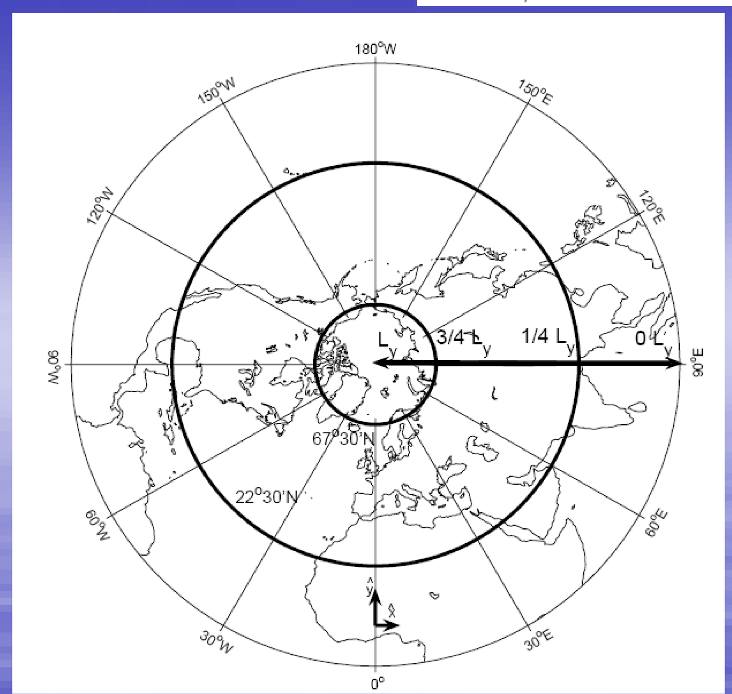
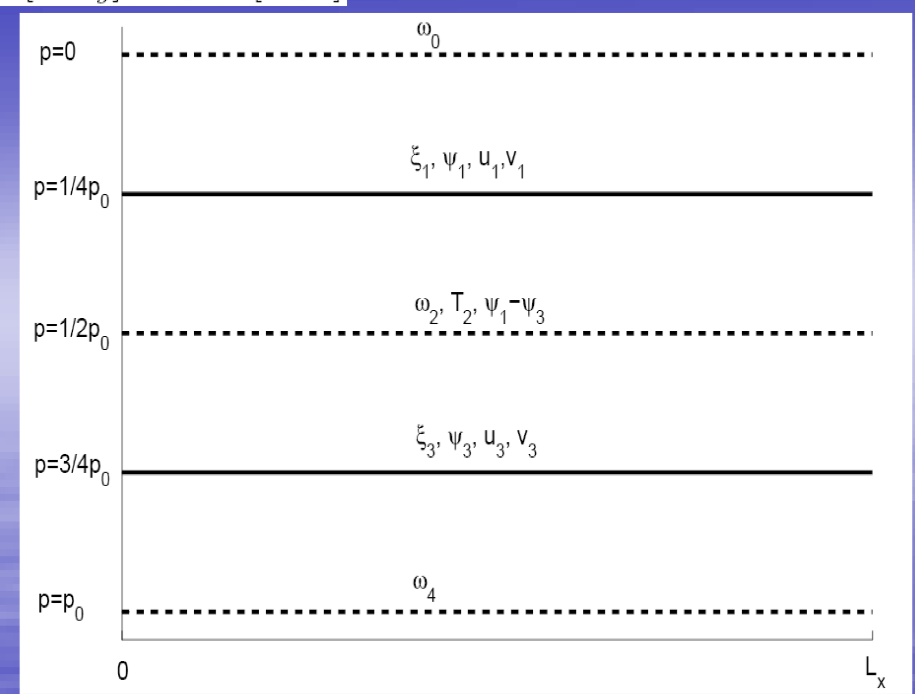
Statistical Properties of a Minimal Climate Model: What beyond mean field theory?

