

Alternating sign matrices and tournaments

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1 Introduction

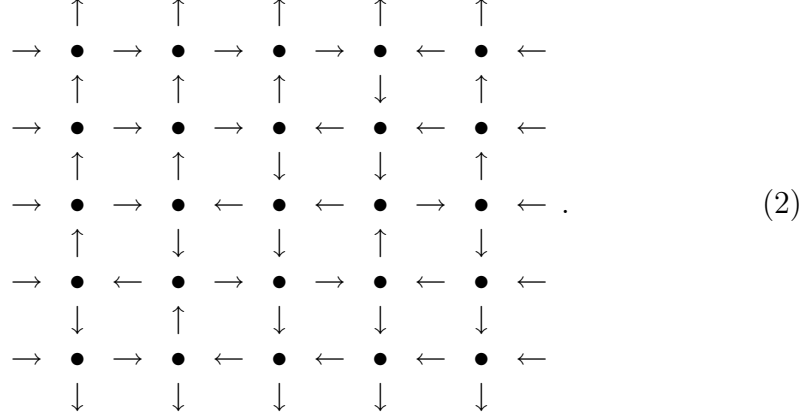
An alternating sign matrix of order n is an n -by- n matrix with entries in $\{-1, 0, 1\}$ such that each row and column has sum 1 and the nonzero entries in each row and column alternate in sign. For example

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}. \quad (1)$$

The problem of enumerating alternating sign matrices has led to many deep results and difficult conjectures. For a history of alternating sign matrices see [1].

There are several combinatorial and physical structures equivalent to alternating sign matrices. One of the most fertile has been the square-ice model originally used in statistical mechanics. The square ice graph corresponding

to (1) is



This directed graph has inward edges at the left and right and outward edges at the top and bottom. Both the in-degree and out-degree of all its degree four vertices equal two. These vertices are of six types, which we label according to their incident in-edges. *Horizontal* vertices have two horizontal in-edges, *vertical* vertices have two vertical in-edges and *northwest* vertices have in-edges pointing in the north (up) and west (left) directions. We define *southwest*, *northeast* and *southeast* vertices similarly. The correspondence between alternating sign matrices and square-ice graphs is characterized as follows: horizontal vertices correspond to entries 1, vertical vertices to entries -1 and all other vertices to entries 0.

Bressoud [2] has asked for a bijective proof of the following identity

$$\sum_{T \in \mathcal{T}_n} \lambda^{U(T)} \prod_{i=1}^n x_i^{\omega(i)} = \sum_{A \in \mathcal{A}_n} \lambda^{SW(A)} (1 + \lambda)^{V(A)} \prod_{i=1}^n x_i^{SW_i(A) + SE_i(A) + V_i(A)} \quad (3)$$

due essentially to Robbins and Rumsey [5]. Here $\mathcal{A}_n$ is the set of n -by- n alternating sign matrices, $SW(A)$, $SE(A)$ and $V(A)$ denote respectively the numbers of southwest, southeast and vertical vertices in the square-ice graph corresponding to $A \in \mathcal{A}_n$, and $SW_i(A)$, $SE_i(A)$ and $V_i(A)$ denote the numbers of vertices of the appropriate type in the i -th column of this graph. Also $\mathcal{T}_n$ is the set of tournaments on the set $\{1, 2, \dots, n\}$, that is the set of orientations on the complete graph with vertex set $\{1, 2, \dots, n\}$. Given $T \in \mathcal{T}_n$ we denote by $\omega(i)$ the out-degree of the vertex i in T , and we let $U(T)$ be the number of upsets in T , that is the number of edges oriented from i to j with $i > j$.

We give a bijective proof of the following refinement of (3):

$$\prod_{1 \leq i < j \leq n} (x_i + y_j) = \sum_{A \in \mathcal{A}_n} \prod_{i=1}^n x_i^{SE_i(A)} y_i^{SW_i(A)} (x_i + y_i)^{V_i(A)}. \quad (4)$$

This equation is equivalent to [3, Theorem 3.1]. Putting $y_i = \lambda x_i$ in (4) gives (3) since the factor $x_i + \lambda x_j$ on the left corresponds to the edge between i and j . This edge is either oriented from i to j and so contributes x_i to the product, or is oriented from j to i and so contributes λx_j to the product. The latter case corresponds to an upset and so the correct power of λ appears in the product.

2 Triangles

We use an alternative combinatorial model for alternating sign matrices, namely complete monotone triangles. A complete monotone triangle of order n is a triangular array of integers with n rows and k entries in the k -th row. Each row is strictly increasing and the final row contains the entries $1, 2, \dots, n$ in ascending order. Each entry not in the final row lies weakly between the entries immediately to the left and right in the row below. Here is an example of a complete monotone triangle of order 5:

$$\begin{array}{cccccc}
 & & & & 4 & \\
 & & & 3 & & 4 \\
 & & 2 & & 3 & & 5 \\
 & 1 & & 3 & & 4 & & 5 \\
 1 & & 2 & & 3 & & 4 & & 5
 \end{array} . \tag{5}$$

To produce a complete monotone triangle from an alternating sign matrix we form a new matrix by taking its cumulative column sums. For instance (1) has cumulative column sums

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} . \tag{6}$$

This new matrix has k 1s in the k -th row with the remaining entries 0. We form a triangle by writing in the k -th row of the triangle, the positions of the k 1s in the column sum matrix. Applying this to (6) gives (5). This gives a bijection between alternating sign matrices and complete monotone triangles of the same order.

