

Coefficients of products of powers of eta functions

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1 Introduction

The infinite product

$$(q)_\infty = \prod_{j=1}^{\infty} (1 - q^j)$$

has been much studied. In particular setting $q = \exp(2\pi iz)$ where z is a complex number with positive imaginary part, we have the Dedekind eta function

$$\eta(z) = q^{1/24} \prod_{j=1}^{\infty} (1 - q^j) = \exp(\pi iz/12) \prod_{j=1}^{\infty} (1 - q^j).$$

In [2] Cooper, Hirschhorn and Lewis studied the coefficients of the series $(q)_\infty^r$, $(q)_\infty^r (q^2)_\infty^s$, $(q)_\infty^r (q^3)_\infty^s$ and $(q)_\infty^r (q^4)_\infty^s$ for various values of r and s . For the series $(q)_\infty^r$ they recover results of Newman [5] originally proved using modular forms. Their methods are elementary applications of the Macdonald identities of ranks 1 and 2 [4].

Cooper et al. also state three conjectures, based on numerical evidence, giving further instances of exponents r and s for which their results appear to be valid. We prove two of the cases, $(r, s) = (3, -1)$ and $(r, s) = (-1, 3)$, of their Conjecture 1. Indeed we prove slightly more since the conjecture in [2] only concerns primes $p \equiv 7$ or $11 \pmod{24}$.

Theorem 1 Define sequences $(b(m))_{m=0}^{\infty}$ and $(c(m))_{m=0}^{\infty}$ by

$$\sum_{m=0}^{\infty} b(m)q^m = \frac{(q)_{\infty}^3}{(q^2)_{\infty}} \quad \text{and} \quad \sum_{m=0}^{\infty} c(m)q^m = \frac{(q^2)_{\infty}^3}{(q)_{\infty}}.$$

Let p be a prime number. Then for the following values of p and ε

$$\varepsilon = \begin{cases} 1 & \text{if } p \equiv 7, 13 \text{ or } 23 \pmod{24}, \\ -1 & \text{if } p \equiv 11, 17 \text{ or } 19 \pmod{24} \end{cases}$$

we have

$$b(p^2m + (p^2 - 1)/24) = \varepsilon b(m) \quad \text{and} \quad c(p^2m + 5(p^2 - 1)/24) = \varepsilon c(m).$$

The proof of this result occupies the remainder of the paper. Our method is elementary and makes no use of the theory of modular forms. We require the following classical identities: Euler's pentagonal number formula

$$(q)_{\infty} = \sum_{r=-\infty}^{\infty} (-1)^r q^{r(3r+1)/2} \quad (1)$$

and the product formulas for the classical theta functions θ_2 and θ_4

$$\frac{(q)_{\infty}^2}{(q^2)_{\infty}} = \sum_{r=-\infty}^{\infty} (-1)^r q^{r^2}, \quad (2)$$

$$2q^{1/4} \frac{(q^4)_{\infty}^2}{(q^2)_{\infty}} = \sum_{\substack{r=-\infty \\ r \equiv 1 \pmod{2}}}^{\infty} q^{r^2/4}. \quad (3)$$

We also use the ideal theory of the imaginary quadratic field $\mathbf{Q}(\sqrt{-6})$.

2 The L -function

We construct an L -function associated to a particular character of the group of generalized ideal classes of the imaginary quadratic field $\mathbf{Q}(\sqrt{-6})$. Its coefficients, as a Dirichlet series, will be closely related to the $b(m)$ and $c(m)$.

A good reference for the properties of quadratic fields we use is [1].