Valerio Lucarini [valerio.lucarini@unibo.it] (1), Renato Vitolo (2), Antonio Speranza (3)
(1) Department of Physics, University of Bologna,Bologna, Italy
(2) School of Engineering, Computing and Mathematics, Univ. of Exeter, Exeter, UK
(3) Department of Mathematics, Univ. of Camerino, Camerino, Italy

•Lucarini, V., Speranza, A. Vitolo, R. (2007): **Parametric smoothness and self-scaling of the statistical properties of a minimal climate model**: What beyond the mean field theories?, Physica D, 234, 105-123
• Lucarini, V., Speranza, A. Vitolo, R. **Self-Scaling of the Statistical Properties of a Minimal Model of the Atmospheric Circulation**,, pp. 197-219, in 20 Years of Nonlinear Dynamics in Geosciences, Eds. J. Elsner, A. Tsonis, Springer (New York, USA) (2007)

$x \in \mathbf{R}/2\pi L_x, \quad y \in [0, L_y], \quad p \in [0, p_0]$





QG model: Two layers, mid-latitude β-channel, Eckman pumping, Newtonian Forcing

ABSTRACT

A quasi-geostrophic intermediate complexity model of the mid-latitude atmospheric circulation is considered, featuring simplified baroclinic conversion and barotropic convergence processes. The model undergoes baroclinic forcing towards a given latitudinal temperature profile controlled by the forced equator-to-pole temperature difference T_E., and features thermal diffusion and viscous-like dissipation. As T_E increases, a transition takes place from a stationary regime - Hadley equilibrium - to a periodic regime, and eventually to a chaotic regime where evolution takes place on a strange attractor. The dependence of the attractor dimension, metric entropy, and bounding box volume in phase space is studied by varying T_E. All the considered properties have a smooth dependence on T_E, which results into power law scaling properties. These properties are coherent with the chaotic hypothesis proposed by Gallavotti and Cohen, which entails an effective structural stability for the attractor of the system. Also, for the statistical properties of global observables having physical relevance and responding to global balances, like the total energy of the system and the averaged zonal wind, power-law scalings with respect to T_E are detected. If verified as being general enough, scaling laws could be of great help in setting up a theory for the overall statistical properties of the general circulation of the atmosphere and in guiding - on a heuristic basis – both data analysis and realistic simulations, going beyond the unsatisfactory mean field theories and brute force approaches. A leading example for this would be the possibility of estimating the sensitivity of the output of the system with respect to changes in the parameters.

INTRODUCTION

Historically, the problem of **General Atmospheric Circulation** (GAC) has been essentially approached in terms of analyzing the **time-mean circulation**, interpreting it as a **fixed point** in the phase-space, and trying to parameterize the processes which could ***maintain* it stationary**. Such view is **unsatisfactory** for both mathematical and physical reasons.

• In dynamical systems, **the stability properties of the *time mean state* say nothing about the properties of the full nonlinear system**. It is **impossible** to create a **self-consistent theory** of the time-mean circulation relying only on the time-mean fields.
• It is **impossible** to apply the **fluctuation-dissipation theorem** for a chaotic dissipative system such as the climate system. It is then impossible to parameterize correctly a Climate Change theory, due to the **non-equivalence between the external and internal fluctuations**. Internal and external flcutuations are not equivalent, because **internal fluctuations** occur along the unstable manifold and the **external fluctuations** move the system out of the attractor with probability 1.
• The adoption of models of **ever increasing resolution** will not necessarily lead to the *final* understanding of the GAC (a sort of *brute force* approach), since in the limit of infinite resolution for any numerical model of fluid flow the numerical convergence to the statistical properties of the real fluid **is not guaranteed.**

Our work focuses on the following **question**: what may come next and what can we do for setting up a **new theory of the GAC**? Which statistical properties may be relevant? We are moving along the lines of **Lorenz** in the construction of **minimal climate models** aimed at a basic understanding of the fully nonlinear physical mechanisms and mathematical properties of the real system. We are searching for a **paradigmatic framework** for future research.

HADLEY EQUILIBRIUM, BIFURCATIONS, CHAOS

We take as numerical laboratory a **quasi-geostrophic (QG)** model of *intermediate* complexity for the mid-latitudes atmospheric circulation. The number of variables is between 48 and 384. The model is vertically discretized into **two layers**, which is the minimum for **baroclinic conversion** to take place, and latitudinally discretized by a Fourier half-sine pseudo-spectral expansion up to order *JT*. We have used *JT* = 8, 16, 32, 64, yielding a hierarchy of QG models having increasing phase space dimension. These models are **almost-linear**: the eddy field is truncated to **one wavenumber** in the longitudinal (zonal) direction, so that the evolution equation of the **waves** is **linear** in terms of the **time-varying zonal flow**. This provides a dynamical meaning for the separation between zonal and eddy flow, where the zonal wind acts effectively as an integrator.