The entries not in the bottom row of a complete monotone triangle fall into three types. Consider an entry j and let i and k be its neighbours in the row below, so that

$$\begin{array}{ccc}
 & j & \\
 i & & k
 \end{array}$$

is part of the triangle. Then either

(i) $i = j < k$,

(ii) $i < j = k$, or

(iii) $i < j < k$.

Entries of types (i), (ii) and (iii) correspond respectively to southeast, southwest and vertical vertices in column j of the associated square-ice graph. We introduce a refinement of the notion of complete monotone triangle, namely that of *oriented complete monotone triangle*. An oriented complete monotone triangle is a complete monotone triangle with an orientation (left or right) given to each entry not in the bottom row. Entries of type (i) must be oriented to the right and entries of type (ii) must be oriented to the left. Entries of type (iii) may be oriented in either direction. We shall depict an oriented complete monotone triangle by replacing each right-oriented entry j by a symbol x_j and each left-oriented entry j by a symbol y_j . Thus in the configuration

$$\begin{array}{c} x_j \\ z_k \end{array}$$

where z_k denotes either x_k , y_k or k (in the bottom row) we must have $j < k$ while in

$$\begin{array}{c} y_j \\ z_i \end{array}$$

we must have $i < j$.

As an example, one of the oriented complete monotone triangles arising from (5) is

$$\begin{array}{ccccccccc} & & & & y_4 & & & & \\ & & & & y_3 & & x_4 & & \\ & & & y_2 & & x_3 & & y_5 & \\ & x_1 & & y_3 & & y_4 & & y_5 & \\ 1 & 2 & 3 & 4 & 5 & & & & \end{array} \quad (7)$$

As the type (iii) vertices in a complete monotone triangle correspond to the -1 entries in the alternating sign matrix, an alternating sign matrix with r -1 s gives rise to 2^r oriented complete monotone triangles. Denote the set of oriented complete monotone triangles of order n by $\mathcal{O}_n$.

We define the *weight* of an oriented complete monotone triangle to be the product of the entries in the rows other than the last. Thus the weight of the triangle in (7) is $x_1 y_2 x_3 y_3^2 x_4 y_4^2 y_5^2$. It is clear that the right side of (4) is the sum of the weights of all elements of $\mathcal{O}_n$.

We define the weight of a tournament $T \in \mathcal{T}_n$ as the products of the weights of its edges where the weight of an edge oriented from i to j is x_i if $i < j$ and y_i if $i > j$. Then the left side of (4) is the sum of the weights of all elements of $\mathcal{T}_n$. We shall prove the following theorem.

Theorem 1 *There is a weight-preserving bijection between $\mathcal{O}_n$ and $\mathcal{T}_n$.*

The equation (4) is an immediate corollary. The remainder of the paper is devoted to the construction of the weight-preserving bijection $\Phi : \mathcal{O}_n \rightarrow \mathcal{T}_n$ and its inverse $\Psi : \mathcal{T}_n \rightarrow \mathcal{O}_n$.

We first indicate how to display a tournament in a triangle in a similar fashion to an oriented complete monotone triangle. We introduce new indeterminates a_{ij} and b_{ij} where $1 \leq i < j \leq n$. For $1 \leq k < n$, let $\mathcal{V}_k = \{a_{i,j}, b_{i,j} : j - i = k\}$. Also let $\mathcal{X} = \{x_i, y_j : 1 \leq i, j \leq n\}$. Each tournament corresponds to a set of $\frac{1}{2}n(n-1)$ of indeterminates in $\bigcup_{k=1}^{n-1} \mathcal{V}_k$ where we take a_{ij} when there is an edge oriented from i to j and we take b_{ij} when there is an edge oriented from j to i . We arrange these symbols in $n-1$ rows of a triangle where we put those entries in $\mathcal{V}_{n-j}$ in the j -th row in ‘ascending order’. The i -th entry in row j thus has subscript $i(i+n-j)$. We fill the n -th row with the numbers $1, 2, \dots, n$ in ascending order. For example,

$$\begin{array}{cccccc}
 & & & & a_{15} & \\
 & & & b_{14} & & b_{25} \\
 & & b_{13} & & b_{24} & & b_{35} & & \\
 & b_{12} & & b_{23} & & a_{34} & & a_{45} & \\
 1 & & 2 & & 3 & & 4 & & 5
 \end{array} \quad . \quad (8)$$

We now identify tournaments with triangles of this form. We define the bijections Φ and Ψ by taking a triangle of one type and converting it by a series of moves to one of the other type. We pass through a series of intermediate triangles which in general contain variables x_i and y_i and variables a_{ij} and b_{ij} . We define a *triangle* of order n as an n -rowed triangle with k entries in the k -th row and with $1, 2, \dots, n$ in the last row and entries drawn from $\mathcal{X} \cup \bigcup_{k=1}^{n-1} \mathcal{V}_k$ in the first $(n-1)$ -rows. Each of our primitive moves either permutes the entries of the triangle or replaces an a_{ij} by x_i or an x_i by an a_{ij} or a b_{ij} by y_j or a y_j by a b_{ij} . We define the *weight* of a triangle as the product of the weights of its entries above its bottom row where x_i , y_j , a_{ij} and b_{ij} have weights x_i , y_j , x_i and y_j respectively. This weighting agrees with that of oriented monotone triangles and tournaments, and will be preserved under each primitive move.

We interpret x_j as an outgoing edge from vertex j to some as yet unknown larger vertex. Similarly y_j represents an outgoing edge from vertex j to an

unknown smaller vertex. Also a_{ij} denotes an edge oriented from i to j ($i < j$) and b_{ij} denotes an edge oriented from j to i ($i < j$). At intermediate stages of our algorithm, the a_{ij} s and b_{ij} s that occur are still provisional, as for instance a_{ij} may be replaced by x_i and then by a_{ik} for $k \neq j$.

