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CHAOS AND QUASI-PERIODICITY IN DIFFEOMORPHISMS OF THE SOLID TORUS

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Abstract. This paper focuses on the parametric abundance and the 'Cantorial' persistence under perturbations of a recently discovered class of strange attractors for diffeomorphisms, the so-called quasi-periodic Hénon-like. Such attractors were first detected in the Poincaré map of a periodically driven model of the atmospheric flow: they were characterised by marked quasi-periodic intermittency and by $\Lambda_1 > 0, \Lambda_2 \approx 0$, where Λ_1 and Λ_2 are the two largest Lyapunov exponents. It was also conjectured that these attractors coincide with the closure of the unstable manifold of a hyperbolic invariant circle of saddle-type.

This type of attractor is here investigated in a model map of the solid torus, constructed by a skew coupling of the Hénon family of planar maps with the Arnol′d family of circle maps. It is proved that Hénon-like strange attractors occur in certain parameter domains. Numerical evidence is produced, suggesting that quasi-periodic circle attractors and quasi-periodic Hénon-like attractors persist in relatively large subsets of the parameter space. We also discuss two problems in the numerical identification of so-called strange nonchaotic attractors and the persistence of all these classes of attractors under perturbation of the skew product structure.

1. Introduction.

The study of the geometric structure and statistical properties of attractors of maps is a central problem in Dynamical Systems theory. Several mathematical characterisations have been found since the 1990's. A basic example is provided by the Hénon family of maps [21]:

$$H_{a,b} : \mathbb{R}^2 \to \mathbb{R}^2, \quad (x, y) \mapsto (1 - ax^2 + y,\, bx), \tag{1}$$

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where a and b are real parameters. Benedicks and Carleson [2, 3] proved that there exists a set of parameter values $\mathfrak{S}$, with positive Lebesgue measure, such that for all $(a, b) \in \mathfrak{S}$ map (1) has a strange attractor coinciding with the closure $\mathrm{Cl}\,(W^u(p))$ of the unstable manifold of a saddle fixed point p. Here $\mathrm{Cl}\,(-)$ denotes the topological closure. Similar techniques were then used to prove occurrence of strange attractors in families of maps, near homoclinic tangencies in two or higher dimensions [29, 35, 40, 43], and near tangencies in the saddle-node critical case [16]. A general set-up was established in [46] to prove existence of strange attractors with one positive Lyapunov exponent in families of two-dimensional maps. The strange attractors considered in these references are called *Hénon-like* [12, 16, 29, 43]. Essentially, all the above results are obtained by perturbation of 1-dimensional maps. There is limited understanding of the structure of attractors of larger topological dimension. As positive exceptions we mention Viana [44] and Pumariño-Tatjer [36].

A 'new' class of attractors has been conjectured to occur in a Poincaré map of an atmospheric model subject to periodic forcing [9, 10, 45]. Figure 1 (A) shows an attractor of such a Poincaré map, which is a diffeomorphism of $\mathbb{R}^3 = \{x, y, z\}$. A cross-section Σ displays a folded structure (Figure 1 (B)), similar to that of the Hénon attractor. The image of Σ under the Poincaré map is a folded curve looking like a planar Hénon attractor (box in Figure 1 (A)). This suggests that the attractor in Figure 1 (A) is contained in the closure of the unstable manifold $W^u(\mathscr{C})$ of a quasi-periodic invariant circle $\mathscr{C}$ of saddle type, where $W^u(\mathscr{C})$ is two-dimensional. Pronounced quasi-periodic intermittency characterises the dynamics on attractors of this type: this can be detected in the power spectrum [9, 10, 45]. Loosely speaking, the dynamics on such attractors is 'quasi-periodic product Hénon': for this reason, we coin the term *quasi-periodic Hénon-like* attractors. Parameter ranges where quasi-periodic Hénon-like attractors occur are typically preceded and interspersed by the occurrence of Hénon-like attractors and periodicity.

This paper aims to understand the structure of quasi-periodic Hénon-like attractors and their onset through bifurcations from the Hénon-like case. We consider families of maps which, loosely speaking, contain a component for chaoticity and a component for periodic/quasi-periodic transitions. This leads us to the following model:

$$\begin{pmatrix} x \\ y \\ \theta \end{pmatrix} \mapsto \begin{pmatrix} 1 - (a + \varepsilon \sin(2\pi\theta))x^2 + y \\ bx \\ \theta + \alpha + \delta \sin(2\pi\theta) \end{pmatrix}, \tag{2}$$

defined on the solid torus $\mathbb{R}^2 \times \mathbb{S}^1$, where $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ is the circle. Map (2) is a skew product where the Hénon map (1) is driven by the Arnol′d family of maps of the circle [1]:

$$A_{\alpha,\delta} : \mathbb{S}^1 \to \mathbb{S}^1, \quad \theta \mapsto \theta + \alpha + \delta \sin(2\pi\theta). \tag{3}$$

For the Arnol′d family, hyperbolic periodic attractors occur in the interior of countably many *resonance tongues* in the (α, δ)-parameter plane (black in Figure 2 (A)), constituting a large open set. Quasi-periodic dynamics occurs for a positive Lebesgue measure subset of the complement of the tongues [1, 15]. For the Hénon family, hyperbolic periodic dynamics occurs in a countable union of strips with non-empty interior in the (a, b)-parameter plane (Figure 2 (B)). strange attractors occur for a positive measure subset of the complement of the open periodicity strips [3]. Four types of dynamics are obtained for map (2) with $\varepsilon = 0$, by combining the above described 'typical' dynamics for the Arnol′d and Hénon maps, see Table 1. The main

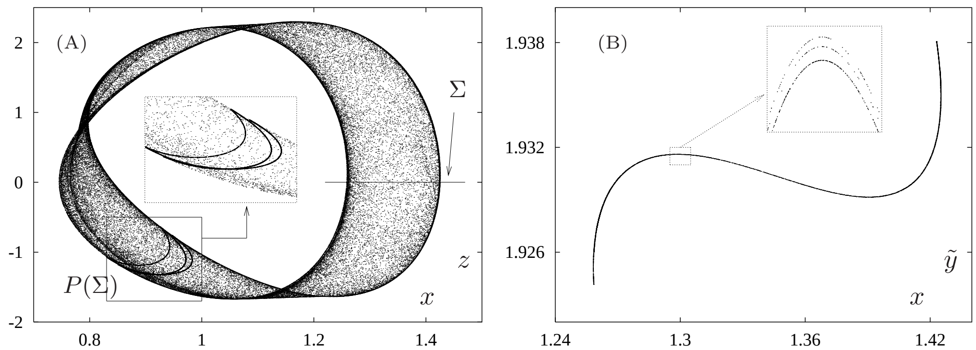


FIGURE 1. (A) Strange attractor of a Poincaré map of an atmospheric model [9], projection on the (x, z)-plane. (B) Projection on the $(x, \tilde{y})$-plane of a 'slice' Σ given by points of the attractor with $|z| < 0.0001$ (see (A)), where $\tilde{y} = y - 0.133 * z$. The central box in (A) shows the image $P(\Sigma)$ under the Poincaré map.

question addressed here is: what are the persistent features of these four types of dynamics for small perturbations $\varepsilon << 1$? We outline our results:

Hénon dynamics \ Arnol′d dynamics	periodic ($\Lambda_A < 0$): black in Fig. 2 (A)	quasi-periodic ($\Lambda_A = 0$): yellow in Fig. 2 (A)
periodic ($\Lambda_H < 0$): black in Fig. 2 (B)	case 1a	case 1b
chaotic ($\Lambda_H > 0$): red in Fig. 2 (B)	case 2a	case 2b

TABLE 1. Four dynamical behaviours for map (2) with $\varepsilon = 0$, arising from combining the dynamics of the Hénon (1) and Arnol′d (3) maps. Λ_A and Λ_H denote the maximal Lyapunov exponent of a typical orbit for the Arnol′d and Hénon family, respectively.

Case 1a: is trivial: this combination gives rise to a hyperbolic periodic attractor in the solid torus $\mathbb{R}^2 \times \mathbb{S}^1$ for map (2) in the uncoupled case $\varepsilon = 0$. This is persistent for $|\varepsilon| \ll 1$ and corresponds to the black regions in Figure 2 (C).

Case 1b: corresponds, in the uncoupled case $\varepsilon = 0$, to a normally hyperbolic quasi-periodic invariant circle attractor for some iterate of map (2): we refer to this as a periodically invariant quasi-periodic circle attractor for map (2). We prove that this attractor has certain persistence properties for $|\varepsilon| \ll 1$, using centre manifold [23] and KAM theory [5, 6] (Section 2.1). Attractors of this type occur in a subset of the yellow domain in Figure 2 (C) located near $\varepsilon = 0$. A different type of attractors occurs for larger ε: namely phenomena related to the so-called *strange nonchaotic attractors* [17, 19, 20, 24, 25, 31, 32] take place.

Case 2a: one of the main results of this paper is that map (2) has a Hénon-like attractor which persists for $|\varepsilon| \ll 1$ (Section 2.2). This type of attractors is found in the red domains of Figure 2 (C), see Figure 3 for illustrations.

