

# Combinatorial proofs of $q$ -series identities

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## 1 Introduction

We provide combinatorial proofs of six of the ten  $q$ -series identities listed in [3, Theorem 3]. Andrews, Jiménez-Urroz and Ono prove these identities using formal manipulation of identities arising in the theory of basic hypergeometric series. Our proofs are purely combinatorial, based on interpreting both sides of the identities as generating functions for certain partitions. One of these identities arose in the work of Zagier [6] and the author has already given a combinatorial proof of the same [4].

## 2 Notation

For convenience we summarize the notations we use for partitions and sets of partitions.

The generic partition is denoted by  $\lambda$ . The number partitioned by  $\lambda$  is  $N_\lambda$ . The parts of  $\lambda$  are  $\lambda_1 \geq \lambda_2 \geq \dots$ . In particular the largest part of  $\lambda$  is  $\lambda_1$ . We let  $n_\lambda$  denote the number of parts in  $\lambda$ . Also we let  $n_\lambda(d)$  be the number of parts in  $\lambda$  which equal  $d$ , so that  $n_\lambda = \sum_d n_\lambda(d)$ . We let  $d_\lambda$  denote the number of distinct parts in  $\lambda$ .

The set of all partitions is denoted by  $\mathcal{P}$ . The set of all partitions into distinct parts is denoted by  $\mathcal{D}$ .

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### 3 The identities

Each of the identities in Theorem 3 of [3] has the following structure:

$$\sum_{N=0}^{\infty} \left[ \prod_{j=1}^{\infty} b_j(q) - \prod_{j=1}^N b_j(q) \right] = \prod_{j=1}^{\infty} b_j(q) \sum_{d=1}^{\infty} c_d(q) + G(q). \quad (1)$$

Here the  $b_j(q)$  are power series in  $q$  tending  $q$ -adically to 1 as  $j \rightarrow \infty$ , the  $c_d(q)$  are power series in  $q$  tending  $q$ -adically to 0 as  $d \rightarrow \infty$  and  $G(q)$  is an explicitly given power series. Our strategy is to interpret

$$\sum_{N=0}^{\infty} \left[ \prod_{j=1}^{\infty} b_j(q) - \prod_{j=1}^N b_j(q) \right] \quad \text{and} \quad \prod_{j=1}^{\infty} b_j(q) \sum_{d=1}^{\infty} c_d(q)$$

as weighted generating functions for two classes of partitions, and to give a weight preserving “almost-bijection” between the two classes. There may be exceptional partitions which are not paired under the almost-bijection; these account for the series  $G(q)$ .

**Theorem 1** *The identity (1) is valid for the following values of  $b_j(q)$ ,  $c_d(q)$  and  $G(q)$ :*

| Case  | $b_j(q)$                      | $c_d(q)$               | $G(q)$                                                             |
|-------|-------------------------------|------------------------|--------------------------------------------------------------------|
| (i)   | $1/(1 - q^j)$                 | $q^d/(1 - q^d)$        | 0                                                                  |
| (ii)  | $(1 + q^j)/(1 - q^j)$         | $2q^d/(1 - q^{2d})$    | 0                                                                  |
| (iii) | $(1 - q^{2j-1})/(1 - q^{2j})$ | $(-1)^d q^d/(1 - q^d)$ | 0                                                                  |
| (iv)  | $1 - q^j$                     | $q^d/(1 - q^d)$        | $\sum_{r=1}^{\infty} (-1)^r [(3r - 1)q^{r(3r-1)} + 3rq^{r(3r+1)}]$ |
| (v)   | $(1 - q^j)/(1 + q^j)$         | $2q^d/(1 - q^{2d})$    | $4 \sum_{r=1}^{\infty} (-1)^r r q^{r^2}$                           |
| (vi)  | $(1 - q^{2j})/(1 - q^{2j+1})$ | $(-1)^d q^d/(1 - q^d)$ | $(1 - q) \sum_{r=1}^{\infty} r q^{r(r+1)/2}$                       |

These correspond to parts 1–3 and 8–10 of [3, Theorem 3]. The remaining parts 4–7 have a mock theta function as  $G(q)$  and they appear to lie beyond the methods of this paper.

To prove the theorem in each case we start with a rearrangement of the left side of (1).