For convenience we suppose that our triangles have an empty “row 0” above the first row. We also use the symbols z_i to denote indifferently x_i , y_i or i (appearing in the final row) and c_{ij} to denote indifferently a_{ij} or b_{ij} .

We need to define the notion of an *admissible* triangle. A triangle with no entry in $\bigcup_{k=1}^{n-1} \mathcal{V}_k$ will be admissible if and only if it lies in $\mathcal{O}_n$ and a triangle with no entry in $\mathcal{X}$ will be admissible if and only if it lies in $\mathcal{T}_n$. At each stage of our algorithm, admissibility will be preserved.

A triangle is admissible if and only if the following conditions hold:

- (i) it has an admissible ranking,
- (ii) it has the monotone diagonal property,
- (iii) it has the monotone row property.

We explain the conditions in turn. We say that a row of a triangle is an $\mathcal{X}$ -row if all its entries lie in $\mathcal{X}$ and that it is a $\mathcal{V}$ -row if all its entries lie in some $\mathcal{V}_k$. (We regard the notional row 0 as both a $\mathcal{X}$ -row and a $\mathcal{V}$ -row). An *admissible* ranking of a triangle is an assignment of non-negative integers, called ranks, to each row of the triangle (including row 0) with the following properties:

- the n -th row has rank 0 and the $(n - 1)$ -th row has rank 1,
- if the r -th row ($r > 0$) has rank k then either row r is a $\mathcal{X}$ -row and row $r - 1$ also has rank k , or row r has all its entries in $\mathcal{V}_k$ and row $r - 1$ has rank $k + 1$.

It follows that the ranks in an admissible ranking decrease weakly as one descends the triangle and that the ranks of the $\mathcal{V}$ -rows form the sequence $d, d - 1, \dots, 2, 1$ for some $d \geq 0$. The rank of a $\mathcal{V}$ -row with entries in $\mathcal{V}_k$ is k .

If a triangle has an admissible ranking then it is plain that the ranking is uniquely determined. We call such a triangle a *ranked* triangle.

To explain the monotone diagonal and monotone row properties we need the notion of the left and right values of an entry in a ranked triangle. Let u be an entry in a ranked triangle. It is assigned a *left value* $l(u)$ and a *right value* $r(u)$. Note that these values will in general depend on the rank of the row in which u lies, not simply on the indeterminate u itself. When $u = a_{ij}$ or b_{ij} we define $l(u) = i$ and $r(u) = j$. If $u = x_i$ lies in a row with rank t

we define $l(u) = i$ and $r(u) = i + t$ while if $u = y_j$ lies in a row with rank t we define $l(u) = j - t$ and $r(u) = j$. Finally if $u = i$ is a number in the last row we let $l(u) = l(v) = i$. In all cases $r(u) - l(u)$ is the rank of the row containing u .

To define the monotone diagonal property consider three entries in a ranked triangle appearing in the following arrangement

$$\begin{array}{ccc} & v & \\ u & & w \end{array} . \quad (9)$$

An arrangement (9) has the *monotone diagonal property* if

- $l(u) \leq l(v)$ and $r(v) \leq r(w)$,
- if $u = y_j$ for some j then $l(u) < l(v)$, and
- if $w = x_j$ for some j then $r(v) < r(w)$.

A ranked triangle has the monotone diagonal property if all arrangements (9) within have the monotone diagonal property. The first of these conditions states that the left values increase weakly when traversing northeast diagonals and the right values increase weakly when traversing southeast diagonals. If the second condition is relevant, then the two rows represented in (9) have the same rank, and we can rewrite the condition as $j < r(v)$. Similarly the final condition is equivalent to $l(v) < j$.

If T is a ranked triangle having the monotone diagonal property then the left value of the j -th entry in each row is at least j and that the right value of the k -th entry from the right in each row is at most $n + 1 - k$.

We say that an arrangement (9) has the *monotone row property* if

- $l(u) < l(w)$.

We say that a ranked triangle has the *monotone row property* if every arrangement (9) within has the monotone row property.

Lemma 1 *A triangle of order n all of whose rows but the last are $\mathcal{X}$ -rows is admissible if and only if it lies in $\mathcal{O}_n$.*

Proof Suppose that $T \in \mathcal{O}_n$. Then T clearly has an admissible ranking with each row but the last having rank 1. Let us consider the arrangement (9). We divide into cases according to whether u , v and w are x s or y s or they lie in the last row. The cases where $v = y_j$ are similar to those where $v = x_j$ so we look at these in detail:

- (i) $u = x_i, w = x_k$. Then $i \leq j < k$. Hence $l(u) = i \leq j = l(v)$ and $r(v) = j + 1 < k + 1 = r(w)$. Also $l(u) = i < k = l(w)$.
- (ii) $u = x_i, w = y_k$. Then $i \leq j < k$. Hence $l(u) = i \leq j = l(v)$ and $r(v) = j + 1 \leq k = r(w)$. We consider the entry between u and w in the next row:

$$\begin{array}{ccc} & x_j & \\ x_i & & y_k \\ & z_l & \end{array} .$$

We have $i < l < k$ so that $i + 1 < k$ and so $l(u) = i < k - 1 = l(w)$.

