

A Probabilistic Proof of the Andrews-Gordon Identities

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1 Introduction

We adopt the standard q -series notation:

$$(u)_\infty = \prod_{j=0}^{\infty} (1 - uq^j)$$

and

$$(u)_n = \frac{(u)_\infty}{(uq^n)_\infty}$$

for each integer n . Also we let

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{(q)_n}{(q)_m (q)_{n-m}}$$

denote the q -binomial coefficient.

The Andrews-Gordon identities are:

Theorem 1 *For integers $k \geq 2$ and $1 \leq i \leq k$,*

$$\sum_{\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{k-1} \geq 0} \frac{q^{\lambda_1^2 + \lambda_2^2 + \dots + \lambda_{k-1}^2 + \lambda_i + \lambda_{i+1} + \dots + \lambda_{k-1}}}{(q)_{\lambda_1 - \lambda_2} \cdots (q)_{\lambda_{k-2} - \lambda_{k-1}} (q)_{\lambda_{k-1}}} = \prod_{\substack{r=1 \\ r \neq 0, \pm i \pmod{2k+1}}}^{\infty} \frac{1}{1 - q^r}. \quad (1)$$

This power series formulation is due to Andrews [1]. Gordon [5] originally proved that the right side of these identities were generating functions for certain classes of partitions defined by conditions on the differences of their parts. Andrews [2] gave another proof of this.

In [4] Fulman provided a probabilistic proof of the “extreme” cases of the Andrews-Gordon identities, namely, for $i = 1$ and $i = k$. When $k = 2$ these are the celebrated Rogers-Ramanujan identities. We extend Fulman’s method to prove Theorem 1 in full generality.

2 The proof

We shall write

$$AGS(i, k) = \sum_{\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{k-1} \geq 0} \frac{q^{\lambda_1^2 + \lambda_2^2 + \dots + \lambda_{k-1}^2 + \lambda_i + \lambda_{i+1} + \dots + \lambda_{k-1}}}{(q)_{\lambda_1 - \lambda_2} \cdots (q)_{\lambda_{k-2} - \lambda_{k-1}} (q)_{\lambda_{k-1}}}$$

and

$$AGP(i, k) = \prod_{\substack{r=1 \\ r \neq 0, \pm i \pmod{2k+1}}}^{\infty} \frac{1}{1 - q^r}$$

for the left side and right sides of (1) (“Andrews-Gordon sum” and “Andrews-Gordon product”).

For simplicity’s sake we alter Fulman’s notation slightly, replacing his q by $1/q$ and his generic partition λ by its conjugate λ' . Fulman defines a Markov chain on $\{0, 1, 2, \dots\}$ with transition probabilities

$$K_u(a, b) = \frac{u^b q^{b^2} (q)_a (uq)_a}{(q)_{a-b} (q)_b (uq)_b}$$

for $a \geq b$. All other transition probabilities vanish. (We assume that $0 < q < 1$ and $0 < u \leq 1$.) Note we have adopted our change of notation and also indicated the dependence on the parameter u .

We run this Markov chain starting at an initial state $\lambda_0 = a$ to produce a sequence of states $\lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \dots$. With probability 1, $\lambda_n = 0$ for all sufficiently large n , so the λ_i form a partition. By considering $\Pr(\lambda_k = 0)$ in the cases $u = 1$ and $u = q$ respectively, and letting $a \rightarrow \infty$, Fulman proves the $i = k$ and $i = 1$ cases of (1). To prove (1) in general we effectively run the Markov chain a fixed number s of times with $u = 1$ then run it indefinitely with $u = q$.