EQUATIONS OF THE MODEL

$$\partial_t \Delta_H \tau - \frac{2}{H_2^2}\frac{\partial}{\partial t}\tau + J\left(\tau, \Delta_H\phi + \beta y + \frac{2}{H_2^2}\phi\right) + J(\phi, \Delta_H \tau) = \frac{2\nu_E}{H_2^2}\Delta_H(\phi - \tau) - \frac{2\kappa}{H_2^2}\Delta_H\tau + \frac{2\nu_N}{H_2^2}(\tau - \tau^\star),$$
$$\partial_t \Delta_H \phi + J(\phi, \Delta_H\phi + \beta y) + J(\tau, \Delta_H \tau) = -\frac{2\nu_E}{H_2^2}\Delta_H(\phi - \tau).$$

$$\phi(x, y, t) = -\int_{\pi/2}^{y} U(z,t)\,\mathrm{d}z + A\exp(\mathrm{i}\chi x) + \mathrm{c.c.},$$
$$\tau(x, y, t) = -\int_{\pi/2}^{y} m(z,t)\,\mathrm{d}z + B\exp(\mathrm{i}\chi x) + \mathrm{c.c.},$$

***ϕ* barotropic component; *τ* baroclinic component**

$$X = \sum_{j=1}^{JT} X_j \sin\left(\frac{\pi j y}{L_y}\right)$$

***JT* components in y**
1 wave in x, L_x=2π/χ

The model undergoes baroclinic forcing towards a given latitudinal temperature profile controlled by the forced equator-to-pole temperature difference T_E, and features thermal diffusion and viscous-like dissipation. As T_E increases, a transition takes place from a stationary regime - Hadley equilibrium - to a periodic regime, and eventually to a chaotic regime with a strange attractor. The route to the **strange attractor** involves a **Hopf bifurcation** of the Hadley equilibrium (baroclinic instability), a finite number of **periodic doublings**, followed by **quasi-periodic** breakdown of the **invariant torus**.

In the chaotic regime, the 1st **Lyapunov exponent λ_1**, the **predictability time $t_p = 1/\lambda_1$**, the **metric entropy** *h*, the **Lyapunov dimension D_L**, the **bounding box Volume** V_{BB} are computed for various T_E.

• λ_1 increases sublinearly with T_E. If T_E>14, t_p<10 (~12 days). In the range T_E≥12, $t_p \propto (T_E - T^{crit}_E)^\gamma$, with γ ranging around [-0.85,-0.8]. The metric entropy has a linear dependence *with respect to* T_E. The larger T_E, the larger the number of *active D.O.F.*, and the faster the *forgetting* of the initial condition.

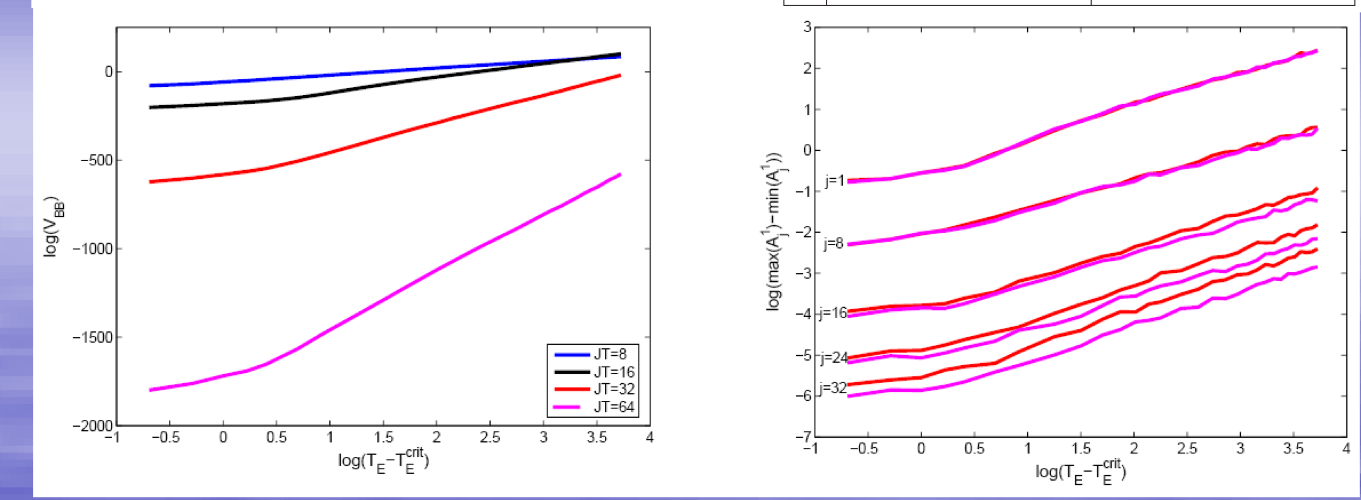
• It is possible to distinguish three characteristic regimes in the behavior of the function $D_L(T_E)$:
1. For small values of T_E, $D_L \propto (T_E - T^{crit}_E)^\gamma$*, with* γ ranging between [0.5,0.7]
2. For larger values of T_E a linear scaling regime of D_L is found, with a linear coefficient ~1.2.
3. For T_E larger than a *JT*-depending threshold, D_L increase sublinearly with T_E.