**Lemma 1** *Let  $a_n(q)$  ( $n = 1, 2, \dots$ ) be power series in the indeterminate  $q$  with  $a_n \rightarrow 0$  in the  $q$ -adic topology. Then*

$$\sum_{N=0}^{\infty} \left[ \prod_{j=1}^{\infty} (1 + a_j(q)) - \prod_{j=1}^N (1 + a_j(q)) \right] = \sum_{n=1}^{\infty} n a_n(q) \prod_{j=1}^{n-1} (1 + a_j(q)). \quad (2)$$

**Proof** The product

$$\prod_{j=1}^{\infty} (1 + a_j(q))$$

is the sum of all terms  $a_S(q) = \prod_{j \in S} a_j(q)$  where  $S$  runs through the finite subsets of  $\mathbf{N} = \{1, 2, \dots\}$ . Similarly

$$\prod_{j=1}^N (1 + a_j(q))$$

is the sum of all terms  $a_S(q)$  where  $S$  runs through the finite subsets of  $\{1, 2, \dots, N\}$ . Hence the difference

$$\prod_{j=1}^{\infty} (1 + a_j(q)) - \prod_{j=1}^N (1 + a_j(q))$$

is the sum of all  $a_S(q)$  over all finite  $S$  having an element strictly larger than  $N$ . Each  $a_S(q)$  occurs in this difference for exactly  $\max(S)$  distinct  $N$ , where  $\max(S)$  is the largest element of  $S$ . Hence

$$\sum_{N=0}^{\infty} \left[ \prod_{j=1}^{\infty} (1 + a_j(q)) - \prod_{j=1}^N (1 + a_j(q)) \right] = \sum_{\substack{S \subseteq \mathbf{N} \\ 0 < |S| < \infty}} \max(S) a_S(q).$$

But

$$a_n(q) \prod_{j=1}^{n-1} (1 + a_j(q)) = \sum_{\substack{S \subseteq \mathbf{N} \\ \max(S)=n}} a_S(q)$$

and so (2) follows. □

### **Proof of Theorem 1**

Case (i) By Lemma 1, (1) is equivalent to

$$\sum_{n=1}^{\infty} \frac{nq^n}{1 - q^n} \prod_{j=1}^{n-1} \frac{1}{1 - q^j} = \prod_{j=1}^{\infty} \frac{1}{1 - q^j} \sum_{d=1}^{\infty} \frac{q^d}{1 - q^d}. \quad (3)$$

The coefficient of  $q^m$  in

$$\frac{q^n}{1 - q^n} \prod_{j=1}^{n-1} \frac{1}{1 - q^j}$$

is the number of partitions of  $m$  with largest part  $n$ . Thus

$$\sum_{n=1}^{\infty} \frac{nq^n}{1 - q^n} \prod_{j=1}^{n-1} \frac{1}{1 - q^j} = \sum_{\lambda \in \mathcal{P}} \lambda_1 q^{N_\lambda}.$$

Now

$$\frac{1}{1-q^d} \frac{q^d}{1-q^d} = \sum_{m=1}^{\infty} m q^{dm}$$

and so

$$\frac{q^d}{1-q^d} \prod_{j=1}^{\infty} \frac{1}{1-q^j} = \sum_{\lambda \in \mathcal{P}} n_{\lambda}(d) q^{N_{\lambda}}.$$

Hence

$$\prod_{j=1}^{\infty} \frac{1}{1-q^j} \sum_{d=1}^{\infty} \frac{q^d}{1-q^d} = \sum_{\lambda \in \mathcal{P}} n_{\lambda} q^{N_{\lambda}}.$$

As the number of parts in a partition equals the largest part of its conjugate, then

$$\sum_{\lambda \in \mathcal{P}} n_{\lambda} q^{N_{\lambda}} = \sum_{\lambda \in \mathcal{P}} \lambda_1 q^{N_{\lambda}}$$

which proves (3).

Case (ii) In this case, using Lemma 1, (1) becomes

$$\sum_{n=1}^{\infty} \frac{2nq^n}{1-q^n} \prod_{j=1}^{n-1} \frac{1+q^j}{1-q^j} = 2 \prod_{j=1}^{\infty} \frac{1+q^j}{1-q^j} \sum_{d=1}^{\infty} \frac{q^d}{1-q^{2d}}. \quad (4)$$

Note that

$$\frac{1+q^j}{1-q^j} = 1 + 2 \sum_{m=1}^{\infty} q^{mj}.$$

Then

$$\frac{2q^n}{1-q^n} \prod_{j=1}^{n-1} \frac{1+q^j}{1-q^j} = \sum_{\substack{\lambda \in \mathcal{P} \\ \lambda_1 = n}} 2^{d_{\lambda}} q^{N_{\lambda}}.$$

Hence

$$\sum_{n=1}^{\infty} \frac{2nq^n}{1-q^n} \prod_{j=1}^{n-1} \frac{1+q^j}{1-q^j} = \sum_{\lambda \in \mathcal{P}} \lambda_1 2^{d_{\lambda}} q^{N_{\lambda}}.$$

On the other side note that

$$\frac{1+q^d}{1-q^d} \frac{q^d}{1-q^{2d}} = \frac{q^d}{(1-q^d)^2} = \sum_{m=1}^{\infty} m q^{md}.$$

It follows that

$$\frac{2q^d}{1-q^{2d}} \prod_{j=1}^{\infty} \frac{1+q^j}{1-q^j} = \sum_{\lambda \in \mathcal{P}} n_{\lambda}(d) 2^{d_{\lambda}} q^{N_{\lambda}}$$

and so

$$2 \prod_{j=1}^{\infty} \frac{1+q^j}{1-q^j} \sum_{d=1}^{\infty} \frac{q^d}{1-q^{2d}} = \sum_{\lambda \in \mathcal{P}} n_{\lambda} 2^{d_{\lambda}} q^{N_{\lambda}}.$$

As the number of distinct parts in a permutation is invariant under conjugation, (4) follows. (Note that in the Ferrers diagram of a partition, the number of distinct parts is the number of *extreme* boxes—those boxes which are both at the end of a row and the bottom of a column. Their number is manifestly invariant under conjugation).