Instead of K_u we look at the related lower triangular matrix M_u with entries

$$M_u(a, b) = \frac{u^b q^{b^2}}{(q)_{a-b}}$$

for $a \geq b$. We consider the $(a, 0)$ entry of $M_1^s M_q^{k-s}$, where $0 \leq s < k$, and let $a \rightarrow \infty$. This entry is

$$\sum_{a \geq \lambda_1 \geq \lambda_2 \cdots \geq \lambda_{k-1}} \frac{q^{\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_{k-1}^2 + \lambda_{s+1} + \lambda_{s+2} + \cdots + \lambda_{k-1}}}{(q)_{a-\lambda_1} (q)_{\lambda_1 - \lambda_2} \cdots (q)_{\lambda_{k-2} - \lambda_{k-1}} (q)_{\lambda_{k-1}}}$$

whose limit, as $a \rightarrow \infty$, is

$$Q(s, k) = \frac{1}{(q)_\infty} \sum_{\lambda_1 \geq \lambda_2 \cdots \geq \lambda_{k-1}} \frac{q^{\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_{k-1}^2 + \lambda_{s+1} + \lambda_{s+2} + \cdots + \lambda_{k-1}}}{(q)_{\lambda_1 - \lambda_2} \cdots (q)_{\lambda_{k-2} - \lambda_{k-1}} (q)_{\lambda_{k-1}}} = \frac{AGS(s+1, k)}{(q)_\infty}.$$

Fulman [4, Corollary 1 to Theorem 3] derives a formula equivalent to

$$M_u^r(a, b) = \sum_{n=b}^a \frac{(-1)^{n-b} u^{rn} q^{rn^2 + \binom{n-b}{2}} (1 - uq^{2n})(uq)_{n+b-1}}{(q)_{a-n} (q)_{n-b} (uq)_{a+n}}. \quad (2)$$

We need to consider the case $b = 0$ separately since the numerator of the $n = 0$ term in (2) acquires a factor $(uq)_{-1} = 1/(1-u)$ which makes no sense when we substitute $u = 1$. We have

$$\begin{aligned} M_u^r(a, 0) &= \frac{(1-u)(uq)_{-1}}{(q)_a (uq)_a} + \sum_{n=1}^a \frac{(-1)^n u^{rn} q^{rn^2 + \binom{n}{2}} (1 - uq^{2n})(uq)_{n-1}}{(q)_{a-n} (q)_n (uq)_a} \\ &= \frac{1}{(q)_a (uq)_a} + \sum_{n=1}^a \frac{(-1)^n u^{rn} q^{rn^2 + \binom{n}{2}} (1 - uq^{2n})(uq)_{n-1}}{(q)_{a-n} (q)_n (uq)_a}. \end{aligned}$$

We can substitute $u = 1$ in this expression without difficulty to get

$$M_1^r(a, 0) = \frac{1}{(q)_a^2} + \sum_{n=1}^a \frac{(-1)^n q^{rn^2 + \binom{n}{2}} (1 - q^{2n})(q)_{n-1}}{(q)_{a-n} (q)_n (q)_a}.$$

We have, for $a > 0$,

$$\begin{aligned} &M_1^s M_q^{k-s}(a, 0) \\ &= M_1^s(a, 0) M_q^{k-s}(0, 0) + \sum_{c=1}^a M_1^s(a, c) M_q^{k-s}(c, 0) \\ &= \frac{1}{(q)_a^2} + \sum_{\substack{c, m, n \\ a \geq n \geq c \geq m \geq 0, n > 0}} \frac{(-1)^{n-c} q^{sn^2 + \binom{n-c}{2}} (1 - q^{2n})(q)_{n+c-1}}{(q)_{a-n} (q)_{n-c} (q)_{a+n}} \\ &\quad \times \frac{(-1)^m q^{(k-s)(m^2+m) + \binom{m}{2}} (1 - q^{2m+1})(q^2)_{m-1}}{(q)_{c-m} (q)_m (q^2)_{c+m}} \\ &= \frac{1}{(q)_a^2} + \sum_{\substack{m, n \\ a \geq n \geq m \geq 0, n > 0}} \frac{(-1)^m q^{sn^2 + (k-s)(m^2+m) + \binom{m}{2}} (1 - q^{2n})(1 - q^{2m+1})}{(q)_{a-n} (q)_{a+n}} S(m, n) \end{aligned}$$

where

$$S(m, n) = \sum_{c=m}^n \frac{(-1)^{n-c} q^{\binom{n-c}{2}} (q)_{n+c-1}}{(q)_{n-c} (q)_{c-m} (q)_{c+m+1}}.$$