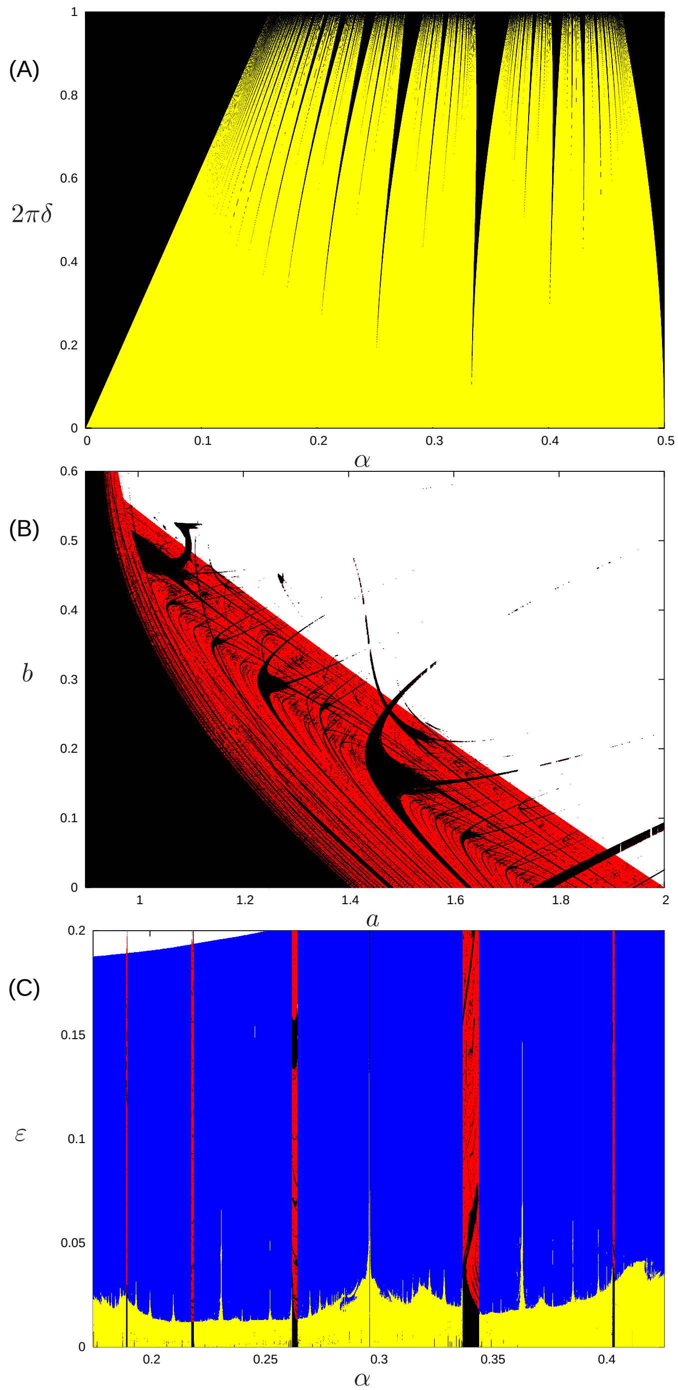


FIGURE 2. Organisation of: (A) the (α, δ)-parameter plane of the Arnol′d family (3); (B) the (a, b)-parameter plane of the Hénon family (1); (C) the (α, ε)-parameter plane of map (2), for $a = 1.25, b = 0.3$, $\delta = 0.6/(2\pi)$. Attractor types are classified according to their Lyapunov exponents, see codes 1-4 in Table 2 and see Appendix A for the algorithm. Initial point is the origin for (B): other initial points might converge to different types of attractors in case of multistability.

code	colour	Lyapunov exponents	attractor type	case
1	black	$0 > \Lambda_1$	periodic attractor	1a
2	yellow	$0 = \Lambda_1 > \Lambda_2$	quasi-periodic invariant circle or other types of attractor (see text for details)	1b
3	red	$\Lambda_1 > 0 > \Lambda_2$	Hénon-like strange attractor	2a
4	blue	$\Lambda_1 > 0 = \Lambda_2$	quasi-periodic Hénon-like strange attractors	2b
	white		no attractor detected	

TABLE 2. Colour coding for Figure 2, attractors are classified by their Lyapunov exponents $\Lambda_1 \geq \Lambda_2 \geq \Lambda_3$. Rightmost column: cases discussed in the Introduction for family (2).

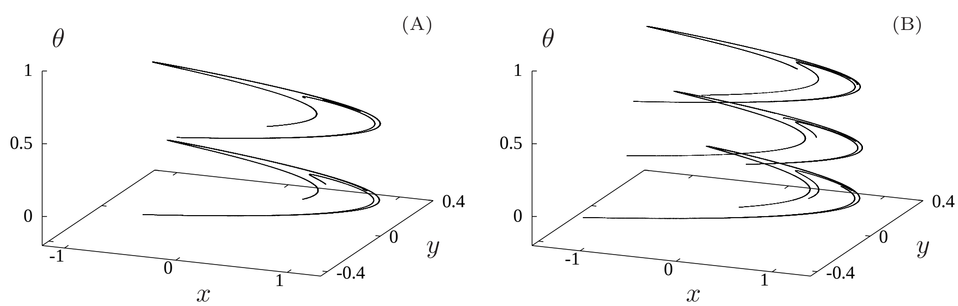


FIGURE 3. Hénon-like strange attractors of the model family (2), for $a = 1.3$, $b = 0.3$, $\varepsilon = 0.2$. (A) Parameters are $(\alpha, \delta) = (0.51, 0.116)$, in a resonance tongue of period two for the Arnol′d family (3). (B) Same as (A) for $\alpha = 0.33793$, in an Arnol′d tongue of period three.

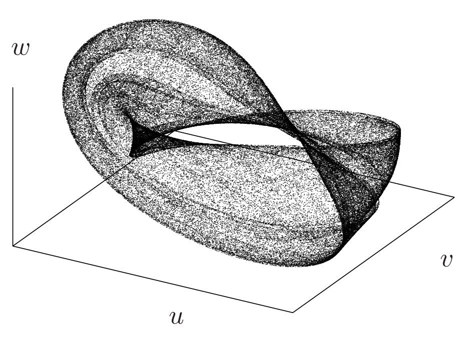


FIGURE 4. Quasi-periodic Hénon-like strange attractor of the model family (2), projection on the (u, v, w)-space, with $u = (r + 4)\cos(2\pi\theta)$, $v = (r + 4)\sin(2\pi\theta)$, with $r = x\cos(2\pi\theta) + 10y\sin(2\pi\theta)$, and $w = -x\sin(2\pi\theta) + 10y\cos(2\pi\theta)$. Parameter values: $a = 1.85$, $b = -0.2$, $\delta = 0$, $\alpha = (\sqrt{5} - 1)/2$, $\varepsilon = 0.1$.

Case 2b: we prove that a quasi-periodic Hénon-like attractor occurs for map (2) in the uncoupled case $\varepsilon = 0$ (Section 2.3). This attractor has the form $\mathrm{Cl}\,(W^u(\mathscr{C}))$, where $\mathscr{C}$ is a normally hyperbolic quasi-periodic invariant circle of saddle type. We provide numerical evidence suggesting that this phenomenon persists for $|\varepsilon| \ll 1$: this corresponds to the blue regions in Figure 2 (C)), see Figure 4 for an illustration.

We present a combination of theoretical and numerical results concerning the issues of persistence of the above four cases, and we also address the following question: which of the above features are persistent under perturbation of the skew product structure of (2), and what changes occur for the non-persistent features?

Our theoretical results are formulated for more general families, of which (2) is a special case, see Section 2 (all longer proofs are postponed to Section 4). Numerical evidence, conjectural results and open problems are discussed in Section 3.

2. **Analytical study.**

Our theoretical results are formulated in the next three subsections, while longer proofs are postponed to Sections 4.1 and 4.2.

2.1. **Invariant circles of saddle type and basins of attraction.**

We generalise an unpublished result of Tangerman and Szewc [34, Appendix 3] to families of maps obtained by perturbing the product of a planar map times a rotation on $\mathbb{S}^1$. Consider a dissipative (i.e., area contracting) diffeomorphism

$$K = (K_1, K_2) : \mathbb{R}^2 \to \mathbb{R}^2, \tag{4}$$

of class C^n with $n \geq 1$. Denote by $R_\alpha : \mathbb{S}^1 \to \mathbb{S}^1$ the rigid rotation $R_\alpha(\theta) = \theta + \alpha$. We define the following family of diffeomorphisms of the solid torus $\mathbb{R}^2 \times \mathbb{S}^1$:

$$P_{\alpha,\varepsilon} : \mathbb{R}^2 \times \mathbb{S}^1 \to \mathbb{R}^2 \times \mathbb{S}^1, \quad (x, y, \theta) \mapsto \big(K_1(x, y) + P_1,\ K_2(x, y) + P_2,\ \theta + \alpha + P_3\big), \tag{5}$$

where P_j, for $j = 1, \ldots, 3$, is a smooth function of $(x, y, \theta, \alpha, \varepsilon)$ such that $P_j = 0$ for $\varepsilon = 0$. Model map (2) is a special case of (5), however the latter is not a skew product, given the full coupling of the two components. A hyperbolic fixed point p of K (4) corresponds to a normally hyperbolic invariant circle $\mathscr{C}_{\alpha,0} = \{p\} \times \mathbb{S}^1$ for the map $P_{\alpha,\varepsilon}$ at $\varepsilon = 0$.

