

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