

Franklin's argument proves an identity of Zagier

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4 September 2000

Abstract

Recently Zagier proved a remarkable q -series identity. We show that this identity can also be proved by modifying Franklin's classical proof of Euler's pentagonal number theorem.

Mathematics Subject Classification (2000): 05A17 11P81

1 Introduction

We use the standard q -series notation:

$$(a)_n = \prod_{k=1}^n (1 - aq^{k-1})$$

where n is a nonnegative integer or $n = \infty$. Euler's pentagonal number theorem states that

$$(q)_\infty = 1 + \sum_{r=1}^{\infty} (-1)^r (q^{r(3r-1)/2} + q^{r(3r+1)/2}). \quad (1)$$

Recently Zagier proved the following remarkable identity

Theorem 1

$$\sum_{n=0}^{\infty} [(q)_\infty - (q)_n] = (q)_\infty \sum_{k=1}^{\infty} \frac{q^k}{1 - q^k} + \sum_{r=1}^{\infty} (-1)^r [(3r-1)q^{r(3r-1)/2} + 3rq^{r(3r+1)/2}]. \quad (2)$$

This is [4, Theorem 2] slightly rephrased.

Equation (1) has a combinatorial interpretation. The coefficient of q^N in $(q)_\infty$ equals $d_e(N) - d_o(N)$ where $d_e(N)$ (respectively $d_o(N)$) is the number of partitions of N into an even (respectively odd) number of distinct parts. Franklin [3] showed that

$$d_e(N) - d_o(N) = \begin{cases} (-1)^r & \text{if } N = \frac{1}{2}r(3r \pm 1) \text{ for a positive integer } r, \\ 0 & \text{otherwise.} \end{cases}$$

His proof was combinatorial. He set up what was almost an involution on the set of partitions of N into distinct parts. This “involution” reverses the parity of the number of parts. However there are certain partitions for which his map is not defined. These exceptional partitions occur precisely when $N = \frac{1}{2}r(3r \pm 1)$, and so account for the nonzero terms on the right of (1).

We show that Zagier’s identity has a similar combinatorial interpretation, which, miraculously, Franklin’s argument proves at once.

2 Proof of Theorem 1

We begin by recalling Franklin’s “involution”. Let $\mathcal{D}_N$ denote the set of partitions of N into distinct parts and let $\mathcal{D} = \bigcup_{N=0}^{\infty} \mathcal{D}_N$. For $\lambda \in \mathcal{D}_N$ let $N_\lambda = N$, n_λ be the number of parts in λ and m_λ be the largest part of λ (if λ is the empty partition of 0 let $m_\lambda = 0$). Then

$$(q)_\infty = \sum_{\lambda \in \mathcal{D}} (-1)^{n_\lambda} q^{N_\lambda}. \quad (3)$$

Let λ be a non-empty partition in $\mathcal{D}$. Denote its smallest part by a_λ . If the parts of λ are $\lambda_1 > \lambda_2 > \lambda_3 > \dots$ let $b = b_\lambda$ denote the largest b such that $\lambda_b = \lambda_1 + 1 - b$ (so that $\lambda_k = \lambda_1 + 1 - k$ if and only if $1 \leq k \leq b$). If $\lambda \in \mathcal{D}$ is not *exceptional* (we shall explain this term shortly), then we define a new partition λ' as follows. If $a_\lambda \leq b_\lambda$ we obtain λ' by removing the smallest part from λ and then adding 1 to the largest a_λ parts of this new partition. If $a_\lambda > b_\lambda$ we obtain λ' by subtracting 1 from the b_λ largest parts of λ and then appending a new part b_λ to this new partition.

The exceptional partitions are those for which this procedure breaks down. We regard the empty partition as exceptional, also we regard those for which $n_\lambda = b_\lambda$ and $a_\lambda = b_\lambda$ or $b_\lambda + 1$. If λ is not exceptional, then neither is λ' and $\lambda'' = \lambda$ and $(-1)^{n_{\lambda'}} = -(-1)^{n_\lambda}$. Thus on the right side of (3) the contributions from non-exceptional partitions cancel. The non-empty exceptional partitions are of two forms: for each positive integer r we have $\lambda = (2r - 1, 2r - 2, \dots, r + 1, r)$ for which $n_\lambda = r$, $m_\lambda = 2r - 1$ and

$N_\lambda = \frac{1}{2}r(3r-1)$, and we have $\lambda = (2r, 2r-1, \dots, r+2, r+1)$ for which $n_\lambda = r$, $m_\lambda = 2r$ and $N_\lambda = \frac{1}{2}r(3r+1)$. Thus from (3) we deduce (1).