Proposition 1. (NORMALLY HYPERBOLIC INVARIANT CIRCLE) *Suppose that K has a hyperbolic fixed point $p = (x_0, y_0)$. Then for all $\alpha \in [0, 1]$ the map $P_{\alpha,0}$ has a normally hyperbolic invariant circle $\mathscr{C}_{\alpha,0} = \{p\} \times \mathbb{S}^1$. The manifold $\mathscr{C}_{\alpha,0}$ is r-normally hyperbolic for all integers r with $1 \leq r \leq n$. Moreover, for all $r < n$ there exists an $\varepsilon_r > 0$ such that for all $\varepsilon < \varepsilon_r$ and all $\alpha \in [0, 1]$, $P_{\alpha,\varepsilon}$ has a normally hyperbolic invariant circle $\mathscr{C}_{\alpha,\varepsilon}$ of class C^r, which is C^r-close to $\mathscr{C}_{\alpha,0}$.*

Proof. The dynamics of $P_{\alpha,0}$ on $\mathscr{C}_{\alpha,0}$ is parallel with rotation number α. This implies that $\mathscr{C}_{\alpha,0}$ is an r-normally hyperbolic invariant manifold for all $r \leq n$ and, therefore, it is of class C^n. So $\mathscr{C}_{\alpha,0}$ (as well as its stable and unstable manifolds), is persistent under C^n-small perturbations. This directly follows from [23, Theorem 1.1]. □

Proposition 1 allows us to construct a basin of attraction with nonempty interior for the invariant set $\mathrm{Cl}\,(W^u(\mathscr{C}_{\alpha,\varepsilon}))$, provided that p is a saddle point, while the one-dimensional unstable manifold $W^u(p) \subset \mathbb{R}^2$ of the map K, see (4), does not escape to infinity. For $(x, y, \theta) \in \mathbb{R}^2 \times \mathbb{S}^1$, denote by $\omega(x, y, \theta)$ the ω-limit set of (x, y, θ) under $P_{\alpha,\varepsilon}$.

Theorem 2.1. (ATTRACTOR CONTAINED IN $\mathrm{Cl}\,(W^u(\mathscr{C}))$) *Fix integers n and r such that $n \geq 2$ and $1 \leq r < n$. Choose $\varepsilon < \varepsilon_r$ as in Proposition 1 and let $\alpha \in [0,1]$. Suppose that $K : \mathbb{R}^2 \to \mathbb{R}^2$ is of class C^n and satisfies:*

1. *K has a saddle fixed point $p \in \mathbb{R}^2$ and a transversal homoclinic point $q \in W^s(p) \cap W^u(p)$.*
2. *K is uniformly dissipative: there exists $\kappa < 1$ such that $|\det(DK(x,y))| \leq \kappa$ for all $(x,y) \in \mathbb{R}^2$.*
3. *$W^u(p)$ is contained in a bounded subset of $\mathbb{R}^2$.*

Then there exists an $\varepsilon^* < \varepsilon_r$ such that for all $\varepsilon < \varepsilon^*$ there exists an open, nonempty bounded set $U \subset \mathbb{R}^2 \times \mathbb{S}^1$ such that for all $(x,y,\theta) \in U$

$$\omega(x,y,\theta) \subset \mathrm{Cl}\,(W^u(\mathscr{C}_{\alpha,\varepsilon}))\,. \tag{6}$$

Remark 1. By taking iterates of the map $P_{\alpha,\varepsilon}$, Theorem 2.1 can be adapted to the case where p is a saddle periodic point. In this context we have the inclusion (6), where $\mathscr{C}_{\alpha,\varepsilon}$ is a periodically invariant circle, *i.e.* a circle which is invariant under some iterate of $P_{\alpha,\varepsilon}$.

Under the conditions of Theorem 2.1, the invariant set $\mathrm{Cl}\,(W^u(\mathscr{C}_{\alpha,\varepsilon}))$ attracts all orbits with initial state in an open set U. This holds for an open set of ε-values. In general, however, $\mathrm{Cl}\,(W^u(\mathscr{C}_{\alpha,\varepsilon}))$ is not an attractor, since it might be non-topologically transitive. This occurs, for example, if $\mathrm{Cl}\,(W^u(\mathscr{C}_{\alpha,\varepsilon}))$ contains a periodic attractor.

In the next Theorem we prove that at least the circle $\mathscr{C}_{\alpha,\varepsilon}$ is quasi-periodic (and, hence, topologically transitive) for a set of parameter values having large relative measure.

Theorem 2.2. (NORMALLY HYPERBOLIC QUASI-PERIODIC CIRCLES) *Let $P_{\alpha,\varepsilon}$ be a C^n-family of diffeomorphisms as in (5), where $n \geq 5$. Choose ε_r as in Proposition 1. Then there exists an $\varepsilon^{**} < \varepsilon_r$ such that for all $\varepsilon < \varepsilon^{**}$ the following holds.*

1. *There exists a set $D_\varepsilon \subset [0,1]$ with Lebesgue measure $\mathrm{meas}(D_\varepsilon) > 0$ such that for $\alpha \in D_\varepsilon$ the restriction of $P_{\alpha,\varepsilon}$ to the circle $\mathscr{C}_{\alpha,\varepsilon}$ is smoothly conjugate to an irrational rigid rotation.*
2. *$\mathrm{meas}(D_\varepsilon)$ tends to 1 for $\varepsilon \to 0$.*

Proofs of Theorems 2.1 and 2.2 are given in Section 4.1. Theorems 2.2 and 2.1 have straightforward generalisations for the case where p is a hyperbolic periodic point.

2.2. **Hénon-like attractors in a family of skew product maps.** The invariant set $\mathrm{Cl}\,(W^u(\mathscr{C}))$ is attracting under quite general circumstances (Theorem 2.1). In general $\mathrm{Cl}\,(W^u(\mathscr{C}))$ may be not topologically transitive: in this case it is not considered an attractor (see below for precise definitions). For the particular case of map (2), we show that $\mathrm{Cl}\,(W^u(\mathscr{C}))$ contains Hénon-like attractors for a positive measure set of parameter values. A few basic definitions are first recalled.

Definition 2.3. [16, 29, 43] Let $F : M \to M$ be a C^1-diffeomorphism, where M is an m-dimensional smooth manifold.

1. An F-invariant set $\mathscr{A} \subset M$ is called *topologically transitive* if there exists a point $z \in \mathscr{A}$ such that the orbit $\mathrm{Orb}(z) = \{F^j(z)\}_{j\geq 0}$ of z under F is dense in $\mathscr{A}$.
2. A set $\mathscr{A} \subset M$ is called an *attractor* if it is topologically transitive, compact, F-invariant and if the stable set (basin of attraction) $W^s(\mathscr{A})$ has nonempty interior.

3. An attractor $\mathscr{A}$ is called *strange* if there exist constants $\kappa > 0$, $\lambda > 1$, a dense orbit $\mathrm{Orb}(z) \subset \mathscr{A}$ and a vector $v \in T_z M$ such that
$$\|DF^n(z)v\| \geq \kappa\lambda^n \qquad \text{for } n \geq 0.$$
4. The attractor $\mathscr{A}$ is called *Hénon-like* if there exist a saddle periodic orbit $\mathrm{Orb}(p) = \{p, F(p), \dots, F^n(p)\}$, a point z in the unstable manifold $W^u(\mathrm{Orb}(p))$, constants $\kappa > 0$, $\lambda > 1$, and tangent vectors $v, w \in T_z M$, with $w \neq 0$, such that
$$i) \qquad \mathscr{A} = \mathrm{Cl}\left(W^u(\mathrm{Orb}(p))\right), \tag{7}$$
$$ii) \qquad \mathrm{Orb}(z) \text{ is dense in } \mathscr{A}, \tag{8}$$
$$iii) \qquad \|DF^n(z)v\| \geq \kappa\lambda^n \qquad \text{for } n \geq 0, \tag{9}$$
$$iv) \qquad \|DF^n(z)w\| \to 0 \qquad \text{as } n \to \pm\infty, \tag{10}$$
where $\mathrm{Cl}(\cdot)$ denotes topological closure.

Hénon-like attractors are strange, since they admit a dense orbit with a positive Lyapunov exponent (by conditions (8) and (9)). Hénon-like attractors are also *non-uniformly hyperbolic*: indeed they contain *critical points* (condition (10)), that is, points belonging to a dense orbit for which a nonzero tangent vector w exists, which is contracted both by positive and by negative iteration of the derivative DF.