- (iii) $u = y_i, w = x_k$. Then $i \leq j < k$. Hence $l(u) = i - 1 < j = l(v)$ and $r(v) = j < k = r(w)$. Also $l(u) + 1 = i < k = l(w)$.
- (iv) $u = y_i, w = y_k$. Then $i \leq j < k$. Hence $l(u) = i - 1 < j = l(v)$ and $l(v) = j \leq k - 1 = l(w)$. Also $l(u) = i - 1 < k - 1 = l(w)$.
- (v) $u = i, w = k$ (in the last row). Then $i \leq j < k$. Hence $l(u) = i \leq j = l(v)$ and $r(v) = j + 1 \leq k = r(w)$.

In each case, the monotone diagonal and monotone row properties are satisfied. Hence T is an admissible triangle.

Now suppose that T is an admissible triangle with each row but the last an $\mathcal{X}$ -row. Then all rows but the last are ranked 1. Again consider the arrangement (9) and divide into cases as above. As before we consider in detail only the cases where $v = x_j$ as the cases where $v = y_j$ are similar.

- (i) $u = x_i, w = x_k$. Then $i = l(u) \leq l(v) = j$ and $j = r(v) - 1 < r(w) - 1 = k$. Hence $i < k$.
- (ii) $u = x_i, w = y_k$. Then $i = l(u) \leq l(v) = j$ and $j = r(v) - 1 \leq r(w) - 1 < r(w) = k$. Hence $i < k$.
- (iii) $u = y_i, w = x_k$. Then $i = l(u) + 1 < l(v) + 1 = j + 1$ and so $i \leq j$. Also $j = r(v) - 1 < r(w) - 1 = k$. Hence $i < k$.
- (iv) $u = y_i, w = y_k$. Then $i = l(u) + 1 < l(v) + 1 = j + 1$ and so $i \leq j$. Also $j = r(v) - 1 \leq r(w) - 1 = k - 1$ and so $j < k$. Hence $i < k$.
- (v) $u = i, w = k$ (in the last row). Then $i = l(u) \leq l(v) = j$ and $j = r(v) - 1 \leq r(w) - 1 = k - 1$ and so $j < k$.

It follows that T is an oriented complete monotone triangle. $\square$

Lemma 2 *A triangle of order n all of whose rows but the last are $\mathcal{V}$ -rows is admissible if and only if it lies in $\mathcal{T}_n$.*

Proof If the triangle T with no $\mathcal{X}$ -rows is admissible then the rank of the k -th row must be $t = n - k$. Its entries must contain each of the k possible subscripts of the form i_{i+t} in increasing order, by the monotone row property. It must therefore be a tournament.

Conversely suppose that T is a tournament. In any arrangement (9) if $l(u) = i$ then $r(u) = i + t$ where t is the rank of the lower row and then $l(v) = i$, $r(v) = i + t + 1$, $l(w) = i + 1$ and $r(w) = i + t + 1$. It is then plain that the monotone diagonal and monotone row properties hold, so that T is admissible. $\square$

3 The algorithm

We define two operations on admissible triangles. Each operates on a pair of consecutive rows: rows $d - 1$ and d where $1 \leq d < n$. The operation of *raising* can be applied when row d is a $\mathcal{X}$ -row and row $d - 1$ is a $\mathcal{V}$ -row (including the case when $d - 1 = 0$). Rows $d - 1$ and d then have the same rank t . We start by making the following swaps in rows $d - 1$ and d whenever possible

$$\begin{array}{ccc} & b_{jk} & \longrightarrow & x_j \\ x_j & & & b_{jk} \end{array}$$

and

$$\begin{array}{ccc} a_{jk} & \longrightarrow & y_k \\ y_k & & a_{jk} \end{array}.$$

We then replace any a_{jk} remaining in row $d - 1$ by x_j and any b_{jk} remaining in row $d - 1$ by y_k . We then replace all x_i in row d by $a_{i_{i+t}}$ and all y_j in row r by b_{j-tj} . For instance

$$\begin{array}{cccc} & b_{13} & a_{57} & a_{68} \\ x_1 & & y_5 & y_7 & x_7 \end{array}$$

first becomes

$$\begin{array}{cccc} & x_1 & y_7 & a_{68} \\ b_{13} & & y_5 & a_{57} & x_7 \end{array}$$

and finally becomes

$$\begin{array}{cccc} & x_1 & y_7 & x_6 \\ b_{13} & & b_{35} & a_{57} & a_{79}. \end{array}$$

It is not immediately obvious that when we introduce an a_{ii+t} or a b_{j-tj} that $i+t \leq n$ or $j-t \geq 1$. For the moment we shall ignore this difficulty. Later we will show that raising preserves admissibility. As the left and right values of each entry in an admissible triangle lie between 1 and n inclusive, then we must have $i+t \leq n$ or $j-t \geq 1$ as appropriate.

We denote this raising operation by R_d . The rank of row $d-1$ has now been raised to $t+1$ while the ranks of the remaining rows are left unchanged. Also the weight of the new triangle is the same as that of the old.

Proposition 1 *When applied to an admissible triangle, the raising operator R_d produces an admissible triangle.*

Proof Let T be an admissible triangle with row $d-1$ and d having rank t and with row $d-1$ a $\mathcal{V}$ -row and row d an $\mathcal{X}$ -row. Let $T' = R_d(T)$.

First note that the operation of raising does not alter the left and right values of the entries in row d . In row $d-1$ suppose the entry v is replaced by v' . If $v' = x_i$ then $l(v') = l(v)$ and $r(v') = r(v) + 1$, while if $v' = y_j$ then $r(v') = r(v) - 1$ and $l(v') = l(v)$.

