Zagier's conjecture

In a series of lectures in Exeter in April 1998 (under the auspices of the LMS), Don Zagier stated a conjecture concerning the "function"

$$f(x) = \sum_{m=0}^{\infty} \prod_{j=1}^{m} (1 - x^j) = 1 + (1-x) + (1-x)(1-x^2) + (1-x)(1-x^2)(1-x^3) + \cdots. \quad (*)$$

Zagier's conjecture strengthens an earlier conjecture of Kontsevich. As it stands, this seems not to define a function of x; if $|x| < 1$ or $|x| > 1$ the terms fail to converge to zero. Neither is it a formal power series. However, if a root of unity ζ is substituted for x, all but a finite number of terms vanish and the sum makes sense. The same is true for the formal derivatives of $(*)$. Zagier's conjecture concerns the values of $f^{(m)}(\zeta)$ for roots of unity ζ.

Let ζ be an n-th root of unity. Then consider the expression $f(\zeta e^{-t})$. This does make sense as a power series in t; for if $m \geq rn$, then

$$\prod_{j=1}^{m} (1 - \zeta^j e^{-jt}) \qquad (\dagger)$$

contains r factors $(1 - e^{knt})$ each of which is divisible by t. Hence $(\dagger)$ is divisible by t^r, and so the series for $f(\zeta e^{-t})$ adds to a genuine formal power series in t. One can express the coefficients in this series in terms of the $f^{(m)}(\zeta)$ and *vice versa* if one is so minded.

Define $A_m(\zeta)$ as follows: set

$$e^{-t/24} f(\zeta e^{-t}) = \sum_{m=0}^{\infty} \frac{A_m(\zeta)}{m!} \left(\frac{t}{24}\right)^m.$$

Now take a number N which is divisible by $12n$; hence $\zeta^{N/12} = 1$. Define $B_m(\zeta)$ as follows: set

$$\frac{1}{2} \sum_{r=1}^{N} \left(\frac{12}{r}\right) \zeta^{(r^2-1)/24} \frac{e^{-rt}}{1 - e^{-Nt}} = \sum_{m=0}^{\infty} (-1)^m \frac{B_m(\zeta) t^{2m+1}}{(2m+1)!}$$

where $(12/r)$ is a Jacobi symbol. (For even r we take $(12/r) = 0$, while for odd r we have $(12/r) = (3/r)$). It is easy to see that the left side is independent of the choice of N. Zagier's conjecture is that $A_m(\zeta) = B_m(\zeta)$ for all m and ζ.

Zagier has verified by calculation that the conjecture holds for many m and ζ, but has no proof that works in infinitely many instances. The interesting special cases of $m = 0$ and of $\zeta = 1$ are open.

There is clearly some relation to Euler's celebrated pentagonal number formula. This states that

$$\prod_{j=1}^{\infty} (1 - x^j) = \sum_{m=1}^{\infty} \left(\frac{12}{m}\right) x^{(m^2-1)/24}.$$

Also the sum in the definition of $B_n(\zeta)$ can be thought of as a way of interpreting the infinite sum

$$\frac{1}{2} \sum_{r=1}^{\infty} \left(\frac{12}{r}\right) \zeta^{(r^2-1)/24} e^{-rt}$$

as a formal power series.

For $\zeta = 1$ we have

$$\begin{aligned}\sum_{m=0}^{\infty} (-1)^m \frac{B_m(1)t^{2m+1}}{(2m+1)!} &= \frac{e^{-t} - e^{-5t} - e^{-7t} + e^{-11t}}{2(1 - e^{-12t})} \\ &= \frac{e^{5t} - e^{t} - e^{-t} + e^{-5t}}{2(e^{6t} - e^{-6t})} \\ &= \frac{e^{2t} - e^{-2t}}{2(e^{3t} + e^{-3t})} \\ &= \frac{\sinh 2t}{2\cosh 3t}\end{aligned}$$

and so

$$\sum_{m=0}^{\infty} \frac{B_m(1)t^{2m+1}}{(2m+1)!} = \frac{\sin 2t}{2\cos 3t}.$$

Thus $B_m(1)$ is the coefficient of $t^{2m+1}/(2m+1)!$ in $\sin 2t/2\cos 3t$. These coefficients are known as Glaisher's T-numbers and form sequence A002436 in Sloane's On-Line Encyclopedia of Integer Sequences.

After some manipulation, we find that that

$$B_0(\zeta) = \frac{1}{4N} \sum_{r=1}^{N} \left(\frac{12}{r}\right) \zeta^{(r^2-1)/24}(r^2 - Nr + N^2/6).$$

Zagier's conjecture implies that this equals $f(\zeta)$.