Let $K = \mathbf{Q}(\sqrt{-6})$, and let $\mathcal{O} = \mathbf{Z}[\sqrt{-6}]$ be its ring of integers. The field K has class number 2. We study the ideals of $\mathcal{O}$. If $\mathcal{I}$ is an ideal of $\mathcal{O}$ then we let $\bar{\mathcal{I}}$ denote the ideal consisting of the complex conjugates of elements of $\mathcal{I}$. The primes 2 and 3 ramify in K . We have $2\mathcal{O} = \wp_2^2$ and $3\mathcal{O} = \wp_3^2$ where $\wp_2 = \langle 2, \sqrt{-6} \rangle$ and $\wp_3 = \langle 3, \sqrt{-6} \rangle$ are non-principal ideals of $\mathcal{O}$. All

other primes are unramified in K . If the Legendre symbol $(-6/p) = 1$, that is if $p \equiv 1, 5, 7$ or $11 \pmod{24}$ then p splits: $p\mathcal{O} = \wp_p \overline{\wp_p}$ for prime ideals $\wp_p \neq \overline{\wp_p}$. On the other hand if $(-6/p) = -1$, that is if $p \equiv 13, 17, 19$ or $23 \pmod{24}$ then p is inert: $p\mathcal{O}$ is a prime ideal of $\mathcal{O}$.

Recall that the norm of a nonzero ideal $\mathcal{I}$ is $N(\mathcal{I}) = |\mathcal{O}/\mathcal{I}|$. We also have $\mathcal{I}\overline{\mathcal{I}} = \langle N(\mathcal{I}) \rangle$. It follows that the norm is multiplicative: $N(\mathcal{I}\mathcal{J}) = N(\mathcal{I})N(\mathcal{J})$.

Lemma 1 *Let $\mathcal{I}$ be an ideal of $\mathcal{O}$ with $N(\mathcal{I})$ coprime to 6. Then $N(\mathcal{I}) \equiv 1$ or $7 \pmod{24}$ if $\mathcal{I}$ is principal and $N(\mathcal{I}) \equiv 5$ or $11 \pmod{24}$ if $\mathcal{I}$ is non-principal.*

Proof If $\mathcal{I} = \langle a + b\sqrt{-6} \rangle$ is principal then $N(\mathcal{I}) = a^2 + 6b^2$. For $N(\mathcal{I})$ to be coprime to 6 we need $a \equiv \pm 1 \pmod{6}$ and it follows that $a^2 \equiv 1 \pmod{24}$. But $b^2 \equiv 0$ or $1 \pmod{4}$ and so $N(\mathcal{I}) \equiv 1$ or $7 \pmod{24}$.

Suppose that $\mathcal{I}$ is non-principal. Since $\wp_2$ is also non-principal and K has class number 2, it follows that $\wp_2\mathcal{I} = \langle a + b\sqrt{-6} \rangle$ is principal. Also $N(\wp_2\mathcal{I}) = 2N(\mathcal{I}) = a^2 + 6b^2$. Hence $a = 2c$ is even and $N(\mathcal{I}) = 2c^2 + 3b^2$. For $N(\mathcal{I})$ to be coprime to 6 we need b to be odd and $a \equiv \pm 1 \pmod{3}$. Then $b^2 \equiv 1 \pmod{8}$ so that $3b^2 \equiv 3 \pmod{24}$. Also $c^2 \equiv 1 \pmod{3}$ and $c^2 \equiv 0$ or $1 \pmod{4}$ so that $2c^2 \equiv 2$ or $8 \pmod{24}$. Consequently $N(\mathcal{I}) \equiv 5$ or $11 \pmod{24}$. $\square$

We consider the generalized ideal class group modulo the ideal $\mathcal{M} = 4\wp_3$. For the theory of generalized ideal classes see [3, Chapter VI §1] but our treatment is self-contained. We define an equivalence relation on the set of ideals of $\mathcal{O}$ which are coprime to $\mathcal{M}$ as follows. Set $\mathcal{I} \sim \mathcal{J}$ if there exist $\alpha, \beta \in \mathcal{O}$ with $\alpha \equiv \beta \equiv 1 \pmod{\mathcal{M}}$ and with $\alpha\mathcal{I} = \beta\mathcal{J}$. It is readily verified that this is an equivalence relation. The equivalence class of an ideal $\mathcal{I}$ is denoted by $[\mathcal{I}]$. Again it is readily verified that the set of equivalence classes $[\mathcal{I}]$ form a group $\text{Cl}_{\mathcal{M}}(K)$ under the operation $[\mathcal{I}][\mathcal{J}] = [\mathcal{I}\mathcal{J}]$. There is a natural surjective homomorphism from $\text{Cl}_{\mathcal{M}}(K)$ to the class group of K , so that the $[\mathcal{I}]$ for principal ideals $\mathcal{I}$ form an index 2 subgroup $\text{Cl}_{\mathcal{M}}(K)_0$ of $\text{Cl}_{\mathcal{M}}(K)$.