We now consider the occurrence of Hénon-like strange attractors in the skew product family
$$T_{\alpha,\delta,a,\varepsilon} : \mathbb{R}^2 \times \mathbb{S}^1 \to \mathbb{R}^2 \times \mathbb{S}^1, \qquad \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} \mapsto \begin{pmatrix} 1 - ax^2 + \varepsilon f \\ \varepsilon g \\ A_{\alpha,\delta}(\theta) \end{pmatrix}. \tag{11}$$
Here $(\alpha, \delta, a, \varepsilon)$ are parameters, while f and g are functions of $(a, x, y, \theta, \varepsilon, \alpha, \delta)$. Notice that the family (2) takes the form (11) after a rescaling $y \mapsto \sqrt{|b|}y$ and by choosing $b = \mathcal{O}(\varepsilon)$: this makes sense, since we focus on small values of b. We perturb away from cases where (α, δ) is in one of the Arnol′d resonance tongues, see Figure 2 (A). We recall that the restriction of (11) to $\mathbb{S}^1$ is the Arnol′d family of circle maps (3). Map (11) is a generalisation of the planar Hénon-like families considered in [29, 43]: these are C^3-small diffeomorphisms obtained as perturbations of the Logistic family
$$Q_a : \mathbb{R} \to \mathbb{R}, \qquad x \mapsto 1 - ax^2. \tag{12}$$
The x- and y-components of $T_{\alpha,\delta,a,\varepsilon}$ also depend on the circle dynamics by the perturbative terms f and g. The only requirement on f and g is that their C^3-norms are bounded on compact sets. Occurrence of Hénon-like attractors is proved in the family $T_{\alpha,\delta,a,\varepsilon}$ for all parameter values belonging to a set of positive (Lebesgue) measure. For all values in this set, the parameters (α, δ) are such that the dynamics of the Arnol′d family $A_{\alpha,\delta}$ (3) is of Morse-Smale type: there exist periodic points θ^s and θ^r in $\mathbb{S}^1$, such that θ^s is attracting and θ^r repelling for $A_{\alpha,\delta}$. By $\mathfrak{A}^{q/n}$ we denote the open resonance tongue in the (α, δ)-plane where these periodic points have rotation number q/n [1, 15] and the width of the tongue in α behaves as δ^n [8], compare with Figure 2 (A). The parameter space under consideration is the set of all $(\alpha, \delta, a, \varepsilon) \in \mathbb{R}^4$ such that
$$\alpha \in [0, 1],\ \delta \in \big[0, 1/(2\pi)\big),\ a \in [0, 2],\ |\varepsilon| < 1. \tag{13}$$

The attractors that we obtain coincide with the closure of the one-dimensional unstable manifold of a hyperbolic periodic orbit $p = (x_0, y_0, \theta^s) \in \mathbb{R}^2 \times \mathbb{S}^1$ of saddle type:

$$\mathscr{A} = \mathrm{Cl}\left(W^u\left(\mathrm{Orb}(p)\right)\right).$$

For the statement of the result we need a few definitions and notations.

Definition 2.4.
1. A map $M : J \to J$, where $J \subset \mathbb{R}$ is an interval, is called topologically mixing if for any open intervals J_1, $J_2 \subset J$ there exists n_0 such that
$$M^n(J_1) \cap J_2 \neq \emptyset \quad \text{for all } n \geq n_0.$$
2. The interval $K_a = [Q_a^2(0), Q_a(0)]$ is called the core or the restrictive interval of the Logistic family Q_a (12).

It is well-known that $Q_a([0,1]) = Q_a(K_a) = K_a$ for all a, where K_a is the core of Q_a (12), see *e.g.* [27, Section II.5]. For a given integer $n > 1$, denote by $\Phi(n)$ the set of all integers q such that q and n are relatively prime, where $1 \leq q < n$. For $n = 1$ we put $\Phi(n) = \{1\}$.

Theorem 2.5. (Hénon-like attractors in (11)) *Choose $a^* \in (1,2)$ such that the quadratic map Q_{a^*} in (12) is topologically mixing on its core $K = [1-a^*, 1]$ and its critical point $c = 0$ is preperiodic (that is, $Q^q_{a^*}(c)$ is a periodic point of Q_{a^*} for some integer q). Let $n \geq 1$ be an integer. There exist a periodic point p_0 of the n-th iterate $Q^n_{a^*}$ and positive constants $\bar{\varepsilon}_n$, $\bar{a}_n$ and χ_n such that the following holds.*

1. *For all $(\alpha, \delta, a, \varepsilon)$ as in (13), with*
$$(\alpha, \delta) \in \cup_{q \in \Phi(n)} \mathrm{Cl}\left(\mathfrak{A}^{q/n}\right), \quad |a - a^*| < \bar{a}_n, \quad |\varepsilon| < \bar{\varepsilon}_n \tag{14}$$
the map $T_{\alpha,\delta,a,\varepsilon}$ has a saddle periodic point p, which is the analytic continuation of p_0 and such that the unstable manifold $W^u(\mathrm{Orb}(p))$ is one-dimensional.
2. *For all $(\alpha, \delta, \varepsilon)$ as in (14) there exists a set $\mathfrak{S}_{\alpha,\delta,\varepsilon}$ with*
$$\mathfrak{S}_{\alpha,\delta,\varepsilon} \subset [a^* - \bar{a}_n, a^* + \bar{a}_n], \qquad \mathrm{meas}(\mathfrak{S}) > \chi_n$$
such that for all $a \in \mathfrak{S}_{\alpha,\delta,\varepsilon}$ the closure $\mathrm{Cl}\left(W^u(\mathrm{Orb}(p))\right)$ is a Hénon-like attractor of $T_{\alpha,\delta,a,\varepsilon}$.

Corollary 1. *The set of parameter values for which $T_{\alpha,\delta,a,\varepsilon}$ has a Hénon-like attractor contains the set*

$$\mathfrak{S} = \bigcup_{n \in \mathbb{N}} \left\{ (\alpha, \delta, a, \varepsilon) \;\middle|\; (\alpha, \delta) \in \cup_{q \in \Phi(n)} \mathrm{Cl}\left(\mathfrak{A}^{q/n}\right), \quad |\varepsilon| < \bar{\varepsilon}_n, \quad a \in \mathfrak{S}_{\alpha,\delta,\varepsilon} \right\},$$

and the set $\mathfrak{S}$ has positive Lebesgue measure

$$\mathrm{meas}(\mathfrak{S}) \geq 2 \sum_{n=1}^{\infty} \bar{\varepsilon}_n \chi_n \sum_{q \in \Phi(n)} \mathrm{meas}\, \mathfrak{A}^{q/n}.$$

Our proof of Theorem 2.5 is given in Section 4.2. It is based on a result of Díaz-Rocha-Viana [16, Theorem 5.2], and relies on the following facts:

1. For (α, δ) inside any tongue $\mathfrak{A}^{q/n}$, the asymptotic dynamics of $T_{\alpha,\delta,a,\varepsilon}$ is described by an $\mathcal{O}(\varepsilon)$-perturbation of the n-th iterate Q^n_a.
2. For all n the map Q^n_a is a generic n-modal family, in the sense of [16, Section 5.2], also see the definition given in Section 4.2. To show this, we use that Q_{a^*} is a Misiurewicz map [28], and, therefore, it is Collet-Eckmann (see *e.g.* [27, Section V.4]). See Section 4.2 for details.

As we said above, family (2) takes the form (11) after a rescaling $y \mapsto \sqrt{|b|}y$ and by choosing $b = \mathcal{O}(\varepsilon)$. Therefore, by restricting the parameter δ to sufficiently small values, both Theorem 2.5 and Theorem 2.1 may be applied to (2).

Corollary 2. *Let a^* and p_0 satisfy the hypotheses of Theorem 2.5. Then there exists a positive measure set of parameter values such that the family (2) has Hénon-like attractors, contained in the closure of the unstable manifold of a periodically invariant circle.*

Proof. Take a^* and p_0 as in the hypotheses of Theorem 2.5. Then for all δ and for ε and b sufficiently small, Theorem 2.5 applies. Moreover, for $(\varepsilon, \delta) = (0, 0)$ the circle $\mathscr{C} = \{p_0\} \times \mathbb{S}^1$ is periodically invariant under map (2). In particular, the conditions of Theorem 2.1 are satisfied for b sufficiently small, since:

1. The periodic point $(p_0, 0)$ of $H_{a,0}$ has an analytic continuation $\bar{p}(b)$ for all b sufficiently small, and p_0 is chosen such that $\bar{p}(b)$ has transversal homoclinic points, see Proposition 3.
2. $\det(DH_{a,b}(x, y)) = b$;
3. The unstable manifold of all periodic points of $H_{a,b}$ is bounded for b sufficiently small, since the invariant manifolds depend continuously on the map [29, Prop. 7.1].

So for (ε, δ, b) sufficiently small, the conclusions of Theorem 2.1 hold. □

2.3. **Quasi-periodic Hénon-like attractors.** This paper has been partially motivated by the problem to find a diffeomorphism F with a strange attractor $\mathscr{A}$ such that

$$\mathscr{A} = \mathrm{Cl}\left(W^u(\mathscr{C})\right), \tag{15}$$

where $\mathscr{C}$ is an F-invariant circle of saddle type with irrational rotation number, so with quasi-periodic dynamics. In this context, the role of the saddle periodic orbit in (7) is played by a quasi-periodic invariant circle of saddle type. By analogy with the definition of Hénon-like strange attractors (see Section 2.2), we are led to the following definition.

Definition 2.6. Let $F : M \to M$ be a C^1-diffeomorphism, where M is an m-dimensional smooth manifold. We say that the F-invariant set $\mathscr{A}$ is a *quasi-periodic Hénon-like attractor* if there exist

1. A quasi-periodic invariant circle $\mathscr{C}$ of saddle type such that $\mathscr{A} = \mathrm{Cl}\left(W^u(\mathscr{C})\right)$.
2. A point $x \in \mathscr{A}$ such that $\mathrm{Orb}(x)$ is dense in $\mathscr{A}$ and
3. a dense set $Z \subset \mathscr{A}$ and constants $\kappa > 0$, $\lambda > 1$ such that for all $z \in Z$ there exist vectors $v, w \in T_zM$ such that conditions (9) and (10) hold.