• The volume of the bounding box V_{BB} increases with T_E. Two sharply distinct and well defined power-law regimes exist: $V_{BB} \propto (T_E - T^{crit}_E)^\gamma$. In the lower range $(T_E - T^{crit}_E)$<1.5, γ ~ 2*XJT*, for larger values of T_E we have that γ ~ 5*XJT*. This is 1/3 and 5/6 of the total number of degrees of freedom.

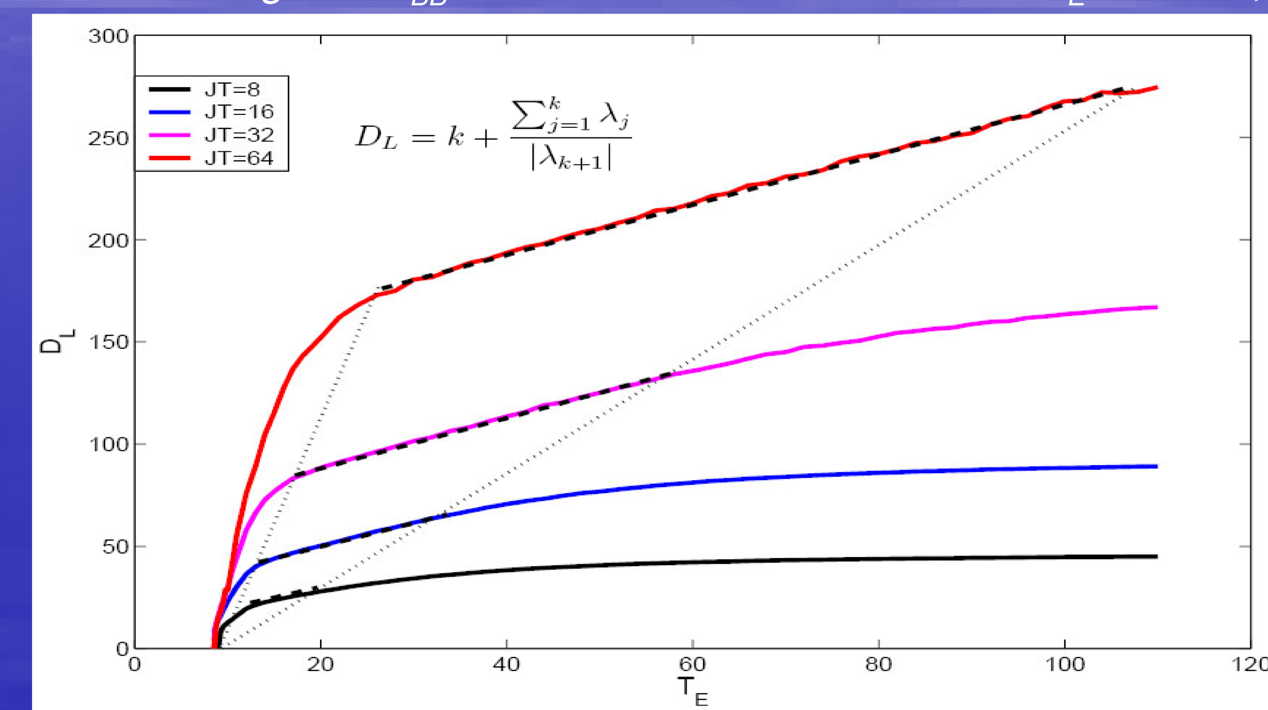
• The physical observables total energy E_{TOT} and wind fields <*U*> and <*m*> also obey power laws:
1. $E_{TOT} \propto (T_E - T^{crit}_E)^\gamma$, with γ ~ 1.9 for $(T_E - T^{crit}_E)$<1.5 and γ ~ 1.52 for higher values of T_E
2. **<U>=<m>** $\propto (T_E - T^{crit}_E)^\gamma$, with γ ~ 0.875 for $(T_E - T^{crit}_E)$<1.5 and γ ~ 0.58 for higher values of T_E
3. Even if a sort of baroclinic adjustment is in action, the system is not kept near marginal stability