Lemma 2 *The groups $\text{Cl}_{\mathcal{M}}(K)_0$ and $\text{Cl}_{\mathcal{M}}(K)$ are isomorphic respectively to $\mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ and $\mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z}$.*

Proof Let $\mathcal{I} = \langle a + b\sqrt{-6} \rangle$ be a principal ideal of $\mathcal{O}$ coprime to $\mathcal{M}$. As $\mathcal{M} = \wp_2^4\wp_3$, then $\mathcal{I}$ is coprime to $\wp_2\wp_3 = \sqrt{-6}\mathcal{O}$. Thus $a + b\sqrt{-6}$ is a unit in the ring $\mathcal{O}/\sqrt{-6}\mathcal{O} \cong \mathbf{Z}/6\mathbf{Z}$ and so $a + b\sqrt{-6} \equiv \pm 1 \pmod{\sqrt{-6}\mathcal{O}}$. Since the unit group of $\mathcal{O}$ is $\{\pm 1\}$ then $\mathcal{I}$ has a unique generator $\gamma(\mathcal{I}) =$

$\pm(a + b\sqrt{-6}) \equiv 1 \pmod{\sqrt{-6}\mathcal{O}}$. If $[\mathcal{I}] = [\mathcal{J}]$ then $\alpha \langle \gamma(\mathcal{I}) \rangle = \beta \langle \gamma(\mathcal{J}) \rangle$ where $\alpha \equiv \beta \equiv 1 \pmod{\mathcal{M}}$. It follows that $\alpha\gamma(\mathcal{I}) = \beta\gamma(\mathcal{J})$ and so $\gamma(\mathcal{I}) \equiv \gamma(\mathcal{J}) \pmod{\mathcal{M}}$. Conversely suppose that $\gamma(\mathcal{I}) \equiv \gamma(\mathcal{J}) \pmod{\mathcal{M}}$. There is $\xi \in \mathcal{O}$ with $\xi\gamma(\mathcal{I}) \equiv 1 \pmod{\mathcal{M}}$. Also $\xi\gamma(\mathcal{J}) \equiv \xi\gamma(\mathcal{I}) \equiv 1 \pmod{\mathcal{M}}$. As $(\xi\gamma(\mathcal{J}))\mathcal{I} = \xi\mathcal{I}\mathcal{J} = (\xi\gamma(\mathcal{I}))\mathcal{J}$ then $[\mathcal{I}] = [\mathcal{J}]$.

Hence the class $[\mathcal{I}]$ for principal $\mathcal{I}$ is determined by the congruence class of its canonical generator $\gamma(\mathcal{I})$ (modulo $\mathcal{M}$). This class is one of the following:

1	1
$1 + \sqrt{-6}$	A
$1 + 2\sqrt{-6}$	$7A^2$
$1 + 3\sqrt{-6}$	$7A^3$
7	7
$7 + \sqrt{-6}$	A^3
$7 + 2\sqrt{-6}$	A^2
$7 + 3\sqrt{-6}$	$7A$

The group is generated by the class A of $1 + \sqrt{-6}$ and the class of 7. These elements have orders 4 and 2 respectively and the second column of the table gives an expression for each element in terms of these generators. Hence, as claimed, $\text{Cl}_{\mathcal{M}}(K)_0$ is isomorphic to $\mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$.