Case (iii) By Lemma 1, (1) is equivalent in this case to

$$\sum_{n=1}^{\infty} n \frac{-q^{2n-1} + q^{2n}}{1 - q^{2n}} \prod_{j=1}^{n-1} \frac{1 - q^{2j-1}}{1 - q^{2j}} = \prod_{j=1}^{\infty} \frac{1 - q^{2j-1}}{1 - q^{2j}} \sum_{d=1}^{\infty} (-1)^d \frac{q^d}{1 - q^d}. \quad (5)$$

Let  $\mathcal{P}^o$  denote the set of partitions with no repeated odd parts, and let  $n_{\lambda}^o$  denote the number of odd parts of  $\lambda$ . Then

$$\prod_{j=1}^{\infty} \frac{1 - q^{2j-1}}{1 - q^{2j}} = \sum_{\lambda \in \mathcal{P}^o} (-1)^{n_{\lambda}^o} q^{N_{\lambda}}$$

while

$$\prod_{j=1}^n \frac{1 - q^{2j-1}}{1 - q^{2j}} = \sum_{\substack{\lambda \in \mathcal{P}^o \\ \lambda_1 \leq 2n}} (-1)^{n_{\lambda}^o} q^{N_{\lambda}}.$$

It follows that

$$\frac{-q^{2n-1} + q^{2n}}{1 - q^{2n}} \prod_{j=1}^{n-1} \frac{1 - q^{2j-1}}{1 - q^{2j}} = \sum_{\substack{\lambda \in \mathcal{P}^o \\ \lambda_1 \in \{2n-1, 2n\}}} (-1)^{n_{\lambda}^o} q^{N_{\lambda}}.$$

Hence

$$\sum_{n=1}^{\infty} n \frac{-q^{2n-1} + q^{2n}}{1 - q^{2n}} \prod_{j=1}^{n-1} \frac{1 - q^{2j-1}}{1 - q^{2j}} = \sum_{\lambda \in \mathcal{P}^o} \lceil \lambda_1/2 \rceil (-1)^{n_{\lambda}^o} q^{N_{\lambda}}.$$

Let us consider the other side. For odd  $d$ ,

$$\frac{(-1)^d q^d}{1 - q^d} (1 - q^d) = -q^d$$

and for even  $d$ ,

$$\frac{(-1)^d q^d}{1 - q^d} \frac{1}{1 - q^d} = \frac{q^d}{(1 - q^d)^2} = \sum_{m=1}^{\infty} m q^{md}.$$

In all cases then

$$\frac{(-1)^d q^d}{1 - q^d} \prod_{j=1}^{\infty} \frac{1 - q^{2j-1}}{1 - q^{2j}} = \sum_{\lambda \in \mathcal{P}^o} n_{\lambda}(d) (-1)^{n_{\lambda}^o} q^{N_{\lambda}}$$

and so

$$\prod_{j=1}^{\infty} \frac{1 - q^{2j-1}}{1 - q^{2j}} \sum_{d=1}^{\infty} \frac{(-1)^d q^d}{1 - q^d} = \sum_{\lambda \in \mathcal{P}^o} n_{\lambda} (-1)^{n_{\lambda}^o} q^{N_{\lambda}}.$$

The desired identity will follow when we construct an involution  $\sigma$  on  $\mathcal{P}^o$  with the properties that  $\sigma$  preserves  $N_{\lambda}$  while interchanging  $n_{\lambda}$  and  $\lceil \lambda_1/2 \rceil$ .

To construct  $\sigma$  we construct a *diagram* for each  $\lambda \in \mathcal{P}^o$ . As an example let  $\lambda = (8, 7, 5, 4, 4, 3, 2, 2, 2, 1)$ . Each part of  $\lambda$  will yield a row in the diagram. An even part  $2k$  will give a row of  $k$  2s, while an odd part  $2k + 1$  will give a row of  $k$  2s followed by a 1. This particular  $\lambda$  gives the diagram

$$\begin{array}{ccccccc} 2 & 2 & 2 & 2 & & & \\ 2 & 2 & 2 & 1 & & & \\ 2 & 2 & 1 & & & & \\ 2 & 2 & & & & & \\ 2 & 2 & & & & & \\ 2 & & & & & & \\ 2 & & & & & & \\ 2 & & & & & & \\ 1 & & & & & & \end{array}.$$

The diagram determines the partition via its row sums. It is a Ferrers diagram with the boxes replaced by 2s and 1s. The 1s can only occur at the end of rows, and for  $\lambda \in \mathcal{P}^o$  they can only occur at the bottom of columns since no odd part is repeated. Any such diagram of 2s and 1s with the 1s occurring only in extreme boxes comes from a partition in  $\mathcal{P}_{\lambda}^o$ . We identify elements of  $\mathcal{P}_{\lambda}^o$  with their diagrams. Now we define  $\sigma$  to be conjugation of diagrams. The number of rows in the diagram of  $\lambda$  is  $n_{\lambda}$  while the number of columns is  $\lceil \lambda_1/2 \rceil$ . Thus  $\sigma$  has the required properties and (5) is proved.