Suppose an arrangement (9) in T becomes

$$\begin{array}{ccc} & v' & \\ u' & & w' \end{array} \quad (10)$$

in T' . We need to check that (10) has the monotone diagonal and monotone row properties. Let the lower row in (10) be row e . The entries in (9) and (10) will be the same unless $e = d-1$, d or $d+1$. Also we only need to show the monotone row property when $e = d-1$ or d .

We can swiftly dispose of the monotone diagonal property when $e = d+1$. In this case $u = u'$ and $w = w'$. Also $l(v) = l(v')$ and $r(v) = r(v')$. The monotone diagonal property for (10) follows immediately from that for (9).

Now consider the case where $e = d-1$. In this case $l(u') = l(u)$ or $l(u) - 1$ and $r(w') = r(w)$ or $r(w) + 1$. Also $v = v'$. Thus $l(u') \leq l(u) \leq l(v) = l(v')$ and $r(v') = r(v) \leq r(w) \leq r(w')$. If $u' = y_j$, then $l(u') = l(u) - 1 < l(v) = l(v')$. Similarly if $w' = x_j$, then $r(v') = r(v) < r(w) + 1 = r(w')$. Also $l(u') < l(w')$ unless $l(w) = l(u) + 1$, $u' = x_i$ and $w' = y_j$ for some i and j . Then $l(u) = i$, $r(u) = j-1$, $l(w) = i+1$, $r(w) = j$ and $j-i = t+1$. Consider the entry in T below u and w :

$$\begin{array}{ccc} & v & \\ u & & w \\ & q & \end{array}$$

From the monotone diagonal property of T , q must be either x_{i+1} or y_{j-1} . Suppose the former. Then w cannot be $b_{i+1,j}$ since that would be swapped with q yielding $w' = x_{i+1}$. Thus $w = a_{i+1,j}$ and for w' to be y_j we need the entry following q in row d in T to be y_j :

$$\begin{array}{ccc} & u & a_{i+1,j} \\ & x_{i+1} & y_j \end{array}.$$

But then in row d , $l(x+1) = i+1$ and $r(x+1) = j-t = i+1$, violating the monotone row property for (9). A similar argument yields a contradiction when $q = y_{j-1}$. Hence (10) also has the monotone diagonal and monotone row properties.

Finally suppose that $d = e$. Now $l(u) = l(u')$ and $l(v') \in \{l(v) - 1, l(v)\}$. Thus $l(u') \leq l(v')$ unless $l(u) = l(v)$ and $v' = y_j$. Then $u = x_{j-t}$ or y_j and $v = a_{j-t,j}$ or $v = b_{j-t,j}$. If $u = y_j$ then $l(u) = j-t = l(v)$ contrary to the monotone diagonal property for (9). Thus $u = x_{j-t}$. If $v = b_{j-t,j}$ then u and v are swapped giving $v' = x_{j-t}$. Thus $v = a_{j-t,j}$. As $v' = y_j$ then y_j must succeed x_{j-t} in row d of T :

$$\begin{array}{ccc} & a_{j-t,j} & \\ x_{j-t} & & y_j \end{array}.$$

But then in row d of T , $l(x_{j-t}) = j-t = l(y_j)$ contrary to the monotone row property of T . Thus $l(u') \leq l(v')$. Similarly $r(v') \leq r(w')$. Thus (9) has the monotone diagonal property. As row d in T' has the same left and right values as the row d of T , then (10) has the monotone row property. We conclude that the triangle T' is admissible. $\square$

The inverse of raising is *lowering*. This time suppose that row d ($1 \leq d < n$) is a $\mathcal{V}$ -row of rank t and that row $d-1$ (including the case $d-1=0$) is a $\mathcal{X}$ -row. Then row $d-1$ has rank $t+1$. We make the swaps

$$\begin{array}{ccc} & x_j & b_{jk} \\ b_{jk} & \longrightarrow & x_j \end{array}$$

and

$$\begin{array}{ccc} y_k & \longrightarrow & a_{jk} \\ a_{jk} & & y_k \end{array}$$

where possible in rows $d-1$ and d . We then replace the a_{jk} and b_{jk} in row $d-1$ by x_j and y_k and the x_j and y_k in row d by $a_{j,j+t}$ and $b_{k-t,k}$. Again, it will follow from the fact that lowering preserves admissibility that $j+t \leq n$ or $k-t \geq 1$ as appropriate. The new triangle has an admissible ranking. All the ranks are unchanged save for row $d-1$ whose rank is lowered from $t+1$ to t . We denote this lowering operator by L_d .

Proposition 2 *When applied to an admissible triangle, the lowering operator L_d produces an admissible triangle.*

Proof Let T be an admissible triangle with row $d - 1$ an $\mathcal{X}$ -row and row d a $\mathcal{V}$ -row of rank t . Then row $d - 1$ of T has rank $t + 1$. Let $T' = L_d(T)$.

The lowering operation does not alter the left and right values in row d . Suppose that the entry v in row $d - 1$ is replaced by v' . If $v = x_j$ then $l(v') = l(v)$ and $r(v') = r(v) - 1$ while if $v = y_j$ then $l(v') = l(v) + 1$ and $r(v') = r(v)$.

Suppose we have an arrangement (9) having lower row e , becoming (10) after lowering. They are identical unless $e \in \{d - 1, d, d + 1\}$.

When $e = d + 1$ the monotone row property of (10) is immediate. Also the monotone diagonal property follows as the values in row d and the entries in row $d + 1$ are unchanged.