Now let us consider $\text{Cl}_{\mathcal{M}}(K)$. The ideal $\wp_5 = \langle 5, 2 + \sqrt{-6} \rangle$ is non-principal of norm 5. Its square is $\wp_5^2 = \langle 1 - 2\sqrt{-6} \rangle$. Let $B = [\wp_5]$. Then $B^2 = [7\mathcal{O}]A^2$ and so B^4 is trivial. Since $[7\mathcal{O}] = A^2B^2$ then $\text{Cl}_{\mathcal{M}}(K)$ is generated by A and B , each of which has order 4. Hence $\text{Cl}_{\mathcal{M}}(K)$ is isomorphic to $\mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z}$. $\square$

A *character* of $\text{Cl}_{\mathcal{M}}(K)$ is a homomorphism from $\text{Cl}_{\mathcal{M}}(K)$ to the multiplicative group of nonzero complex numbers. Such a character χ is determined by its values on the classes A and B which must be fourth roots of unity. Henceforth we consider the particular χ with $\chi(A) = 1$ and $\chi(B) = i$. Its values are:

C	$\chi(C)$	C	$\chi(C)$
$[\mathcal{O}]$	1	$[\wp_5]$	i
$[(1 + \sqrt{-6})\mathcal{O}]$	1	$[(1 + \sqrt{-6})\wp_5]$	i
$[(1 + 2\sqrt{-6})\mathcal{O}]$	-1	$[(1 + 2\sqrt{-6})\wp_5]$	$-i$
$[(1 + 3\sqrt{-6})\mathcal{O}]$	-1	$[(1 + 3\sqrt{-6})\wp_5]$	$-i$
$[7\mathcal{O}]$	-1	$[7\wp_5]$	$-i$
$[(7 + \sqrt{-6})\mathcal{O}]$	1	$[(7 + \sqrt{-6})\wp_5]$	i
$[(7 + 2\sqrt{-6})\mathcal{O}]$	1	$[(7 + 2\sqrt{-6})\wp_5]$	i
$[(7 + 3\sqrt{-6})\mathcal{O}]$	-1	$[(7 + 3\sqrt{-6})\wp_5]$	$-i$

Lemma 3 *Let $\mathcal{I}$ be an ideal of $\mathcal{O}$ which is coprime to $\mathcal{M}$. Then $\chi(\overline{\mathcal{I}}) = \varepsilon(\mathcal{I})\chi(\mathcal{I})$ where $\varepsilon(\mathcal{I}) = \pm 1$. Suppose that $N(\mathcal{I}) \equiv r \pmod{24}$ where $0 \leq r < 24$. Then $r = 1, 5, 7$ or 11 , and the following table gives the values of $\chi(\mathcal{I})$ up to sign, and of $\varepsilon(\mathcal{I})$.*

r	$\chi(\mathcal{I})$	$\varepsilon(\mathcal{I})$
1	± 1	1
5	$\pm i$	1
7	± 1	-1
11	$\pm i$	-1

Proof The ideal $\mathcal{I}$ is a product of prime ideals $\mathcal{P}_j$ with $\mathcal{P}_j \neq \wp_2, \wp_3$. If $N(\mathcal{P}_j)$ is prime then $(-6/N(\mathcal{P}_j)) = 1$ while if $N(\mathcal{P}_j)$ is the square of a prime then $N(\mathcal{P}_j) \equiv 1 \pmod{24}$. Hence $N(\mathcal{P}_j) \equiv 1, 5, 7$, or $11 \pmod{24}$. The set $\{1, 5, 7, 11\}$ forms a subgroup of the multiplicative group modulo 24. Consequently $N(\mathcal{I}) \equiv 1, 5, 7$ or $11 \pmod{24}$.

An ideal $\mathcal{P} \neq \wp_2, \wp_3$ of prime norm is principal if and only if $\chi(\mathcal{P}) = \pm 1$, that is, if and only if $\chi(\mathcal{P})^2 = 1$. Since the class number of K is 2, then it follows that $\mathcal{I}$ is principal if and only if $\chi(\mathcal{I})^2 = 1$, that is, if and only if $\chi(\mathcal{I}) = \pm 1$. We have seen that every ideal of norm congruent to 1 or 7 modulo 24 is principal, and also every ideal of norm congruent to 5 or 11 modulo 24 is non-principal. This determines the second column in the table. Also since $\mathcal{I}$ and $\overline{\mathcal{I}}$ have the same norm, then $\chi(\overline{\mathcal{I}}) = \varepsilon(\mathcal{I})\chi(\mathcal{I})$ where $\varepsilon(\mathcal{I}) = \pm 1$.