Case (iv) This proof appeared in [4]. We outline it for completeness. In this case, using Lemma 1, (1) becomes

$$\begin{aligned} & \sum_{n=1}^{\infty} -nq^n \prod_{j=1}^{n-1} (1 - q^j) - \prod_{j=1}^{\infty} (1 - q^j) \sum_{d=1}^{\infty} \frac{q^d}{1 - q^d} \\ &= \sum_{r=1}^{\infty} [(3r - 1)q^{r(3r-1)/2} + 3rq^{r(3r+1)/2}]. \end{aligned} \quad (6)$$

Then

$$-q^n \prod_{j=1}^{n-1} (1 - q^j) = \sum_{\substack{\lambda \in \mathcal{D} \\ \lambda_1 = n}} (-1)^{n_{\lambda}} q^{N_{\lambda}}$$

and so

$$\sum_{n=1}^{\infty} -nq^n \prod_{j=1}^{n-1} (1 - q^j) = \sum_{\lambda \in \mathcal{D}} \lambda_1 (-1)^{n_\lambda} q^{N_\lambda}.$$

Also as

$$-\frac{q^d}{1 - q^d} (1 - q^d) = -q^d$$

then

$$\frac{-q^d}{1 - q^d} \prod_{j=1}^{\infty} (1 - q^j) = \sum_{\lambda \in \mathcal{D}} n_\lambda (d) (-1)^{n_\lambda} q^{N_\lambda}$$

and so

$$-\prod_{j=1}^{\infty} (1 - q^j) \sum_{d=1}^{\infty} \frac{q^d}{1 - q^d} = \sum_{\lambda \in \mathcal{D}} n_\lambda (-1)^{n_\lambda} q^{N_\lambda}.$$

Hence

$$\sum_{n=1}^{\infty} -nq^n \prod_{j=1}^{n-1} (1 - q^j) - \prod_{j=1}^{\infty} (1 - q^j) \sum_{d=1}^{\infty} \frac{q^d}{1 - q^d} = \sum_{\lambda \in \mathcal{D}} (\lambda_1 + n_\lambda) (-1)^{n_\lambda} q^{N_\lambda}.$$

We now apply Franklin's involution ([5]). For this purpose a partition  $\lambda \in \mathcal{D}$  is *exceptional* if either it is empty, or has the form  $(2r-1, 2r-2, \dots, r+1, r)$  or  $(2r, 2r-1, \dots, r+2, r+1)$  for some positive integer  $r$ . Franklin's involution  $\sigma$  is defined on the nonexceptional partitions in  $\mathcal{D}$ . For  $\lambda \in \mathcal{D}$  let  $s_\lambda$  be its smallest part, and  $t_\lambda$  be the largest integer  $t$  with  $\lambda_t = \lambda_1 + 1 - t > 0$ . Define  $\sigma(\lambda)$  for non-exceptional  $\lambda \in \mathcal{D}$  as follows: if  $s_\lambda \leq t_\lambda$  remove the smallest part from  $\lambda$  and add 1 to each of its  $s_\lambda$  largest parts, while if  $s_\lambda > t_\lambda$  subtract one from each of its  $t_\lambda$  largest parts and create a new part  $t_\lambda$ . The map  $\sigma$  is an involution. It negates  $(-1)^{n_\lambda}$  and it preserves  $N_\lambda$  and  $\lambda_1 + n_\lambda$ . The sum of  $(-1)^{n_\lambda} q^{N_\lambda}$  (respectively  $(\lambda_1 + n_\lambda) (-1)^{n_\lambda} q^{N_\lambda}$ ) over all  $\lambda \in \mathcal{D}$  thus equals the sum over the exceptional  $\lambda$ . We recover the pentagonal number identity

$$\prod_{j=1}^{\infty} (1 - q^j) = \sum_{\lambda \in \mathcal{D}} (-1)^{n_\lambda} q^{N_\lambda} = 1 + \sum_{r=1}^{\infty} (-1)^r [q^{r(3r-1)/2} + q^{r(3r+1)/2}]$$

and the identity

$$\sum_{\lambda \in \mathcal{D}} (\lambda_1 + n_\lambda) (-1)^{n_\lambda} q^{N_\lambda} = \sum_{r=1}^{\infty} (-1)^r [(3r-1)q^{r(3r-1)/2} + 3rq^{r(3r+1)/2}]$$

which we have seen is equivalent to (6).