Suppose that $e = d - 1$. Row $d - 1$ in T' is a $\mathcal{V}$ -row. We have $l(u') \in \{l(u), l(u) + 1\}$ and $l(v') = l(v)$ so that $l(u') \leq l(v')$ unless $l(u) = l(v)$ and $l(u') = l(u) + 1$. In this case $u = y_j$ and so $l(u) < l(v)$ contrary to hypothesis. Hence $l(u') \leq l(v')$ and similarly $r(v') \leq r(w')$. Thus (10) has the monotone diagonal property. As also $l(w') \in \{l(w), l(w) + 1\}$ and $l(u) < l(w)$ then $l(u') < l(v')$ unless $l(u) + 1 = l(w)$, $l(u') = l(u) + 1$ and $l(w') = l(w)$. In this case $u = y_i$ and $w = x_j$ so that by the monotone diagonal property for (9) we have $l(u) < l(v)$ and $r(v) < r(w)$. Consequently $l(v) < l(w)$ and so $l(w) > l(u) + 1$ contrary to hypothesis. Hence (10) also has the monotone row property.

Finally suppose that $e = d$. As $l(v') \in \{l(v), l(v) + 1\}$ then $l(u') = l(u) \leq l(v) \leq l(v')$. Similarly $r(v') \leq r(v) \leq r(w) = r(w')$. Suppose that $u' = y_i$. We must show that $l(u') < l(v')$. This is valid unless $l(u') = l(v')$. If so their common value is $i - t$. As $l(v') = i - t$ and $r(v') = t$ then $v = x_{i-t}$ or y_i . Also $u = a_{i-t}$ or b_{i-t} . We cannot have $v = y_i$ for then $l(v) = i - t - 1$ and $l(u) = i - t$ so that $l(u) \not\leq l(v)$. But if $v = x_{i-t}$ and $u = b_{i-t}$ then u and v are swapped so that $u' = x_{i-t}$. If $v = x_{i-t}$ and $u = a_{i-t}$ the only way that u' can be equal to y_i is if the entry preceding u in the $(d - 1)$ -th row of T is y_i so that we get

$$\begin{array}{ccc} y_i & & x_{i-t} \\ & a_{i-t} & w \end{array}.$$

But in this case the left values of the entries in row $d - 1$ are $i - t - 1$ and $i - t$ which violates the condition that T has the monotone row property. Hence $l(u') < l(v')$. Similarly if $v' = x_j$ then $r(v') < r(w')$ and so (10) has the monotone diagonal property.

As the values of the entries in row d do not change, then (10) has the monotone row property. We conclude that the triangle T' is admissible. $\square$

Lemma 3 *The operations R_d and L_d are inverse: if we operate on an admissible triangle by R_d and then L_d we regain the initial triangle. Similarly if we follow L_d by R_d .*

Proof Suppose that T is an admissible triangle and that $T' = R_d(T)$ is defined. When we raise T by R_d we first swap some pairs of entries and then when we lower T' by L_d these entries are swapped back. If we can show that no other pairs of entries are swapped when lowering we are done, since it is plain that the remaining entries return to their original state. Suppose after raising we have a configuration

$$\begin{array}{c} x_j \\ b_{jk} \end{array}$$

which has not arisen from swapping the entries. This must have come from the configuration

$$\begin{array}{c} a_{jk} \\ y_k \end{array}$$

which is banned by the monotone diagonal property. Similarly a configuration

$$\begin{array}{c} y_k \\ a_{jk} \end{array}$$

cannot arise except by swapping. Hence raising followed by lowering gives the identity. Similarly lowering followed by raising gives the identity. $\square$

We can now describe the bijections Φ and Ψ . We define

$$\Phi = (R_1 \circ R_2 \circ R_3 \circ \cdots \circ R_{n-1}) \circ \cdots \circ (R_1 \circ R_2 \circ R_3) \circ (R_1 \circ R_2) \circ (R_1).$$

It is easy to see that $\Phi(T)$ is well-defined for $T \in \mathcal{O}_n$ and that $\Phi(T) \in \mathcal{T}_n$. Its inverse Ψ is then

$$(L_1) \circ (L_2 \circ L_1) \circ (L_3 \circ L_2 \circ L_1) \circ \cdots \circ (L_{n-1} \circ \cdots \circ L_3 \circ L_2 \circ L_1).$$

As Φ and Ψ are composites of weight-preserving maps, they too are weight-preserving. This completes the proof of Theorem 1

4 An example

As an example let us apply Φ to the oriented complete monotone triangle (7). We indicate the rank of each row in brackets at the end, including the rank

of the empty row 0. Here is (7) in this notation:

$$\begin{array}{cccccc}
 & & & & & [1] \\
 & & & & & [1] \\
 & & & y_4 & & [1] \\
 & & y_3 & & x_4 & [1] \\
 & y_2 & & x_3 & & y_5 & [1] \\
 x_1 & & y_3 & & y_4 & & y_5 & [1] \\
 1 & 2 & 3 & 4 & 5 & & [0]
 \end{array} .$$

We first apply the operator R_1 giving

$$\begin{array}{cccccc}
 & & & & & [2] \\
 & & & & & [1] \\
 & & & b_{34} & & [1] \\
 & & y_3 & & x_4 & [1] \\
 & y_2 & & x_3 & & y_5 & [1] \\
 x_1 & & y_3 & & y_4 & & y_5 & [1] \\
 1 & 2 & 3 & 4 & 5 & & [0]
 \end{array} .$$

Next apply R_2 :

$$\begin{array}{cccccc}
 & & & & & [2] \\
 & & & & & [2] \\
 & & & y_4 & & [1] \\
 & & b_{23} & & a_{45} & [1] \\
 & y_2 & & x_3 & & y_5 & [1] \\
 x_1 & & y_3 & & y_4 & & y_5 & [1] \\
 1 & 2 & 3 & 4 & 5 & & [0]
 \end{array} .$$