Now $\chi(\langle N(\mathcal{I}) \rangle) = \chi(\mathcal{I})\chi(\overline{\mathcal{I}}) = \varepsilon(\mathcal{I})\chi(\mathcal{I})^2$. If $N(\mathcal{I}) \equiv \pm 1 \pmod{24}$ then $\chi(\langle N(\mathcal{I}) \rangle) = 1$ and if $N(\mathcal{I}) \equiv \pm 5 \pmod{24}$ then $\chi(\langle N(\mathcal{I}) \rangle) = -1$. These determine the value of $\varepsilon(\mathcal{I})$. $\square$

If $\mathcal{I}$ is an ideal of $\mathcal{O}$ which is coprime to $\mathcal{M}$ we put $\chi(\mathcal{I}) = \chi([\mathcal{I}])$, otherwise we put $\chi(\mathcal{I}) = 0$. Then $\chi(\mathcal{I}\mathcal{J}) = \chi(\mathcal{I})\chi(\mathcal{J})$ for all ideals $\mathcal{I}$ and $\mathcal{J}$ of $\mathcal{O}$. Define

$$L(s, \chi) = \sum_{\mathcal{I}} \frac{\chi(\mathcal{I})}{N(\mathcal{I})^s}$$

where the summation is over all the nonzero ideals of $\mathcal{O}$. This series is absolutely convergent when $\operatorname{Re}(s) > 1$. However we shall only be concerned with it as a formal generating function, rather than as a function of a complex variable.

Write

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}.$$

We now can determine the $a(n)$ almost completely.

Lemma 4 *The function a is multiplicative: $a(rs) = a(r)a(s)$ whenever r and s are coprime. Thus the values of the $a(m)$ are determined by the values of $a(p^j)$ when p is prime. These are*

$$a(p^j) = \begin{cases} (j+1)\delta(p)^j & \text{if } p \equiv 1 \text{ or } 5 \pmod{24} \\ 1 & \text{if } j \text{ is even and } p \equiv 7, 13 \text{ or } 23 \pmod{24}, \\ (-1)^{j/2} & \text{if } j \text{ is even and } p \equiv 11, 17 \text{ or } 19 \pmod{24}, \\ 0 & \text{otherwise} \end{cases}$$

where $\delta(p) \in \{1, -1\}$ if $p \equiv 1 \pmod{24}$ and where $\delta(p) \in \{i, -i\}$ if $p \equiv 5 \pmod{24}$.

Proof From the unique factorization of ideals in $\mathcal{O}$ and the multiplicativity of χ we have the Euler product for $L(s, \chi)$

$$L(s, \chi) = \prod_{\mathcal{P}} \left(1 + \frac{\chi(\mathcal{P})}{N(\mathcal{P})^s} + \frac{\chi(\mathcal{P})^2}{N(\mathcal{P})^{2s}} + \cdots \right) = \prod_{\mathcal{P}} \left(1 - \frac{\chi(\mathcal{P})}{N(\mathcal{P})^s} \right)^{-1}$$

where the product is over the prime ideals $\mathcal{P}$ of $\mathcal{O}$. Since $\chi(\wp_2) = \chi(\wp_3) = 0$ then

$$L(s, \chi) = \prod_{p \geq 5} L_{(p)}(s, \chi)$$

where p runs over the rational primes and

$$L_{(p)}(s, \chi) = \begin{cases} \left(1 - \frac{\chi(\wp_p)}{p^s} \right)^{-1} \left(1 - \frac{\chi(\overline{\wp_p})}{p^s} \right)^{-1} & \text{if } (-6/p) = 1, \\ \left(1 - \frac{\chi(p\mathcal{O})}{p^{2s}} \right)^{-1} & \text{if } (-6/p) = -1. \end{cases}$$

This Euler product representation shows that a is multiplicative.