Case (v) By Lemma 1 (1) is equivalent to

$$\sum_{n=1}^{\infty} \frac{-2nq^n}{1 + q^n} \prod_{j=1}^{n-1} \frac{1 - q^j}{1 + q^j} - 2 \prod_{j=1}^{\infty} \frac{1 - q^j}{1 + q^j} \sum_{d=1}^{\infty} \frac{q^d}{1 - q^{2d}} = 4 \sum_{r=1}^{\infty} (-1)^r r q^{r^2}. \quad (7)$$

Now

$$\frac{1 - q^j}{1 + q^j} = 1 + 2 \sum_{m=1}^{\infty} (-1)^m q^{mj}$$

and

$$\frac{-2q^n}{1 + q^n} = 2 \sum_{m=1}^{\infty} (-1)^m q^{mn}.$$

Hence

$$\frac{-2q^n}{1 + q^n} \prod_{j=1}^{n-1} \frac{1 - q^j}{1 + q^j} = \sum_{\substack{\lambda \in \mathcal{P} \\ \lambda_1 = n}} (-1)^{n_\lambda} 2^{d_\lambda} q^{N_\lambda}$$

and so

$$\sum_{n=1}^{\infty} \frac{-2nq^n}{1 + q^n} \prod_{j=1}^{n-1} \frac{1 - q^j}{1 + q^j} = \sum_{\lambda \in \mathcal{P}} \lambda_1 (-1)^{n_\lambda} 2^{d_\lambda} q^{N_\lambda}.$$

On the other hand

$$\frac{-2q^d}{1 - q^{2d}} \frac{1 - q^d}{1 + q^d} = \frac{-2q^d}{(1 + q^d)^2} = 2 \sum_{m=1}^{\infty} (-1)^m m q^{md}$$

so that

$$\frac{-2q^d}{1 - q^{2d}} \prod_{j=1}^{\infty} \frac{1 - q^j}{1 + q^j} = \sum_{\lambda \in \mathcal{P}} n_\lambda(d) (-1)^{n_\lambda} 2^{d_\lambda} q^{N_\lambda}$$

and then

$$-2 \prod_{j=1}^{\infty} \frac{1 - q^j}{1 + q^j} \sum_{d=1}^{\infty} \frac{q^d}{1 - q^{2d}} = \sum_{\lambda \in \mathcal{P}} n_\lambda (-1)^{n_\lambda} 2^{d_\lambda} q^{N_\lambda}.$$

We need to prove that

$$\sum_{\lambda \in \mathcal{P}} (\lambda_1 + n_\lambda) (-1)^{n_\lambda} 2^{d_\lambda} q^{N_\lambda} = 4 \sum_{r=1}^{\infty} (-1)^r r q^{r^2}.$$

Declare a partition *exceptional* if it is either empty or is “square”, consisting of exactly  $r$  parts all equal to  $r$ . The sum of  $(\lambda_1 + n_\lambda) (-1)^{n_\lambda} 2^{d_\lambda} q^{N_\lambda}$  over the exceptional  $\lambda$  is  $4 \sum_{r=1}^{\infty} (-1)^r r q^{r^2}$ , so all we need to show is that the sum over the nonexceptional  $\lambda$  vanishes. This time there is no convenient involution, so we need a more subtle approach.

Let  $s_\lambda$  be the smallest part of the partition  $\lambda$  and let  $s'_\lambda$  be the smallest part of its conjugate. In other words  $s'_\lambda$  is the number of parts in  $\lambda$  equal to  $\lambda_1$ . Suppose that  $\lambda$  is not exceptional. If  $s_\lambda \leq s'_\lambda$  define the *right neighbour* of  $\lambda$  as the partition formed by removing the smallest part of  $\lambda$  and adding 1 to its  $s_\lambda$  largest parts. If  $s_\lambda \geq s'_\lambda$  define the *left neighbour* of  $\lambda$  as the partition formed by subtracting 1 from its  $s'_\lambda$  largest parts and creating a

new part  $s'_\lambda$ . If  $\lambda'$  is the left neighbour of  $\lambda$  then  $\lambda$  is the right neighbour of  $\lambda'$  and conversely. Each nonexceptional  $\lambda$  has one or two neighbours. If we define a graph  $G$  with vertex set the non-exceptional partitions and joining each partition to its neighbours then each vertex in  $G$  has degree 1 or 2. Also  $G$  is acyclic since moving to a partition's right neighbour decreases  $n_\lambda$  by 1. Thus the graph  $G$  is a disjoint union of paths of length at least 2. For example the following partitions form a path moving from left to right:  $(4,4,4,3,2,2,2)$ ,  $(5,5,4,3,2,2)$ ,  $(6,6,4,3,2)$  and  $(7,7,4,3)$ . In each path  $N_\lambda$  and  $\lambda_1 + n_\lambda$  are constant while the  $(-1)^{n_\lambda}$  alternate in sign. We also see that if one endpoint has  $d$  distinct parts, then all non-endpoints have  $d+1$  distinct parts and the other endpoint has  $d$  distinct parts. Thus the sums of  $(-1)^{n_\lambda} 2^{d_\lambda}$  and  $(\lambda_1 + n_\lambda)(-1)^{n_\lambda} 2^{d_\lambda}$  over each path vanish. Hence the sums of  $(-1)^{n_\lambda} 2^{d_\lambda} q^{N_\lambda}$  and  $(\lambda_1 + n_\lambda)(-1)^{n_\lambda} 2^{d_\lambda} q^{N_\lambda}$  over the nonexceptional partitions vanish. We thus recover the classical theta function factorization