$$V_{BB} \propto (T_E - T_E^{crit})^\gamma, \quad \gamma = \epsilon N \quad \epsilon \sim \begin{cases} 1/3, & T_E - T_E^{crit} \lesssim 1.5 \\ 5/6, & T_E - T_E^{crit} \gtrsim 1.5 \end{cases}$$

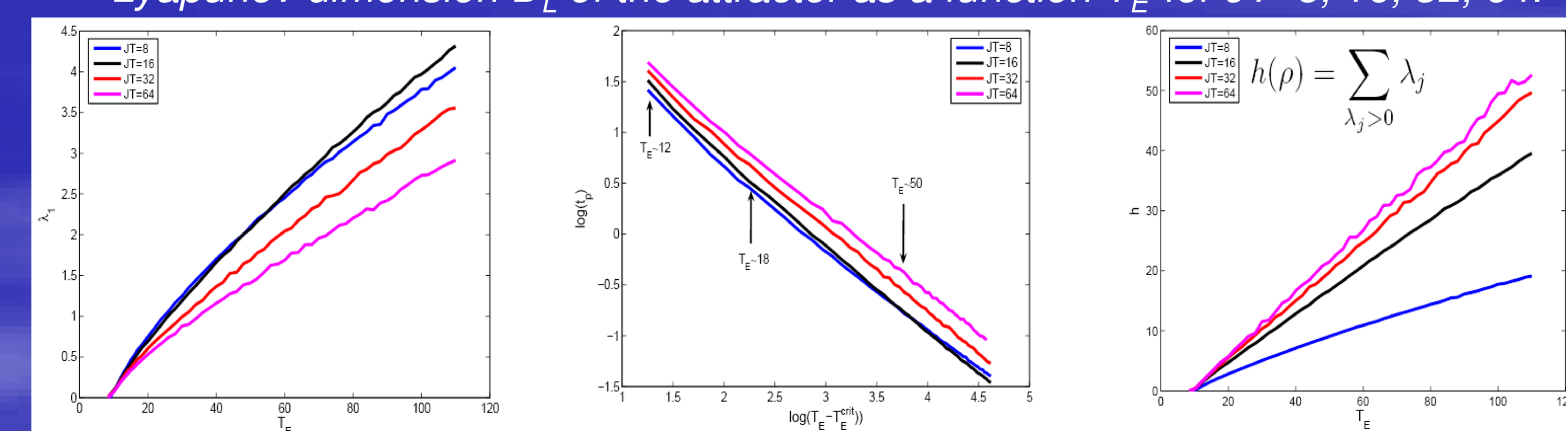
JT	$\gamma[\log(T_E - T_E^{crit}) \le 0.5]$	$\gamma[\log(T_E - T_E^{crit}) \ge 0.5]$
8	40 ± 1	40 ± 1
16	33 ± 3	80 ± 1
32	66 ± 2	160 ± 1
64	133 ± 4	320 ± 1



Volume of the Bounding Box V_{BB} of the attractor as a function of T_E for JT=8, 16, 32, 64.