For $(-6/p) = -1$ we get

$$L_{(p)}(s, \chi) = \sum_{j=0}^{\infty} \frac{\chi(p\mathcal{O})^j}{p^{2js}}.$$

Recall that $\chi(p\mathcal{O}) = 1$ if $p \equiv \pm 1 \pmod{12}$ and $\chi(p\mathcal{O}) = -1$ if $p \equiv \pm 5 \pmod{12}$. This gives the values stated for $a(p^j)$ whenever $p \equiv 13, 17, 19$ or $23 \pmod{24}$.

Suppose then that $(-6/p) = 1$. Then $\wp_p$ is principal if and only if $\overline{\wp_p}$ is, and so either $\chi(\wp_p)$ and $\chi(\overline{\wp_p})$ both lie in $\{\pm 1\}$ or they both lie in $\{\pm i\}$. Hence $\chi(\wp_p) = \pm \chi(\overline{\wp_p})$. Write $\chi(\wp_p) = \varepsilon_p \chi(\overline{\wp_p})$. If $\varepsilon_p = -1$ then

$$L_{(p)}(s, \chi) = \left(1 + \frac{\chi(\wp_p \overline{\wp_p})}{p^{2s}} \right)^{-1} = \left(1 + \frac{\chi(p\mathcal{O})}{p^{2s}} \right)^{-1} = \sum_{j=0}^{\infty} (-1)^j \frac{\chi(p\mathcal{O})^j}{p^{2js}}.$$

If $\varepsilon_p = 1$ define $\delta(p) = \chi(\wp_p) = \chi(\overline{\wp_p})$. Then

$$L_{(p)}(s, \chi) = \left(1 - \frac{\delta(p)}{p^s}\right)^{-2} = \sum_{j=0}^{\infty} (j+1) \frac{\delta(p)^j}{p^{js}}.$$

We can distinguish between these cases by congruence conditions modulo 24. If $p \equiv 1$ or $7 \pmod{24}$ then $\wp_p$ is principal and so $\chi(\wp_p) = \pm 1$. Hence $\varepsilon_p = \chi(\wp_p)\chi(\overline{\wp_p}) = \chi(p\mathcal{O})$ so that $\varepsilon_p = 1$ when $p \equiv 1 \pmod{24}$ and $\varepsilon_p = -1$ when $p \equiv 7 \pmod{24}$. If $p \equiv 5$ or $11 \pmod{24}$ then $\wp_p$ is non-principal and so $\chi(\wp_p) = \pm i$. Hence $\varepsilon_p = -\chi(\wp_p)\chi(\overline{\wp_p}) = -\chi(-p\mathcal{O})$ so that $\varepsilon_p = 1$ when $p \equiv 5 \pmod{24}$ and $\varepsilon_p = -1$ when $p \equiv 11 \pmod{24}$. These give the remaining entries in our table.

Recall that for $p \equiv 1 \pmod{24}$ we have $\delta(p) = \chi(\wp_p) = \pm 1$ and for $p \equiv 5 \pmod{24}$ we have $\delta(p) = \chi(\wp_p) = \pm i$. $\square$

We remark that there is no simple congruence criterion on p that determines $\delta(p)$ for $p \equiv 1$ or $5 \pmod{24}$.

Corollary 1 *Let p be prime. If $p \equiv 7, 13$ or $23 \pmod{24}$ then $a(p^2m) = a(m)$ for all m , while if $p \equiv 11, 17$ or $19 \pmod{24}$ then $a(p^2m) = -a(m)$ for all m .*

Proof We write $m = p^j m'$ where p is coprime to m' . Then we compare

$$a(m) = a(p^j m') = a(p^j) a(m')$$

and

$$a(p^2 m) = a(p^{j+2} m') = a(p^{j+2}) a(m')$$

which follow from the multiplicativity of a . The corollary now follows from the computed values of $a(p^j)$. $\square$

3 Products of eta functions

We now relate the $a(n)$ to the $b(m)$ and $c(m)$ defined in the introduction. These are related to products of Dedekind eta-functions. Although we shall argue purely formally, we suppose where relevant that $q = \exp(2\pi iz)$ where z is a complex variable. However, our arguments do not depend on this interpretation of q as a complex number.