$$\prod_{j=1}^{\infty} \frac{1 - q^j}{1 + q^j} = \sum_{\lambda \in \mathcal{P}} (-1)^{n_\lambda} 2^{d_\lambda} q^{N_\lambda} = 1 + 2 \sum_{r=1}^{\infty} (-1)^r r q^{r^2}.$$

the identity

$$\sum_{\lambda \in \mathcal{P}} (\lambda_1 + n_\lambda) (-1)^{n_\lambda} 2^{d_\lambda} q^{N_\lambda} = 4 \sum_{r=1}^{\infty} (-1)^r r q^{r^2}$$

which is equivalent to (7).

A different combinatorial proof of this theta function factorization appears in [2].

Case (vi) In this last case, applying Lemma 1, (1) becomes

$$\begin{aligned} \frac{1}{1-q} \sum_{n=1}^{\infty} n \frac{-q^{2n} + q^{2n+1}}{1 - q^{2n+1}} \prod_{j=1}^{n-1} \frac{1 - q^{2j}}{1 - q^{2j+1}} &= \prod_{j=1}^{\infty} \frac{1 - q^{2j}}{1 - q^{2j-1}} \sum_{d=1}^{\infty} (-1)^d \frac{q^d}{1 - q^d} \\ &= \sum_{r=1}^{\infty} r q^{r(r+1)/2}. \end{aligned} \quad (8)$$

Let  $\mathcal{P}^e$  denote the set of partitions with no repeated even part and  $n_\lambda^e$  the number of even parts in the partition  $\lambda$ . Then

$$\begin{aligned} \frac{1}{1-q} \prod_{j=1}^{\infty} \frac{1 - q^{2j}}{1 - q^{2j+1}} &= \sum_{\lambda \in \mathcal{P}^e} (-1)^{n_\lambda^e} q^{N_\lambda}, \\ \frac{1}{1-q} \prod_{j=1}^n \frac{1 - q^{2j}}{1 - q^{2j+1}} &= \sum_{\substack{\lambda \in \mathcal{P}^e \\ \lambda_1 \leq 2n+1}} (-1)^{n_\lambda^e} q^{N_\lambda} \end{aligned}$$

and

$$\frac{1}{1-q} \frac{-q^{2n} + q^{2n+1}}{1-q^{2n+1}} \prod_{j=1}^{n-1} \frac{1-q^{2j}}{1-q^{2j+1}} = \sum_{\substack{\lambda \in \mathcal{P}^e \\ \lambda_1 \in \{2n, 2n+1\}}} (-1)^{n_\lambda^\varepsilon} q^{N_\lambda}.$$

Consequently

$$\frac{1}{1-q} \sum_{n=1}^{\infty} n \frac{-q^{2n} + q^{2n+1}}{1-q^{2n+1}} \prod_{j=1}^{n-1} \frac{1-q^{2j}}{1-q^{2j+1}} = \sum_{\lambda \in \mathcal{P}^e} \lfloor \lambda_1/2 \rfloor (-1)^{n_\lambda^\varepsilon} q^{N_\lambda}.$$

On the other hand for  $d$  even

$$-\frac{(-1)^d q^d}{1-q^d} (1-q^d) = -q^d$$

while for  $d$  odd

$$-\frac{(-1)^d q^d}{1-q^d} \frac{1}{1-q^d} = \frac{q^d}{(1-q^d)^2} = \sum_{m=1}^{\infty} m q^{md}.$$

Hence

$$-(-1)^d \frac{q^d}{1-q^d} \prod_{j=1}^{\infty} \frac{1-q^{2j}}{1-q^{2j-1}} = \sum_{\lambda \in \mathcal{P}^e} n_\lambda(d) (-1)^{d_\lambda^\varepsilon} q^{N_\lambda}$$

and so

$$-\prod_{j=1}^{\infty} \frac{1-q^{2j}}{1-q^{2j-1}} \sum_{d=1}^{\infty} (-1)^d \frac{q^d}{1-q^d} = \sum_{\lambda \in \mathcal{P}^e} n_\lambda (-1)^{d_\lambda^\varepsilon} q^{N_\lambda}.$$