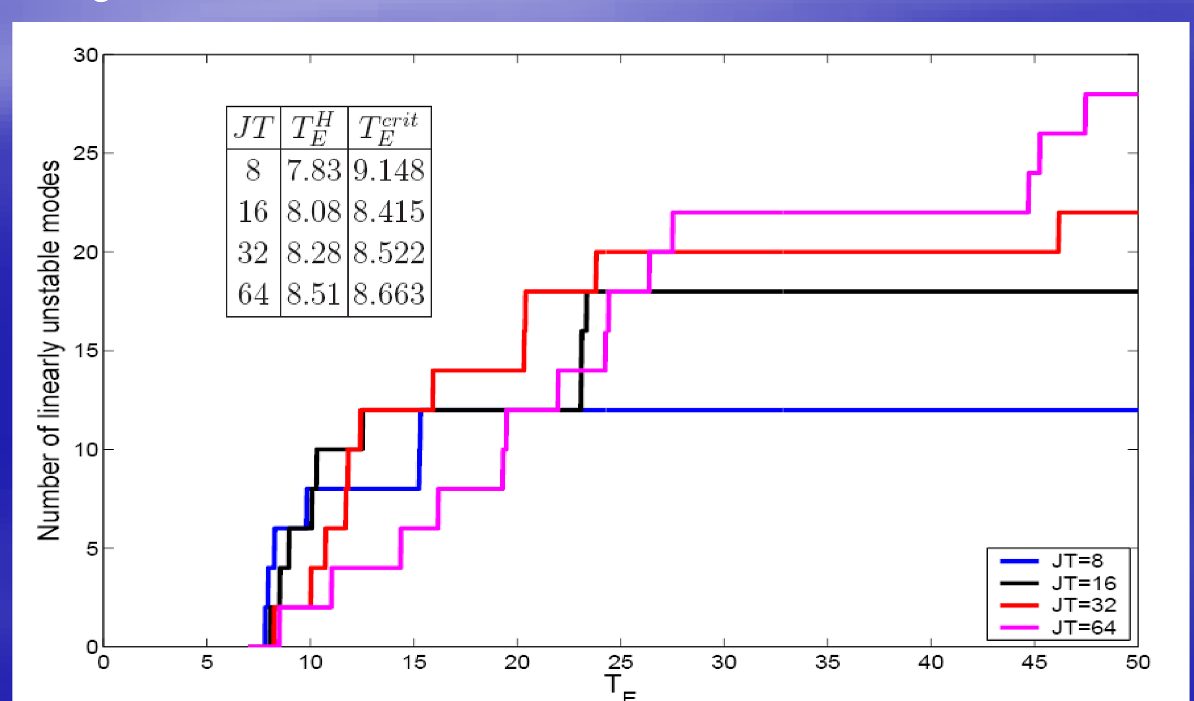
Lyapunov dimension D_L of the attractor as a function T_E for JT=8, 16, 32, 64.