Then R_1 :

$$\begin{array}{cccccc}
 & & & & & [3] \\
 & & & & & [2] \\
 & & & b_{24} & & [1] \\
 & & b_{23} & & a_{45} & [1] \\
 & y_2 & & x_3 & & y_5 & [1] \\
 x_1 & & y_3 & & y_4 & & y_5 & [1] \\
 1 & 2 & 3 & 4 & 5 & & [0]
 \end{array} .$$

Next R_3 :

$$\begin{array}{cccccc}
 & & & & & [3] \\
 & & & & & [2] \\
 & & & b_{24} & & [2] \\
 & & y_3 & & y_5 & [1] \\
 & b_{12} & & a_{34} & & a_{45} & [1] \\
 x_1 & & y_3 & & y_4 & & y_5 & [1] \\
 1 & 2 & 3 & 4 & 5 & & [0]
 \end{array} .$$

Then R_2 :

$$\begin{array}{cccccc}
 & & & y_4 & & \\
 & & b_{13} & & b_{35} & \\
 & b_{12} & & a_{34} & & a_{45} \\
 x_1 & & y_3 & & y_4 & & y_5 \\
 1 & 2 & 3 & 4 & 5 & &
 \end{array}
 \begin{array}{c}
 [3] \\
 [3] \\
 [2] \\
 [1] \\
 [1] \\
 [0]
 \end{array}
 .$$

Now R_1 :

$$\begin{array}{cccccc}
 & & & b_{14} & & \\
 & & b_{13} & & b_{35} & \\
 & b_{12} & & a_{34} & & a_{45} \\
 x_1 & & y_3 & & y_4 & & y_5 \\
 1 & 2 & 3 & 4 & 5 & &
 \end{array}
 \begin{array}{c}
 [4] \\
 [3] \\
 [2] \\
 [1] \\
 [1] \\
 [0]
 \end{array}
 .$$

Next R_4 :

$$\begin{array}{cccccc}
 & & & b_{14} & & \\
 & & b_{13} & & b_{35} & \\
 & x_1 & & y_4 & & y_5 \\
 b_{12} & & b_{23} & & a_{34} & & a_{45} \\
 1 & 2 & 3 & 4 & 5 & &
 \end{array}
 \begin{array}{c}
 [4] \\
 [3] \\
 [2] \\
 [2] \\
 [1] \\
 [0]
 \end{array}
 .$$

Then R_3 :

$$\begin{array}{cccccc}
 & & & b_{14} & & \\
 & & x_1 & & y_5 & \\
 & b_{13} & & b_{24} & & b_{35} \\
 b_{12} & & b_{23} & & a_{34} & & a_{45} \\
 1 & 2 & 3 & 4 & 5 & &
 \end{array}
 \begin{array}{c}
 [4] \\
 [3] \\
 [3] \\
 [2] \\
 [1] \\
 [0]
 \end{array}
 .$$

Next R_2 :

$$\begin{array}{cccccc}
 & & & x_1 & & \\
 & & b_{14} & & b_{25} & \\
 & b_{13} & & b_{24} & & b_{35} \\
 b_{12} & & b_{23} & & a_{34} & & a_{45} \\
 1 & 2 & 3 & 4 & 5 & &
 \end{array}
 \begin{array}{c}
 [4] \\
 [3] \\
 [3] \\
 [2] \\
 [1] \\
 [0]
 \end{array}
 .$$

Finally R_1 :

$$\begin{array}{cccccc}
 & & & & & [5] \\
 & & & & & [4] \\
 & & & a_{15} & & [3] \\
 & & b_{14} & & b_{25} & [2] \\
 & b_{13} & & b_{24} & & b_{35} \\
 & & b_{12} & & b_{23} & & a_{34} & & a_{45} & [1] \\
 1 & & 2 & & 3 & & 4 & & 5 & [0]
 \end{array} .$$

Hence Φ takes the oriented complete monotone triangle (7) to the tournament (8).

5 Remarks

We have defined Φ and Ψ by using the operators R_d and L_d in a fixed order. However it is not hard to prove that performing any achievable sequence of $\binom{n}{2}$ raising (or lowering) operations produces the same map.

Proposition 3 *Let $N = \frac{1}{2}n(n-1)$ and suppose that R_{u_j} ($1 \leq j \leq N$) are raising operators such that*

$$\Phi' = R_{u_N} \circ R_{u_{N-1}} \circ \cdots \circ R_{u_2} \circ R_{u_1}$$

is a well-defined map from $\mathcal{O}_n$ to $\mathcal{T}_n$. Then $\Phi' = \Phi$. Similarly if L_{u_j} ($1 \leq j \leq N$) are lowering operators such that

$$\Psi' = L_{u_N} \circ L_{u_{N-1}} \circ \cdots \circ L_{u_2} \circ L_{u_1}$$

is a well-defined map from $\mathcal{T}_n$ to $\mathcal{O}_n$. Then $\Psi' = \Psi$.

Proof We need only prove the assertion concerning raising operators, as that for lowering operators follows by taking inverses. It is easy to see that if $i \neq j$, T is a triangle and both $R_i(T)$ and $R_j(T)$ exist, then $|i-j| \geq 2$ and $R_j \circ R_i(T)$ and $R_i \circ R_j(T)$ exist and are equal. Also if $|i-j| \geq 2$ and $R_i \circ R_j(T)$ exists then $R_i(T)$ exists. Thus if $|i-j| \geq 2$ we have $R_i \circ R_j = R_j \circ R_i$ in the sense that if one side exists so does the other and they are equal.