We thus need to prove that

$$\sum_{\lambda \in \mathcal{P}^e} (\lfloor \lambda_1/2 \rfloor + n_\lambda) (-1)^{d_\lambda^\varepsilon} q^{N_\lambda} = \sum_{r=1}^{\infty} r q^{r(r+1)/2}.$$

We define  $\lambda \in \mathcal{P}^e$  to be *exceptional* if it is empty or it has the form  $((2s-1)^s)$  or  $((2s+1)^s)$  for some positive integer  $s$ . The sum of  $(\lfloor \lambda_1/2 \rfloor + n_\lambda) (-1)^{d_\lambda^\varepsilon} q^{N_\lambda}$  over the exceptional  $\lambda$  is  $\sum_{r=1}^{\infty} r q^{r(r+1)/2}$ . We must show that the corresponding sum over the nonexceptional  $\lambda$  vanishes, and for this we need to define a suitable involution  $\sigma$  on the set of nonexceptional  $\lambda \in \mathcal{P}^e$ .

We represent each  $\lambda \in \mathcal{P}^e$  by a *diagram*. The parts of  $\lambda$  in descending order determine the rows of the diagram in descending order. An odd part  $2k+1$  becomes a row consisting of 1 followed by  $k$  2s. An even part  $2k$  becomes a row consisting of 1 followed by  $k-1$  2s and ending with a 1. For

instance (9, 8, 5, 5, 5, 4, 3, 2, 1, 1) becomes

$$\begin{array}{cccccc}
1 & 2 & 2 & 2 & 2 & \\
1 & 2 & 2 & 2 & 1 & \\
1 & 2 & 2 & & & \\
1 & 2 & 2 & & & \\
1 & 2 & 2 & & & \\
1 & 2 & 1 & & & \\
1 & 2 & & & & \\
1 & 1 & & & & \\
1 & & & & & \\
1 & & & & & 
\end{array}
.$$

A diagram is a Ferrers diagram filled with 1s and 2s. The left column is all 1s while the remainder of the diagram is filled with 2s apart from the extreme boxes, which are filled with either 1s or 2s. (We cannot have a 1 in any column but the first except at the bottom as there are no repeated even parts). The Ferrers diagram has  $n_\lambda$  rows and  $1 + \lfloor \lambda_1/2 \rfloor$  columns. Let  $s_\lambda$  be the smallest part of the partition  $\lambda$  and if  $\lambda \in \mathcal{P}^e$  let  $s''_\lambda$  be the sum of the final column in the associated diagram. We define  $\sigma(\lambda)$  as follows: if  $s_\lambda < s''_\lambda$  or if  $s_\lambda = s''_\lambda$  is even remove the last part of  $\lambda$  and add a new final column with sum  $s_\lambda$  to the diagram of  $\lambda$ . We then get the diagram of the partition  $\sigma(\lambda)$ . If  $s_\lambda > s''_\lambda$  or if  $s_\lambda = s''_\lambda$  is odd remove the last column of the diagram of  $\lambda$  and create a new final row with sum  $s''_\lambda$ . Again, this is the diagram of the partition  $\sigma(\lambda)$ . We can check that  $\sigma$  is an involution preserving  $N_\lambda$  and negating  $(-1)^{n_\lambda}$ . As  $\sigma$  preserves the sum of the number of rows and the number of columns in the diagram it also preserves  $\lfloor \lambda_1/2 \rfloor + n_\lambda$ . We thus recover the classical theta function factorization

$$\prod_{j=1}^{\infty} \frac{1 - q^{2j}}{1 - q^{2j-1}} = \sum_{\lambda \in \mathcal{P}^e} (-1)^{n_\lambda} q^{N_\lambda} = \sum_{r=0}^{\infty} q^{r(r+1)/2}$$

and our desired identity

$$\sum_{\lambda \in \mathcal{P}^e} (\lfloor \lambda_1/2 \rfloor + n_\lambda) (-1)^{n_\lambda} q^{N_\lambda} = \sum_{r=1}^{\infty} r q^{r(r+1)/2}$$

which is equivalent to (8).

This proof of the theta function factorization is essentially equivalent to that in [2].

□

## 4 Mock theta identities

Four of the identities in [3, Theorem 3] involve mock theta functions. The proofs of these identities appear to be beyond the scope of the present methods, but these methods do provide combinatorial interpretations of the identities. We consider one case in detail: the fourth identity. This identity is equivalent to the following identity from Ramanujan's "Lost" Notebook which was proved by Andrews [1]:

$$\sum_{N=0}^{\infty} \left[ \prod_{j=1}^{\infty} (1 + q^j) - \prod_{j=1}^N (1 + q^j) \right] = \prod_{j=1}^{\infty} (1 + q^j) \left[ -\frac{1}{2} + \sum_{d=1}^{\infty} \frac{q^d}{1 - q^d} \right] + \frac{1}{2} \sum_{n=0}^{\infty} \prod_{j=1}^n \frac{q^j}{(1 + q^j)}. \quad (9)$$