If $\lambda \in \mathcal{D}$ is non-exceptional, then either $n_{\lambda'} = n_\lambda - 1$ in which case $m_{\lambda'} = m_\lambda + 1$ or $n_\lambda = n_{\lambda'} + 1$ in which case $m_{\lambda'} = m_\lambda - 1$. In each case $m_{\lambda'} + n_{\lambda'} = m_\lambda + n_\lambda$. It follows that in the sum

$$\sum_{\lambda \in \mathcal{D}} (-1)^{n_\lambda} (m_\lambda + n_\lambda) q^{N_\lambda}$$

the terms corresponding to non-exceptional λ cancel and so we get only the contribution from exceptional λ . Thus

$$\sum_{\lambda \in \mathcal{D}} (-1)^{n_\lambda} (m_\lambda + n_\lambda) q^{N_\lambda} = \sum_{r=1}^{\infty} (-1)^r [(3r-1)q^{r(3r-1)/2} + 3rq^{r(3r+1)/2}]. \quad (4)$$

This sum occurs in (2), which will follow by analysing the left side of (4).

We break this into two sums. The first

$$\sum_{\lambda \in \mathcal{D}} (-1)^{n_\lambda} m_\lambda q^{N_\lambda}$$

is dealt with in [2, Theorem 5.2]. We repeat their argument. The coefficient of q^N in $(q)_\infty - (q)_n$ is the sum of $(-1)^{n_\lambda}$ over all $\lambda \in \mathcal{D}_N$ having a part strictly greater than n . Such a λ is counted for exactly m_λ different n so that

$$\sum_{n=0}^{\infty} [(q)_\infty - (q)_n] = \sum_{\lambda \in \mathcal{D}} (-1)^{n_\lambda} m_\lambda q^{N_\lambda}. \quad (5)$$

For each positive integer k ,

$$\frac{-q^k}{1-q^k} (q)_\infty = (1-q)(1-q^2) \cdots (1-q^{k-1})(-q^k)(1-q^{k+1}) \cdots$$

The coefficient of q^N in this product is the sum of $(-1)^{n_\lambda}$ over all $\lambda \in \mathcal{D}_N$ having k as a part. Such a λ occurs for n_λ distinct k , and summing we conclude that

$$-(q)_\infty \sum_{k=1}^{\infty} \frac{q^k}{1-q^k} = \sum_{\lambda \in \mathcal{D}} (-1)^{n_\lambda} n_\lambda q^{N_\lambda}. \quad (6)$$

Combining (4), (5) and (6) gives (2).

3 Another identity

We can prove another identity by the same method. As before Franklin's argument proves that

$$\sum_{\lambda \in \mathcal{D}} (-1)^{n_\lambda} x^{m_\lambda + n_\lambda} q^{N_\lambda} = 1 + \sum_{r=1}^{\infty} (-1)^r [x^{3r-1} q^{r(3r-1)/2} + x^{3r} q^{r(3r+1)/2}]. \quad (7)$$

We now give an alternative way of expressing the left side of (7). Consider the contribution due to the partitions λ with m_λ equalling a fixed $m \geq 1$. The product

$$(xq)_m = \prod_{j=1}^m (1 - xq^j) = \sum_{\substack{\lambda \in \mathcal{D} \\ m_\lambda \leq m}} (-1)^{n_\lambda} x^{n_\lambda} q^{N_\lambda}$$

and so

$$\sum_{\substack{\lambda \in \mathcal{D} \\ m_\lambda = m}} (-1)^{n_\lambda} x^{m_\lambda + n_\lambda} q^{N_\lambda} = x^m [(xq)_m - (xq)_{m-1}].$$

Hence

$$\begin{aligned} \sum_{\lambda \in \mathcal{D}} (-1)^{n_\lambda} x^{m_\lambda + n_\lambda} q^{N_\lambda} &= 1 + \sum_{m=1}^{\infty} x^m [(xq)_m - (xq)_{m-1}] \\ &= (1 - x) \sum_{r=0}^{\infty} (xq)_r x^r \\ &= \sum_{r=0}^{\infty} (x)_{r+1} x^r. \end{aligned}$$

Hence

$$\sum_{r=0}^{\infty} (x)_{r+1} x^r = 1 + \sum_{r=1}^{\infty} (-1)^r [x^{3r-1} q^{r(3r-1)/2} + x^{3r} q^{r(3r+1)/2}]. \quad (8)$$

This can also be proved by noting that both sides satisfy the recurrence $S(x) = 1 - qx^2 - s^2 x^3 S(qx)$. This appears as exercise 10 in Chapter 2 of [1] whose solution is outlined in [4]. Zagier deduces (2) from (8), essentially by carefully differentiating with respect to x and setting $x = 1$.

4 Acknowledgments

The author wishes to thank George Andrews and Don Zagier for supplying him with copies of [2] and [4].

References

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