The definition of Φ is framed so that

$$\Phi = R_{v_N} \circ R_{v_{N-1}} \cdots \circ R_{v_2} \circ R_{v_1}$$

where for each j , v_j is the least v such that

$$R_v \circ R_{v_{j-1}} \cdots \circ R_{v_2} \circ R_{v_1}$$

is well-defined on $\mathcal{O}_n$.

Suppose that $u_j \neq v_j$ for some j , and take j to be the least such for which this holds. Then $u_j > v_j$. Since R_j increases the rank of row $j - 1$ by 1, the sequences of u_i s and of v_i s are permutations of each other. Hence for some $k > j$ we have $u_k = v_j$ and take k to be the least such. Let $T' = R_{u_{j-1}} \circ \cdots \circ R_1(T)$ where T is any element of $\mathcal{O}_n$. Then row u_j of T' is an $\mathcal{X}$ -row and all previous rows are $\mathcal{V}$ -rows. When we apply a sequence of raising operators to T' we cannot apply an operator R_w for $w < v_j$ until we have applied the operator T_{v_j} since the rows 1 to $v_j - 1$ will remain as $\mathcal{V}$ -rows and row v_j as a $\mathcal{X}$ -row until this occurs. Also $T_{v_{j+1}}$ cannot be applied until T_{v_j} has been since row T_{v_j} will remain an $\mathcal{X}$ -row during this process. Thus if $j \leq i < k$ then $u_i \geq v_j + 2$. Hence

$$\begin{aligned} R_{u_k} \circ R_{u_{k-1}} \circ \cdots \circ R_{u_{j+1}} \circ R_{u_j} &= R_{v_j} \circ R_{u_{k-1}} \circ \cdots \circ R_{u_{j+1}} \circ R_{u_j} \\ &= R_{u_{k-1}} \circ \cdots \circ R_{u_{j+1}} \circ R_{u_j} \circ R_{v_j}. \end{aligned}$$

By using this substitution in the definition of Φ' , we increase the value of j . Repeating this process eventually gives $\Phi' = \Phi$. $\square$

We can define a notion of oriented alternating sign matrix by assigning each -1 in a given alternating sign matrix an orientation (left or right). This is an equivalent notation to oriented complete monotone triangle. There is a natural correspondence between oriented alternating sign matrices of order n and domino tilings of the Aztec diamond of order $n - 1$ (see [4]). We have thus given an implicit bijection between tournaments on vertex set $\{1, 2, \dots, n\}$ and domino tilings of the Aztec diamond of order $n - 1$. In particular we recover the result that there are $2^{\binom{n}{2}}$ such tilings.

We can apply the argument in the proof of Theorem 1 to more general triangles. Instead of having the last row $1, 2, \dots, n$ choose any set S of n positive integers and consider the oriented monotone triangles $\mathcal{O}_S$ with bottom row consisting of the elements of S in ascending order. We wish to describe the set $\Phi(\mathcal{O}_S)$. Let $\mathcal{T}_S$ be the set of admissible triangles with bottom row consisting of the elements of S , and with each entry above the bottom row being an a_{ij} or a b_{ij} . The proof of Theorem 1 immediately adapts to show that Φ and Ψ are inverse bijections between $\mathcal{O}_S$ and $\mathcal{T}_S$. It is immediate that $|\mathcal{T}_S| = 2^{\binom{n}{2}} |\mathcal{T}'_S|$ where $\mathcal{T}'_S$ is the subset of $\mathcal{T}_S$ consisting of triangles lacking entries b_{ij} . This is because each element of $\mathcal{T}_S$ is derived from a unique element of $\mathcal{T}'_S$ by replacing a subset of its a_{ij} entries by the corresponding b_{ij} . Each triangle in $\mathcal{T}'_S$ is determined by the left values of the entries (recall that a_{ij} has left value i). By replacing each a_{ij} in such a triangle by i we get a bijection between $\mathcal{T}'_S$ and the set of *strict monotone triangles* with bottom row S , that is the set of numerical triangles in which

each arrangement

$$\begin{array}{ccc} & j & \\ i & & k \end{array}$$

satisfies $i \leq j < k$. For instance there are 6 strict monotone triangles with bottom row $S = \{1, 2, 4, 5\}$ namely

$$\begin{array}{ccccc} & & 1 & & \\ & 1 & & 2 & \\ 1 & & 2 & & 4 & 5 \end{array}, \begin{array}{ccccc} & & 1 & & 3 \\ & 1 & & 2 & 4 \\ 1 & & 2 & & 4 & 5 \end{array}, \begin{array}{ccccc} & & 2 & & \\ & 1 & & 3 & \\ 1 & & 2 & & 4 & 5 \end{array},$$

$$\begin{array}{ccccc} & & 1 & & \\ & 1 & & 3 & \\ 1 & & 2 & & 4 & 5 \end{array}, \begin{array}{ccccc} & & 2 & & \\ & 1 & & 3 & 4 \\ 1 & & 2 & & 4 & 5 \end{array}, \begin{array}{ccccc} & & 2 & & \\ & 2 & & 3 & \\ 1 & & 2 & & 4 & 5 \end{array}.$$

By adapting the proof in [4, §4] one can show that if $S = \{a_1, a_2, \dots, a_n\}$ with $a_1 < a_2 < \dots < a_n$ then

$$|\mathcal{T}'_S| = \prod_{1 \leq i < j \leq n} \frac{a_j - a_i}{j - i}$$

and so

$$|\mathcal{T}_S| = 2^{\binom{n}{2}} \prod_{1 \leq i < j \leq n} \frac{a_j - a_i}{j - i}.$$

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