Lemma 5 *For each integer $m \geq 0$ we have*

$$a(1 + 24m) = b(m)$$

and

$$a(5 + 24m) = 2ic(m)$$

while $a(n) = 0$ if n is not congruent to 1 or 5 modulo 24.

Proof First of all we show some cancellation occurs in the series for $L(s, \chi)$, and also that this series separates neatly into “real” and “imaginary” parts.

Let $\mathcal{I}$ be an ideal of $\mathcal{O}$. If $N(\mathcal{I}) \equiv 7 \pmod{24}$ then $\chi(\mathcal{I}) = \pm 1$ and $\chi(\mathcal{I})\chi(\bar{\mathcal{I}}) = -1$ so that $\chi(\bar{\mathcal{I}}) = -\chi(\mathcal{I})$. In the sum for $L(s, \chi)$ the terms corresponding to $\mathcal{I}$ and $\bar{\mathcal{I}}$ cancel and so $a(n) = 0$ when $n \equiv 7 \pmod{24}$. If $N(\mathcal{I}) \equiv 11 \pmod{24}$ then $\chi(\mathcal{I}) = \pm i$ and $\chi(\mathcal{I})\chi(\bar{\mathcal{I}}) = 1$ so again $\chi(\bar{\mathcal{I}}) = -\chi(\mathcal{I})$ and so $a(n) = 0$ when $n \equiv 11 \pmod{24}$. Hence

$$L(s, \chi) = \sum_{n \equiv 1 \pmod{24}} \frac{a(n)}{n^s} + \sum_{n \equiv 5 \pmod{24}} \frac{a(n)}{n^s}$$

and the first of these sums has real coefficients while the second has purely imaginary coefficients. In particular, $a(n) = 0$ unless $n \equiv 1$ or $5 \pmod{24}$.

The ideals of $\mathcal{O}$ which have norm congruent to 1 modulo 24 are principal and have the form $\langle a + b\sqrt{-6} \rangle$ where a and b are uniquely determined by the condition that $a \equiv 1 \pmod{6}$. This ideal will have norm congruent to 1 modulo 24 if and only if $b = 2c$ is even. Hence

$$\sum_{n \equiv 1 \pmod{24}} a(n)q^n = \sum_{\substack{a=-\infty \\ a \equiv 1 \pmod{6}}}^{\infty} \sum_{c=-\infty}^{\infty} \chi(\langle a + 2c\sqrt{-6} \rangle) q^{a^2 + 24c^2}.$$

But from the table of values of χ we see that $(\langle a + 2c\sqrt{-6} \rangle) = (-1)^{(a-1)/2+c}$ so that

$$\sum_{n \equiv 1 \pmod{24}} a(n)q^n = \sum_{\substack{a=-\infty \\ a \equiv 1 \pmod{6}}}^{\infty} (-1)^{(a-1)/2} q^{a^2} \sum_{c=-\infty}^{\infty} (-1)^c q^{24c^2}.$$

But on putting $a = 1 + 6r$ we get, using (1)

$$\sum_{\substack{a=-\infty \\ a \equiv 1 \pmod{6}}}^{\infty} (-1)^{(a-1)/2} q^{a^2} = q \sum_{r=-\infty}^{\infty} (-1)^r q^{36r^2 + 12r} = q(q^{24})_{\infty} = \eta(24z).$$

Also

$$\sum_{c=-\infty}^{\infty} (-1)^c q^{24c^2} = \frac{(q^{24})_{\infty}^2}{(q^{48})_{\infty}} = \frac{\eta(24z)^2}{\eta(48z)}$$

by (2).