The definition mimics the positive Lyapunov exponents and non-uniform hyperbolicity requirements in the definition of Hénon-like attractors and also asks for transitivity. As usual, similar definitions can be given with F replaced by a power F^k.

Returning to the skew product context of the model family (2), in the Arnol′d family $A_{\alpha,\delta}$ we fix parameter values (α, δ) such that the dynamics of $A_{\alpha,\delta}$ is quasi-periodic. Recall that the set of all such (α, δ) has positive measure and is nowhere dense [5, Chap. 1]. Next choose parameter values a and b such that the Hénon map (1) has a Hénon-like strange attractor $\mathscr{A}'$, coinciding with the closure of the unstable manifold of a saddle fixed point p: according to [2, 3, 29], such (a, b) form a set of positive measure. Then, at $\varepsilon = 0$ map (2) has an attractor $\mathscr{A} = \mathscr{A}' \times \mathbb{S}^1$

coinciding with the closure of the unstable manifold of the quasi-periodic saddle type invariant circle $\{p\} \times \mathbb{S}^1$. It may be clear that requirement 3 of Definition 2.6 is satisfied by taking $Z = \mathrm{Orb}(z) \times \mathbb{S}^1$, where z is a point satisfying properties 4 *ii), iii)* and *iv)* in Definition 2.3 of Hénon-like attractors. Only item 2 in Definition 2.6 remains to be verified, to prove that quasi-periodic Hénon-like occur in the product case. This is achieved in the following lemma.

Lemma 2.7. (Transitivity of (2), uncoupled) *Let T be a dissipative C^1-diffeomorphism in an open subset $\mathcal{U} \subset \mathbb{R}^2$ such that*

1. *T has a hyperbolic fixed point p of saddle type.*
2. *The closure of the unstable manifold of p is an Hénon-like strange attractor $\mathscr{A}'$.*

Let $R_\alpha : x \mapsto x + \alpha \mod 1$ a rotation over angle $\alpha \in (0,1) \setminus \mathbb{Q}$. Then the product $F = T \times R$ has a dense orbit in $\mathscr{A} = \mathscr{A}' \times \mathbb{S}^1$.

Proof. We claim that it is sufficient to prove the following:

$(*)$ Given two open sets U, V in $\mathscr{A}$, there exists $k \in \mathbb{N}$ such that $F^k(U) \cap V \neq \emptyset$.

Indeed, if (*) is true, then Proposition 2 in [14] applies.

Next, let us prove $(*)$. Without loss of generality, assume that $U = U' \times (r - \delta, r + \delta)$ and $V = V' \times (s - \varepsilon, s + \varepsilon)$ for some $\delta, \varepsilon > 0$, where U', V' are open sets in $\mathscr{A}'$. First, for fixed $\varepsilon > 0$ we note that given $r, s \in \mathbb{S}^1$ there exists an increasing sequence $\{n_1, n_2, \ldots\}$ such that $R_\alpha^{n_j}(r) \in (s - \varepsilon, s + \varepsilon)$, where $0 < n_1 < N$ and $n_{j+1} - n_j < N$ for all j, with N independent of r and s. As $W^u(p)$ is dense in $\mathscr{A}'$, there exists a point $q' \in W^u(p) \cap V'$. Consider a preimage $u = T^{-l}(q')$ such that u and its first N iterates are close to p. By continuity, there are open sets $Z_0, Z_1, \ldots, Z_N$ around $u, T(u), \ldots, T^N(u)$ whose images under $T^l, T^{l-1}, \ldots, T^{l-N}$ are contained in V'.

Now, there exists a point $x \in U' \cap W^u(p)$ belonging to a dense orbit and also having a positive Lyapunov exponent, such that $T^m(x) \in Z_0$ for some $m \in \mathbb{N}$. It is no restriction to assume that, for some $m \in \mathbb{N}$, the image $T^m(U')$ intersects all $Z_j, j = 0, 1, 2, \ldots, N$. Indeed, in the other case the Lyapunov exponent could not be positive.

Since $T^{l-j}(Z_j \cap T^m(U')) \subset V'$ for $j = 0, \ldots, N$, one has

$$T^{l+m-j}(U') \cap V' \neq \emptyset \tag{16}$$

for all $j = 0, \ldots, N$. To arrange that some of the iterates $R_\alpha^{l+m-j}(r)$ lie inside the interval $(s - \varepsilon, s + \varepsilon)$, observe that $l + m$ is in between two consecutive values n_i and n_{i+1} for some n_i as above. This implies that there exists $j \leq N$ such that $l + m - j = n_i$, which, together with (16), yields that $T^{l+m-j}(U) \cap V \neq \emptyset$. □

3. **Dynamical study.**

Section 3.1 discusses the links between the numerically observed dynamics of map (2) and the theoretical results of Section 2. Several dynamical phenomena of map (2) are specific to its skew product structure: this is discussed in Section 3.2.

3.1. **The dynamics of the skew product map.**

Numerical evidence indicates that all four cases discussed in the Introduction (see Table 1) occur in parameter sets of positive measure for map (2), see the Lyapunov diagram in Figure 2 (C).

Case 1b. Theorem 2.2 in Section 2.1 implies the existence of quasi-periodic attractors for a positive measure set in parameter space. In our numerical exploration, an attractor is classified as a quasi-periodic invariant circle if it satisfies two criteria: the maximal Lyapunov exponent Λ_1 is zero and the *variation test for invariant circles* is passed, see Appendix A for a discussion. Both criteria are satisfied for a parameter subset near $\varepsilon = 0$ in Figure 2 (C).

The condition $\Lambda_1 = 0$ is necessary but not sufficient for identifying quasi-periodic invariant circles: there are parameter values for which Λ_1 is numerically zero, but where the variation test is not passed. This parameter subset is rendered in magenta in the top panel of Figure 5. It is located away from $\varepsilon = 0$ and is particularly visible near the resonance with rotation number $2/7$, with narrower domains occurring near other resonances.

This is explained as follows: the quasi-periodic attractors occurring for small ε eventually *seem* to break down as ε increases (because the variation criterion is no longer satisfied). Plots of the attractors resulting from this process look like the so-called *strange nonchaotic attractors* [17, 19, 20, 24, 25, 31, 32]. Strange nonchaotic attractors are fractal invariant sets with a negative maximal Lyapunov exponent. At first instance one might be tempted to classify as strange nonchaotic all those attractors of map (2) for which $\Lambda_1 = 0$ and the variation test is not passed. However, a deeper examination of several such cases shows that they are in fact smooth invariant circles. In these cases, the failure of the variation test is a consequence of too low numerical resolution. This causes a discrepancy between the numerically observed dynamics and the actual dynamics. Although a full discussion of this phenomenon is out of the scope of the present paper, we briefly present two of the underlying mechanisms in Appendix C, also see [20] for a specific investigation.

In summary: not all parameter values for which $\Lambda_1 = 0$ (yellow in Figure 2 (C)) correspond to smooth invariant circles; this explains our description of attractor type for case 1b in Table 2. Also, not all the parameter values for which $\Lambda_1 = 0$ and the variation test is not passed (magenta in Figure 5 top) are strange nonchaotic attractors. This motivates our description of attractor type for code 5 in Table 3.

Case 2a. A complementary situation regarding Theorem 2.2 occurs when the dynamics on the invariant circle is of Morse-Smale type. For the Arnol′d map, this occurs in the resonance tongues of Figure 2 (A). By Corollary 2 in Section 2.2, map (2) has a Hénon-like attractor for a positive measure set of parameter values. This corresponds to the red domains in Figure 2 (C). It is to be noted that the Hénon-like character of the attractors remains conjectural when *specific values of the parameters* are considered: such is the case for the attractors of Figure 3.

Case 2b. Existence of quasi-periodic Hénon-like attractors is only proved for the unperturbed case $\varepsilon = 0$ (Section 2.3). In the numerical context, such attractors are identified by one positive, one negative, and one zero Lyapunov exponent. Numerical evidence indicates that such attractors:

- persist for small (and not so small) values of (ε, δ), see Figures 4, 7;
- occur in a relatively large part of the parameter plane (blue in Figure 2 (C)).

Strange attractors similar to Figure 7 are observed in several numerical studies [31, 17, 19, 32]. A difference is that most of these studies deal with endomorphisms of the interval with a skew product forcing by a rigid rotation. Strange nonchaotic attractors are found in many of the above studies, see Appendix C.

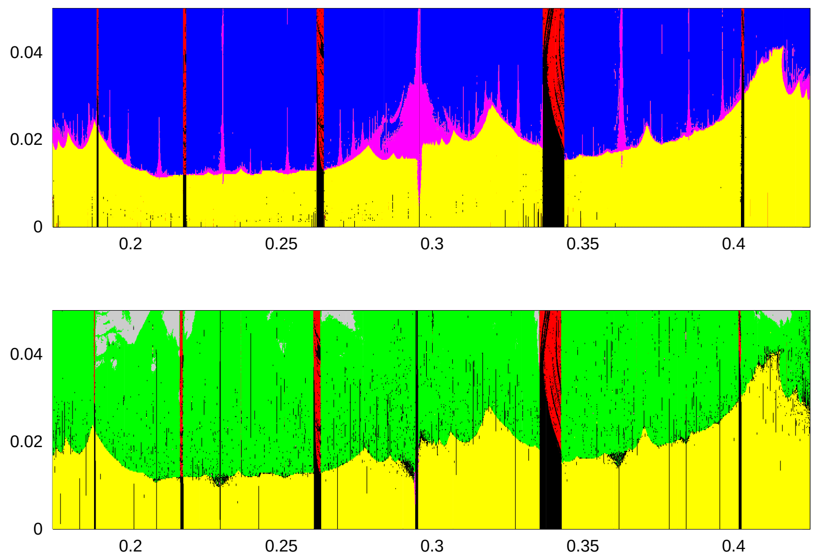


FIGURE 5. Top: magnification near $\varepsilon = 0$ of Figure 2 (C). Parameter values in magenta correspond to candidates for strange nonchaotic attractors, see the text for discussion and explanation. Such magenta domains were shown in blue in Figure 2 (C). Bottom: same as top for Figure 6. The colour coding is given in Tables 2 and 3.