Maximal Lyapunov exp. λ_1 *predictability time t_p* *metric entropy h*

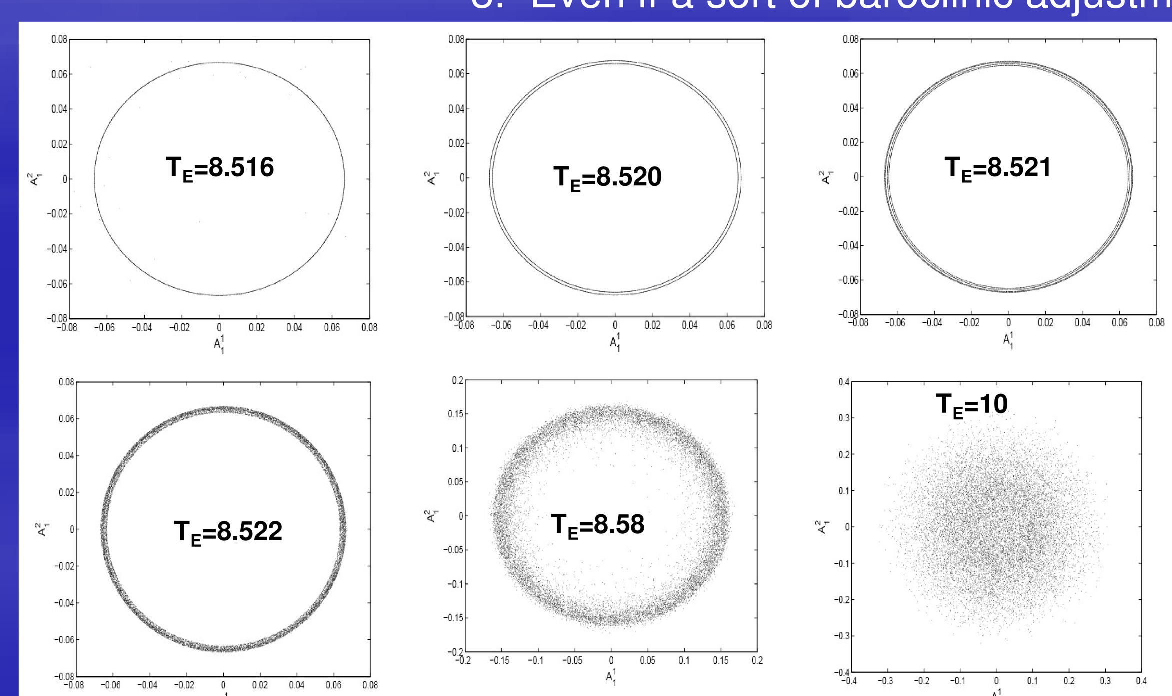
CONCLUSIONS

• The **smoothness** of the parametric dependence of the system properties with respect to changes to the forcing parameter is a matter worth exploring. This coherent with the **chaotic hypothesis** proposed by Gallavotti and Cohen, which entails an effective **structural stability for the attractor of the system**.
• Provided generality applies, the **scaling properties** could be of great help in setting up a theory for the statistical properties of the GAC. A leading example would be the estimation of the sensitivity to of a system.
• A promising physical framework for the analysis of the GAC is to assume that the **wave-wave interaction is mediated by the zonal flow**, which acts as an integrator. Since in the chaotic regime the evolution of the wave components of the fields is determined by the **random matrices** associated to the zonal flow, a sort of bulk ***climate theory*** for the atmospheric disturbances could be constructed determining the statistical properties of such random matrices as a function of the variability of the zonal flow.