The ideals of $\mathcal{O}$ which have norm congruent to 5 modulo 24 are non-principal. For each such ideal $\mathcal{I}$, $\wp_2 \mathcal{I}$ is principal and $\wp_2 \mathcal{I} = \langle a + b\sqrt{-6} \rangle$

with $a + b\sqrt{-6} \in \wp_2$. Hence $a = 2c$ is even, and $N(\mathcal{I}) = N(a + b\sqrt{-6})/2 = 2c^2 + 3b^2$. For this to be congruent to 5 modulo 24 it is necessary and sufficient that b be odd and c be congruent to $\pm 1 \pmod{6}$. We choose a canonical generator for $\wp_2\mathcal{I}$ by specifying that $c \equiv -1 \pmod{6}$. Recall that $\wp_5 = \langle 5, 2 + \sqrt{-6} \rangle$ is non-principal. Also $\wp_2\wp_5 = \langle 2 + \sqrt{-6} \rangle$ and $\wp_2^2 = \langle 2 \rangle$ so that $\wp_2^{-1}\wp_5 = \langle \frac{1}{2}(2 + \sqrt{-6}) \rangle$. Thus $\wp_5\mathcal{I} = \langle \alpha \rangle$ where

$$\alpha = \frac{1}{2}(2 + \sqrt{-6})(2c + b\sqrt{-6}) = 2c - 3b + (b + c)\sqrt{-6}.$$

Note that $2c - 3b \equiv -2 - 3 \equiv 1 \pmod{6}$ and $b + c$ is even. Hence

$$\chi(\langle \alpha \rangle) = (-1)^{(2c-3b-1)/2+(b+c)/2} = (-1)^{(c+1)/2+b}.$$

Thus $\chi(\mathcal{I}) = -i(-1)^{(c+1)/2+b} = i(-1)^{b+(c-1)/2} = -i(-1)^{(c-1)/2}$ as b is odd.

Hence

$$\sum_{n \equiv 5 \pmod{24}} a(n)q^n = -i \sum_{\substack{c=-\infty \\ c \equiv -1 \pmod{6}}}^{\infty} (-1)^{(c-1)/2} q^{2c^2} \sum_{\substack{b=-\infty \\ b \equiv 1 \pmod{2}}}^{\infty} q^{3b^2}.$$

If we put $c = -1 - 6r$ we get

$$\begin{aligned} \sum_{\substack{c=-\infty \\ c \equiv -1 \pmod{6}}}^{\infty} (-1)^{(c-1)/2} q^{2c^2} &= \sum_{r=-\infty}^{\infty} (-1)^{-1-3r} q^{72r^2+24r+2} \\ &= -q^2 \sum_{r=-\infty}^{\infty} (-1)^r q^{2(36r^2+12r)} \\ &= -q^2 (q^{48})_{\infty} = -\eta(48z) \end{aligned}$$

by (1) again. Also

$$\sum_{\substack{b=-\infty \\ b \equiv 1 \pmod{2}}}^{\infty} q^{3b^2} = 2q^3 \frac{(q^{48})_{\infty}^2}{(q^{24})_{\infty}} = 2 \frac{\eta(48z)^2}{\eta(24z)}$$

by (3). We conclude that

$$\sum_{n=1}^{\infty} a(n)q^n = q \frac{(q^{24})_{\infty}^3}{(q^{48})_{\infty}} + 2iq^5 \frac{(q^{24})_{\infty}^3}{(q^{48})_{\infty}} = \frac{\eta(24z)^3}{\eta(48z)} + 2i \frac{\eta(48z)^3}{\eta(24z)}.$$

Hence

$$b(m) = a(1 + 24m) \quad \text{and} \quad 2ic(m) = a(5 + 24m)$$

as claimed. $\square$

We now can prove Theorem 1. If p is prime and $p \equiv 7, 13$ or $23 \pmod{24}$ then

$$b(m) = a(1 + 24m) = a(p^2(1 + 24m)) = b(p^2m + (p^2 - 1)/24)$$

and

$$c(m) = a(5 + 24m) = a(p^2(5 + 24m)) = c(p^2m + 5(p^2 - 1)/24)$$

by Corollary 1. Similarly for primes $p \equiv 11, 17$ or $19 \pmod{24}$ we get

$$b(m) = a(1 + 24m) = -a(p^2(1 + 24m)) = -b(p^2m + (p^2 - 1)/24)$$

and

$$c(m) = a(5 + 24m) = -a(p^2(5 + 24m)) = -c(p^2m + 5(p^2 - 1)/24).$$

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