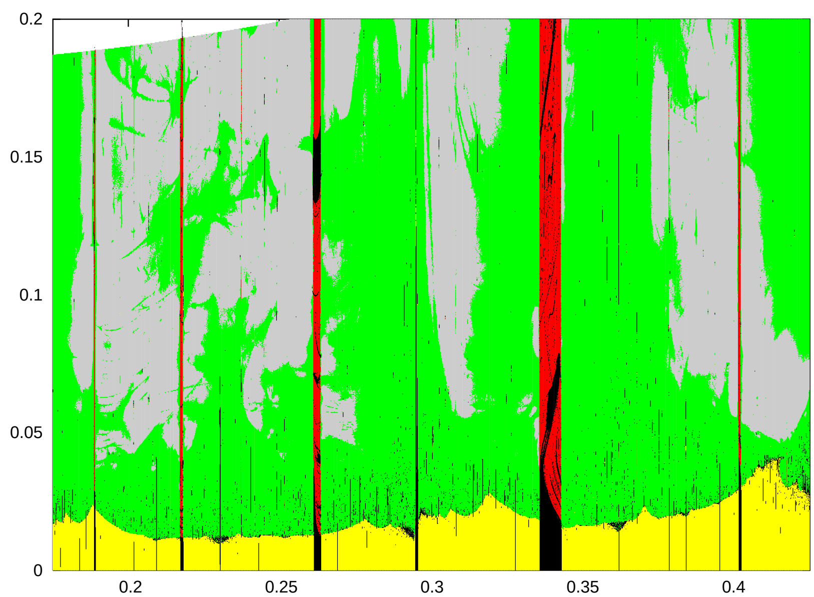
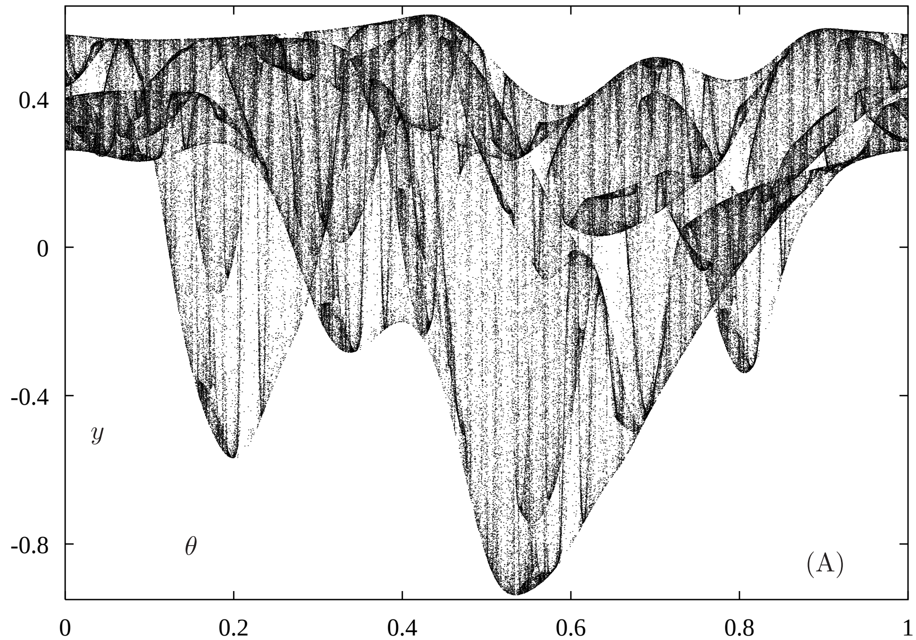


FIGURE 6. Organisation of the (α, ε)-plane for the fully coupled system (17), for $a = 1.25, b = 0.3, \mu = 0.01$ and $\delta = 0.6/(2\pi)$, see Tables 2 and 3 for the colour coding.



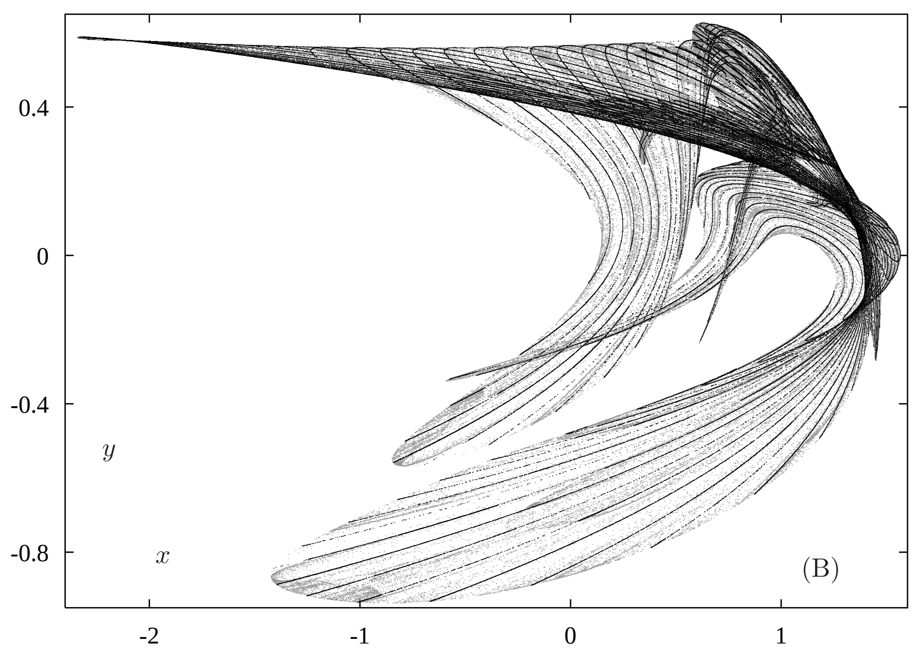


FIGURE 7. Quasi-periodic Hénon-like attractor of the model family (2). (A) Projection on the (θ, y)-plane. (B) Projection on the (x, y)-plane (grey, in the background), with 'slices' of the attractor (black) for $2\pi\theta \in [j/10, j/10 + 0.001]$, $j = 0, 1, \ldots, 62$. Parameter values are fixed at $a = 0.8$, $b = 0.4$, $\delta = 0$, $\alpha = (\sqrt{5} - 1)/2$, $\varepsilon = 0.7$, initial conditions are $x_0 = 1.5$, $y_0 = 0$, $\theta_0 = 0$.

code	colour	criterion	attractor type
5	magenta	$\Lambda_1 = 0$, variation criterion not satisfied	candidate for strange non-chaotic attractor (see text)
6	green	$\Lambda_1 > 0$, $\Lambda_2 < 0$, Λ_2 small, clustering criterion is satisfied	quasi-periodic Hénon-like strange attractor
7	light grey	$\Lambda_1 > 0$, $\Lambda_2 > 0$, Λ_2 small	quasi-periodic Hénon-like strange attractor

TABLE 3. Coding of the additional colours in Figures 5 and 6 (also see Table 2). Additional criteria are used for the identification of the attractor class, see text for details.

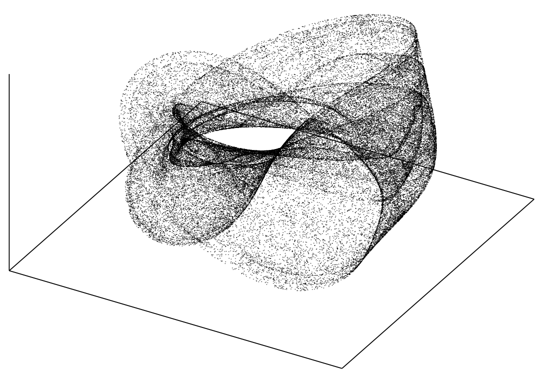


FIGURE 8. Attractor of map $\mathcal{T}$ in (17) for $(\alpha, \varepsilon, \mu) = (0.31, 0.13, 0.01)$, with two positive Lyapunov exponents: $\Lambda_1 \approx 0.29530$ and $\Lambda_2 \approx 0.00016$, projection on variables (u, v, w) similar to Figure 4.