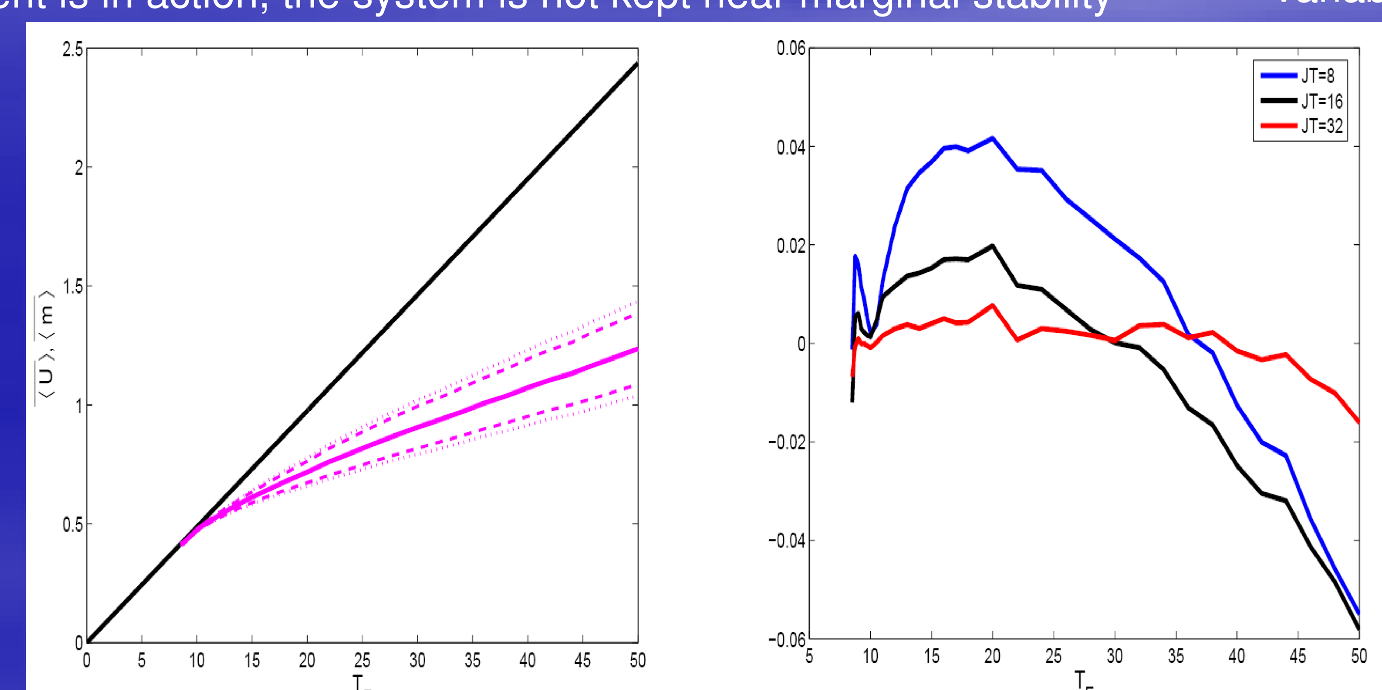


JT	T_E^H	T_E^{crit}
8	7.83	9.148
16	8.08	8.415
32	8.28	8.522
64	8.51	8.663

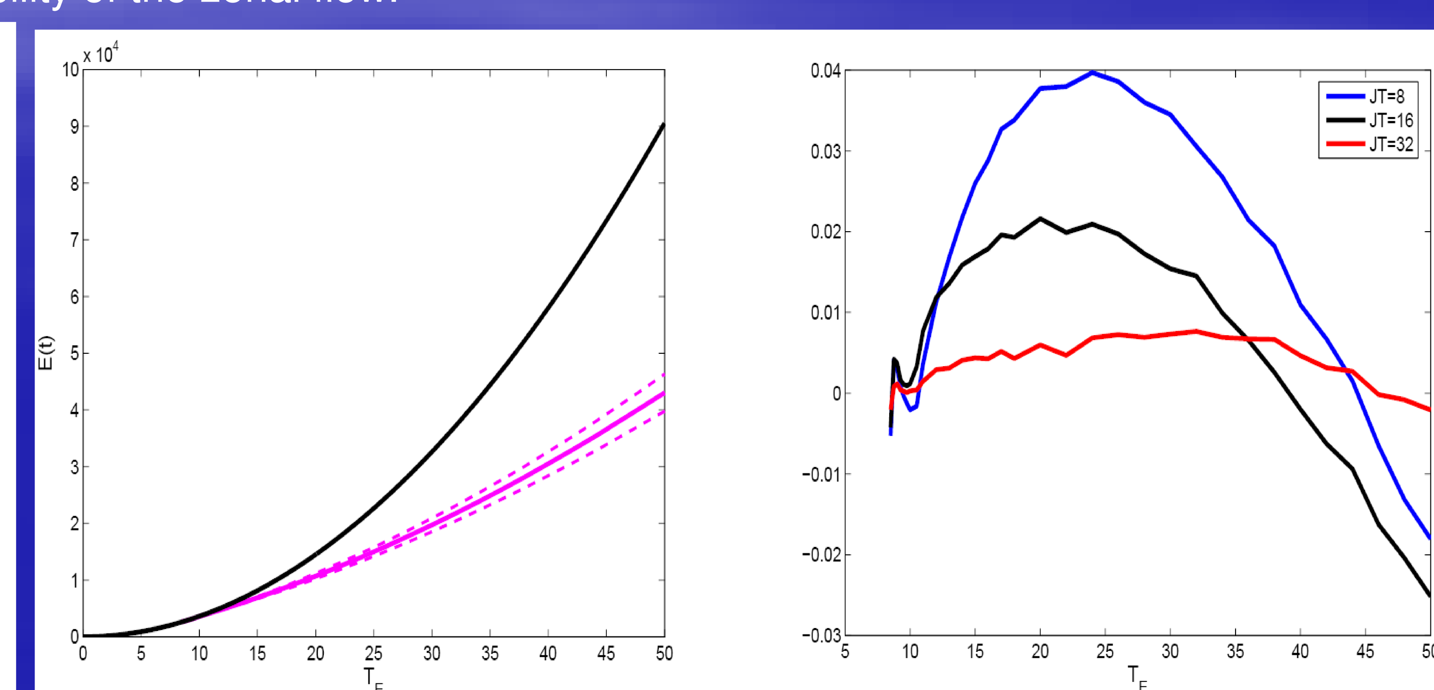
Number of unstable modes of the Hadley equilibrium



Transition to Chaos vs T_E– Poincarè sections



Statistics of the average Total Energy vs T_E



Average total energy vs T_E; energy density: $e(x,y,t) = \frac{\delta p}{g}\left[\left(\vec{\nabla}\phi\right)^2 + \left(\vec{\nabla}\tau\right)^2 + \frac{2}{H_2^2}\tau^2\right]$