By Lemma 1

$$\sum_{N=0}^{\infty} \left[ \prod_{j=1}^{\infty} (1 + q^j) - \prod_{j=1}^N (1 + q^j) \right] = \sum_{n=1}^{\infty} n q^n \prod_{j=1}^{n-1} (1 + q^j).$$

But

$$\sum_{n=1}^{\infty} q^n \prod_{j=1}^{n-1} (1 + q^j) = \sum_{\substack{\lambda \in \mathcal{D} \\ \lambda_1 = n}} q^{N_\lambda}$$

and so

$$\sum_{n=1}^{\infty} n q^n \prod_{j=1}^{n-1} (1 + q^j) = \sum_{\lambda \in \mathcal{D}} \lambda_1 q^{N_\lambda}.$$

Next

$$\frac{q^d}{1 - q^d} (1 + q^d) = q^d + 2 \sum_{m=2}^{\infty} q^{md}.$$

It follows that

$$\frac{q^d}{1 - q^d} \prod_{j=1}^{\infty} (1 + q^j) = \sum_{\lambda \in \mathcal{D}_{d,1}} q^{N_\lambda} + \sum_{\lambda \in \mathcal{D}_{d,2+}} q^{N_\lambda}$$

where  $\mathcal{D}_{d,1}$  is the set of  $\lambda \in \mathcal{D}$  having  $d$  as a part while  $\mathcal{D}_{d,2+}$  is the set of partitions  $\lambda$  having at least two occurrences of the part  $d$  but with no other repeated parts. It follows that

$$\prod_{j=1}^{\infty} (1 + q^j) \frac{q^d}{1 - q^d} = \sum_{\lambda \in \mathcal{D}} n_\lambda q^{N_\lambda} + 2 \sum_{\lambda \in \mathcal{D}_{(1)}} q^{N_\lambda}$$

where  $\mathcal{D}_{(1)}$  denotes the set of partitions having exactly one part with multiplicity at least 2. Of course

$$\prod_{j=1}^{\infty} (1 + q^j) = \sum_{\lambda \in \mathcal{D}} q^{N_\lambda}.$$

For each  $n$

$$\prod_{j=1}^n \frac{q^j}{(1 + q^j)} = \sum_{\lambda \in \mathcal{P}_{(n)}} (-1)^{n_\lambda - n} q^{N_\lambda}$$

where  $\mathcal{P}_{(n)}$  is the set of partitions having no parts greater than  $n$  and having all numbers between 1 and  $n$  as parts. These partitions have a complicated description, but their conjugates have a nicer one:  $\lambda \in \mathcal{P}_{(n)}$  if and only if its conjugate lies in  $\mathcal{D}$  and has exactly  $n$  parts. Hence

$$\sum_{n=0}^{\infty} \prod_{j=1}^n \frac{q^j}{(1 + q^j)} = \sum_{\lambda \in \mathcal{D}} (-1)^{\lambda_1 - n_\lambda} q^{N_\lambda}.$$

Putting all this together, the identity under consideration is equivalent to

$$\sum_{\lambda \in \mathcal{D}} \left[ \lambda_1 - n_\lambda + \frac{1 - (-1)^{\lambda_1 - n_\lambda}}{2} \right] q^{N_\lambda} = 2 \sum_{\lambda \in \mathcal{D}_{(1)}} q^{N_\lambda}$$

or equivalently

$$\sum_{\lambda \in \mathcal{D}} \lceil (\lambda_1 - n_\lambda)/2 \rceil q^{N_\lambda} = \sum_{\lambda \in \mathcal{D}_{(1)}} q^{N_\lambda}.$$

For a partition  $\lambda$  the quantity  $r_\lambda = \lambda_1 - n_\lambda$  is often called its *rank*. The combinatorial interpretation of this identity is that the sum of  $\lceil r_\lambda/2 \rceil$  over all partitions  $\lambda$  of  $n$  into distinct parts equals the number of partitions of  $n$  having exactly one repeated part. For instance here are the partitions  $\lambda$  of 8 into distinct parts.

| $\lambda$ | $r_\lambda$ | $\lceil r_\lambda/2 \rceil$ |
|-----------|-------------|-----------------------------|
| 8         | 7           | 4                           |
| 71        | 5           | 3                           |
| 62        | 4           | 2                           |
| 53        | 3           | 2                           |
| 521       | 2           | 1                           |
| 431       | 1           | 1                           |

The sum of the  $\lceil r_\lambda/2 \rceil$  is 13, and there are 13 partitions of 2 in which exactly one part is repeated namely,  $61^2$ ,  $51^3$ ,  $4^2$ ,  $42^2$ ,  $421^2$ ,  $41^4$ ,  $3^2 2$ ,  $32^2 1$ ,  $321^3$ ,  $31^5$ ,  $2^4$ ,  $21^6$  and  $1^8$ . It would be interesting to have a bijective proof of this fact.

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