3.2. **Perturbing the skew product.** We here analyse the dynamical consequences of the destruction of the skew product structure of map (2): a coupling term μy is added to the angular dynamics, yielding the map

$$\mathcal{T} = \mathcal{T}_{\alpha,\delta,a,b,\varepsilon,\mu} : \mathbb{R}^2 \times \mathbb{S}^1 \to \mathbb{R}^2 \times \mathbb{S}^1, \quad \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} \mapsto \begin{pmatrix} 1 - (a + \varepsilon \sin(2\pi\theta))x^2 + y \\ bx \\ \theta + \alpha + \delta \sin(2\pi\theta) + \mu y \end{pmatrix}. \quad (17)$$

Map (17) has the form (5) and depends on six parameters $(\alpha, \delta, a, b, \varepsilon, \mu)$. Comparison of the Lyapunov diagrams in the skew product case $\mu = 0$ (Figure 2 (C)) and in the fully coupled case $\mu > 0$ (Figure 6) reveals these analogies:

- parameter regions with periodic attractors for $\mu = 0$ are essentially preserved for $\mu \approx 0$; in the latter case, however, more periodic regions are found near resonances.

- parameter regions with Hénon-like attractors (red) are quite similar in the two cases;
- parameter regions with quasi-periodic attractors remain almost unchanged.

The latter two points indicate that the domain of validity of Theorems 2.2 and 2.5 are relatively large. Two remarkable differences occur between the skew and non-skew cases:

- the maximal Lyapunov exponent Λ_1 becomes positive for $\mu \neq 0$, for essentially all parameter values where 'candidates for strange nonchaotic attractors' are found for $\mu = 0$ (see the previous section). This behaviour has been observed varying μ for a sample of values of (α, ε), even when μ is as small as 10^{-12}.
- a (numerically) zero Lyapunov exponent Λ_2 practically never occurs for $\mu \neq 0$: any small perturbation $\mu \neq 0$ has the effect of shifting Λ_2 away from zero, for nearly all parameters where $\Lambda_2 = 0$ for $\mu = 0$ (the blue region of Figure 2 (C)).

Roughly speaking, the domain where $\Lambda_2 = 0$ for the skew product case splits into three parts for $\mu \neq 0$: where $\Lambda_2 = 0$, $\Lambda_2 < 0$ and $\Lambda_2 > 0$ respectively. The first case is virtually absent. In the latter two cases, the value of Λ_2 remains very small, though definitely distinct from zero. The geometric structure of the corresponding attractors shows no appreciable differences. Specifically, provided that μ is small and the other parameters are kept fixed, plots of the attractors typically are very similar

- in the skew product case $\mu = 0$, when $\Lambda_1 > 0$ and $\Lambda_2 = 0$, and
- in the fully coupled case $\mu > 0$, when $\Lambda_1 > 0$ and $\Lambda_2 \approx 0$ (independently of the sign).

Figure 8 shows an attractor with two positive Lyapunov exponents, where $\Lambda_2 > 0$ is small. Parameters are $(\alpha, \varepsilon, \mu) = (0.31, 0.13, 0.01)$, in the light grey region of Figure 6. The attractor shape remains essentially unaltered if parameters are changed to $(\alpha, \varepsilon, \mu) = (0.28, 0.13, 0.01)$ (green region of Figure 6, where $\Lambda_2 < 0$ and is small) or to $(\alpha, \varepsilon, \mu) = (0.28, 0.13, 0.00)$ (blue region in Figure 2, where Λ_2 is numerically zero). We expect that all these attractors are contained in the closure $\mathrm{Cl}\,(W^u(\mathscr{C}))$ of a quasi-periodic invariant circle $\mathscr{C}$ of saddle type. However, further study is needed to clarify the structural differences associated with the different sign of Λ_2.

4. **Proofs.**

4.1. **Basins of attraction and quasi-periodic invariant circles.**

In this section we give proofs of Theorem 2.1 (next section) and Theorem 2.2 (Section 4.1.2).

4.1.1. *The Tangerman-Szewc argument generalised.* Let $K : \mathbb{R}^2 \to \mathbb{R}^2$ be a dissipative diffeomorphism having a saddle fixed point $p = (x_0, y_0)$. Suppose the stable and unstable manifolds $W^s(p)$ and $W^u(p)$ intersect transversally at the homoclinic point $q \in W^s(p) \cap W^u(p)$, see Figure 9. Also assume that $W^u(p)$ is bounded as a subset of $\mathbb{R}^2$. The Tangerman-Szewc Theorem (see *e.g.* [34, Appendix 3]) states that the basin of attraction of the closure of $W^u(p)$ contains the open region U' bounded by the two arcs $\partial^s \subset W^s(p)$ and $\partial^u \subset W^u(p)$ with extremes p and q, see Figure 9. This argument is used to prove existence of strange attractors (in particular, with non-trivial basin of attraction) near homoclinic tangencies of a saddle fixed point of a dissipative diffeomorphism, cf. [29, 43, 46].

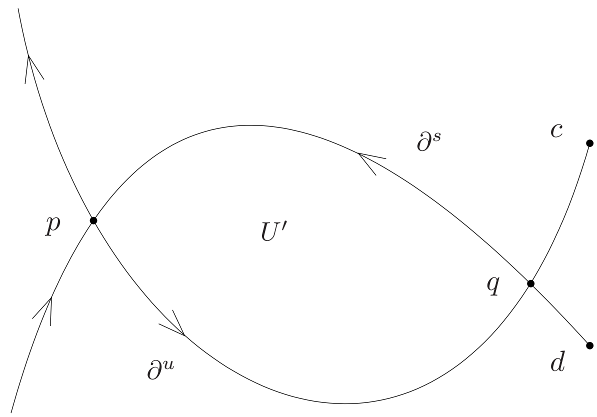


FIGURE 9. Segments ∂^s and ∂^u of the stable and unstable manifold, respectively, of a saddle fixed point p bound a region U, see text for more explanation.

We first prove Theorem 2.1 for $\varepsilon = 0$. This is a straightforward generalisation of the above Tangerman-Szewc Theorem. For small ε, the result is obtained by using persistence of normally hyperbolic invariant manifolds [23, Theorem 1.1] and two transversality lemmas.

***Proof of Theorem* 2.1.** Consider the circle $\mathscr{C}_\alpha = \mathscr{C}_{\alpha,0}$, invariant under map $P_{\alpha,0}$ in (5). The manifolds $W^u(\mathscr{C}_\alpha)$ and $W^s(\mathscr{C}_\alpha)$ are given by $W^u(p)\times\mathbb{S}^1$ and $W^s(p)\times\mathbb{S}^1$, respectively. They intersect transversally at a circle $\mathscr{H} = \{q\}\times\mathbb{S}^1$, consisting of points homoclinic to $\mathscr{C}_\alpha$. Consider the two arcs $\partial^s \subset W^s(p)$ and $\partial^u \subset W^u(p)$ with extremes p and q (Figure 9).They bound an open set $U' \subset \mathbb{R}^2$. Define D^s and D^u to be the portions of stable, and unstable manifold of $\mathscr{C}_\alpha$, respectively, given by

$$D^s = \partial^s \times \mathbb{S}^1 \subset W^s(\mathscr{C}_\alpha) \qquad \text{and} \qquad D^u = \partial^u \times \mathbb{S}^1 \subset W^u(\mathscr{C}_\alpha).$$

Both surfaces D^s and D^u are compact, and their union forms the boundary of the open region $U = U' \times \mathbb{S}^1$, which is topologically a solid torus.

The volume of U decreases under iteration of $P_{\alpha,0}$. Denoting by meas$(\cdot)$ the Lebesgue measure both on $\mathbb{R}^2$ and on $\mathbb{R}^2\times\mathbb{S}^1$, due to condition 2 in Theorem 2.1 we have

$$\text{meas}(P^n_{\alpha,0}(U)) = 2\pi \int_{K^n(U')} dxdy = 2\pi \int_{U'} |\det DK^n|\, dxdy \leq 2\pi\kappa^n \,\text{meas}(U').$$

This implies that the forward evolution of every point $(x, y, \theta) \in U$ approaches the boundary of $P^n_{\alpha,0}(U)$:

$$\text{dist}\big(P^n_{\alpha,0}(x,y,\theta), \partial P^n_{\alpha,0}(U)\big) \to 0 \qquad \text{as } n \to +\infty.$$

Indeed, suppose that this does not hold. Then there exists a $\varrho > 0$ such that for all n there exists $N > n$ such that the ball with centre $P^N_{\alpha,0}(x, y, \theta)$ and radius $\varrho > 0$ is contained inside $P^N_{\alpha,0}(U)$. But this would contradict the fact that meas$(P^n_{\alpha,0}(U)) \to 0$ as $n \to +\infty$.

The boundary of $P^n_{\alpha,0}(U)$ also consists of two portions of stable and unstable manifold of $\mathscr{C}$:

$$\partial P^n_{\alpha,0}(U) = P^n_{\alpha,0}(D^s) \cup P^n_{\alpha,0}(D^u).$$

The diameter of $P^n_{\alpha,0}(D^s)$ tends to zero as $n \to +\infty$, because all points in D^s are attracted to the circle $\mathscr{C}_\alpha$. Since $W^u(\mathscr{C}_\alpha)$ is bounded, all evolutions starting in U are bounded and approach $W^u(\mathscr{C}_\alpha)$, that is,

$$\operatorname{dist}(P^n_{\alpha,0}(x,y,\theta), P^n_{\alpha,0}(D^u)) \to 0 \quad \text{as } n \to +\infty$$

for all $(x,y,\theta) \in U$. This implies that $\omega(x,y,\theta) \subset \operatorname{Cl}(W^u(\mathscr{C}_\alpha))$ for all $(x,y,\theta) \in U$.