I claim that $S(m, n) = 0$ if $m - n \geq 2$. In this case

$$\begin{aligned} S(m, n) &= \frac{1}{(q)_{n-m}} \sum_{c=m}^n (-1)^{n-c} q^{\binom{n-c}{2}} \begin{bmatrix} n-m \\ n-c \end{bmatrix} (q^{c+m+2})_{n-m-2} \\ &= \frac{1}{(q)_{n-m}} \sum_{i=0}^{m-n} (-1)^i q^{\binom{i}{2}} \begin{bmatrix} n-m \\ i \end{bmatrix} (q^{n+m-i+2})_{n-m-2} \\ &= \frac{1}{(q)_{n-m}} \sum_{i=0}^{m-n} (-1)^i q^{\binom{i}{2}} \begin{bmatrix} n-m \\ i \end{bmatrix} f(q^{-i}) \end{aligned}$$

where

$$f(x) = (1 - xq^{n+1})(xq^{n+m+2})_{n-m-2}$$

is a polynomial of degree $n - m - 1$ in x . This vanishes due to the following lemma.

Lemma 1 *Let p be a positive integer, and $f(x)$ be a polynomial of degree $< p$ in x . Then*

$$\sum_{i=0}^p (-1)^i q^{\binom{i}{2}} \begin{bmatrix} p \\ i \end{bmatrix} f(q^{-i}) = 0.$$

Proof It suffices to prove this for $f(x) = x^a$ where $0 \leq a < p$. But then the sum equals

$$\sum_{i=0}^p (-1)^i q^{\binom{i}{2}} \begin{bmatrix} p \\ i \end{bmatrix} q^{-ia} = \prod_{j=0}^{p-1} (1 - q^{j-a}) = 0$$

by the q -binomial theorem [3, Theorem 3.3]. □

We now need to compute $S(m, m)$ for $m > 0$, and $S(m, m+1)$. Firstly

$$S(m, m) = \frac{(q)_{2m-1}}{(q)_{2m+1}} = \frac{1}{(1 - q^{2m})(1 - q^{2m+1})}.$$

Then

$$\begin{aligned} S(m, m+1) &= \frac{(q)_{2m+1}}{(q)_1 (q)_{2m+2}} - \frac{(q)_{2m}}{(q)_1 (q)_{2m+1}} \\ &= \frac{1}{1-q} \left[\frac{1}{1-q^{2m+2}} - \frac{1}{1-q^{2m+1}} \right] \\ &= \frac{-q^{2m+1}}{(1-q^{2m+1})(1-q^{2m+2})}. \end{aligned}$$

Thus

$$\begin{aligned}
Q(s, k) &= \lim_{a \rightarrow \infty} M_1^s M_u^{k-s}(a, 0) \\
&= \frac{1}{(q)_\infty^2} + \sum_{m=1}^{\infty} \frac{(-1)^m q^{sm^2 + (k-s)(m^2+m) + \binom{m}{2}}}{(q)_\infty^2} \\
&\quad - \sum_{m=0}^{\infty} \frac{(-1)^m q^{s(m+1)^2 + (k-s)(m^2+m) + \binom{m}{2} + 2m+1}}{(q)_\infty^2}.
\end{aligned}$$

Replacing m by $-1 - m$ in this last sum yields

$$\begin{aligned}
Q(s, k) &= \sum_{m=-\infty}^{\infty} \frac{(-1)^m q^{(k+1/2)m^2 + (k-s-1/2)m}}{(q)_\infty^2} \\
&= \frac{1}{(q)_\infty^2} \prod_{j=0}^{\infty} (1 - q^{(j-1)(2k+1)+s+1})(1 - q^{j(2k+1)-s-1})(1 - q^{j(2k+1)}) \\
&= \frac{AGP(s, k)}{(q)_\infty},
\end{aligned}$$

using the Jacobi triple product [3, Theorem 2.8]. Hence

$$AGS(s+1, k) = AGP(s+1, k),$$

which proves (1) in general, since $1 \leq s+1 \leq k$.

References

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