To extend this result to small perturbations $P_{\alpha,\varepsilon}$ of $P_{\alpha,0}$, the following transversality lemmas are used.

Lemma 4.1. [37, Chap. 7] *Consider a map $f: V \to M$, where V and M are C^r-differentiable manifolds and f is C^r. Suppose V is compact, $W \subset M$ is a closed C^r-submanifold and f is transversal to W at V (notation: $f \pitchfork W$). Then $f^{-1}(W)$ is a C^r-submanifold of codimension $\operatorname{codim}_V(f^{-1}(W)) = \operatorname{codim}_M(W)$. Further suppose that there is a neighbourhood of $f(\partial_V) \cup \partial_W$ disjoint from $f(V) \cap W$, where ∂_V and ∂_W are the boundaries of V and W. Then any map $g: V \to M$, sufficiently C^r-close to f, is also transversal to W, and the two submanifolds $g^{-1}(W)$ and $f^{-1}(W)$ are diffeomorphic.*

Lemma 4.2. [22, Section 3.2] *Let V_1, V_2, and M be C^r-differentiable manifolds and consider two diffeomorphisms $f_i: V_i \to M$, $i = 1, 2$. Then $f_1 \pitchfork f_2$ if and only if $f_1 \times f_2 \pitchfork \Delta$, where $f_1 \times f_2 : V_1 \times V_2 \to M \times M$ is the product map and $\Delta \subset M \times M$ is the diagonal: $\Delta = \{(y,y) \mid y \in M\}$.*

Fix $r \in \mathbb{N}$ and take $\varepsilon < \varepsilon_r$, where ε_r is given in Proposition 1. Then the map $P_{\alpha,\varepsilon}$ has an r-normally hyperbolic invariant circle $\mathscr{C}_{\alpha,\varepsilon}$ of saddle type. Furthermore, the manifolds $W^u(\mathscr{C}_{\alpha,\varepsilon})$, $W^s(\mathscr{C}_{\alpha,\varepsilon})$, and $\mathscr{C}_{\alpha,\varepsilon}$ are C^r-close to $W^u(\mathscr{C}_\alpha)$, $W^s(\mathscr{C}_\alpha)$, and $\mathscr{C}_\alpha$. We now show that the two manifolds $W^u(\mathscr{C}_{\alpha,\varepsilon})$, $W^s(\mathscr{C}_{\alpha,\varepsilon})$ still intersect transversally. To apply Lemma 4.1 we restrict to two suitable compact subsets $A^u \subset W^u(\mathscr{C}_\alpha)$ and $A^s \subset W^s(\mathscr{C}_\alpha)$ as follows. Consider the segments $\overline{pc} \subset W^u(p)$ and $\overline{pd} \subset W^s(p)$ in Figure 9. Define

$$A^u = \overline{pc} \times \mathbb{S}^1, \qquad A^s = \overline{pd} \times \mathbb{S}^1.$$

In this way, the circle $\mathscr{H}$ is the intersection of the manifolds A^u and A^s, bounded away from their boundaries. Consider the inclusions $i : A^u \to M$ and $j : A^s \to M$, where $M = \mathbb{R}^2 \times \mathbb{S}^1$. By the closeness of $W^u(\mathscr{C}_\alpha)$ to $W^u(\mathscr{C}_{\alpha,\varepsilon})$ there exists a C^r-diffeomorphism $h : A^u \to A^u_\varepsilon \subset W^u(\mathscr{C}_{\alpha,\varepsilon})$ such that the map i is C^r-close to $i_\varepsilon \circ h$, where $i_\varepsilon : A^u_\varepsilon \to M$ is the inclusion [33, Section 2.6]. Similarly, there exists a diffeomorphism $k : A^s \to A^s_\varepsilon \subset W^s(\mathscr{C}_{\alpha,\varepsilon})$ such that the map j is C^r-close $j_\varepsilon \circ k$, where $j_\varepsilon : A^s_\varepsilon \to M$ is the inclusion. By Lemma 4.2 the map $i \times j : A^u \times A^s \to M \times M$ is transversal to the diagonal Δ. For ε small, the map $(i_\varepsilon \circ h) \times (j_\varepsilon \circ k) : A^u \times A^s \to M \times M$ is C^r-close to $i \times j$:

$$\begin{array}{ccc} A^u \times A^s & \xrightarrow{i \times j} & M \times M \\ {\scriptstyle h \times k}\big\downarrow & & \\ A^u_\varepsilon \times A^s_\varepsilon & \xrightarrow{i_\varepsilon \times j_\varepsilon} & M \times M. \end{array}$$

Since Δ is closed and $A^u \times A^s$ is compact, Lemma 4.1 implies that there exists an ε^*, with $0 < \varepsilon^* < \varepsilon_r$, such that $(i_\varepsilon \circ h) \times (j_\varepsilon \circ k) \pitchfork \Delta$ for $\varepsilon < \varepsilon^*$. Furthermore, the submanifolds

$$(i \times j)^{-1}(\Delta) \quad \text{and} \quad \big((i_\varepsilon \circ h) \times (j_\varepsilon \circ k)\big)^{-1}(\Delta)$$

are diffeomorphic. We also have that $\big((i_\varepsilon \circ h) \times (j_\varepsilon \circ k)\big)^{-1}(\Delta)$ is diffeomorphic to $A^u_\varepsilon \cap A^s_\varepsilon$, and $(i \times j)^{-1}(\Delta) = A^u \cap A^s = \mathscr{H}$.

This shows that the intersection $\mathscr{H}_\varepsilon = A^u_\varepsilon \cap A^s_\varepsilon$ is diffeomorphic to $\mathscr{H}$. Define D^u_ε as the part of $W^u(\mathscr{C}_{\alpha,\varepsilon})$ bounded by the invariant circle $\mathscr{C}_{\alpha,\varepsilon}$ and the circle of homoclinic points $\mathscr{H}_\varepsilon$. Define $D^s_\varepsilon = k(D^s)$ similarly. Then the manifolds $D^u_\varepsilon \subset W^u(\mathscr{C}_{\alpha,\varepsilon})$ and $D^s_\varepsilon \subset W^s(\mathscr{C}_{\alpha,\varepsilon})$ form the boundary of an open region $U \subset M$ homeomorphic to a torus. By the closeness of the perturbed manifolds $W^s(\mathscr{C}_{\alpha,\varepsilon})$ and $W^u(\mathscr{C}_{\alpha,\varepsilon})$ to the unperturbed $W^s(\mathscr{C})$ and $W^u(\mathscr{C})$, both U and $W^u(\mathscr{C}_{\alpha,\varepsilon})$ are bounded. Also notice that $P_{\alpha,\varepsilon}$ is dissipative: by taking ε^* small enough, we ensure that $|\det(DF(x,y,\theta))| < \tilde{c} < 1$ for all $\varepsilon < \varepsilon^*$ and (x,y,θ) in U. Therefore, all forward evolutions beginning at points $(x,y,\theta) \in U$ remain bounded. Like in the first part of the proof, one has

$$\omega(x,y,\theta) \subset \mathrm{Cl}\,(W^u(\mathscr{C}_{\alpha,\varepsilon}))$$

for all $(x,y,\theta) \in U$, $\alpha \in [0,1]$ and $\varepsilon < \varepsilon^*$. □

4.1.2. *An application of* KAM *theory.* So far, we did not discuss the dynamics in the saddle invariant circle $\mathscr{C}_{\alpha,\varepsilon}$ of map $P_{\alpha,\varepsilon}$ in (5). Generically, the dynamics on $\mathscr{C}_{\alpha,\varepsilon}$ is of Morse-Smale type. In this case, the circle consists of the union of the unstable manifold of some periodic saddle. Theorem 2.2 describes a complementary case, for which the dynamics is quasi-periodic. Fix $\tau > 2$ and define the set of Diophantine frequencies D_γ by

$$D_\gamma = \left\{ \alpha \in [0,1] \;\middle|\; \left| \alpha - \frac{p}{q} \right| \geq \gamma q^{-\tau} \quad \text{for all } p,\, q \in \mathbb{N},\, q \neq 0 \right\}, \tag{18}$$

where $\gamma > 0$. Since we will apply a version of the KAM Theorem holding for non-conservative, finitely differentiable systems (see [5, Chap. 5] and [6]), a certain amount of smoothness of the circle $\mathscr{C}_{\alpha,\varepsilon}$ is needed, depending on the Diophantine condition specified in (18). Therefore we require that the perturbed family of maps $P_{\alpha,\varepsilon}$ is C^n, for n large enough.

***Proof of Theorem* 2.2.** Consider map $P_{\alpha,0}$ in (5), and let $p = (x_0, y_0)$ be a saddle fixed point of the diffeomorphism K. The invariant circle $\mathscr{C}_{\alpha,0} = \{p\} \times \mathbb{S}^1$ of $P_{\alpha,0}$ can be trivially seen as a graph over $\mathbb{S}^1$:

$$\mathscr{C}_{\alpha,0} = \left\{ (x_0, y_0, \theta) \;\middle|\; \theta \in \mathbb{S}^1 \right\}.$$

Fix $r \in \mathbb{N}$ large enough and $\varepsilon < \varepsilon_r$, where ε_r is taken as in Proposition 1. By the C^r-closeness of $\mathscr{C}_{\alpha,0}$ and $\mathscr{C}_{\alpha,\varepsilon}$ (Proposition 1), the circle $\mathscr{C}_{\alpha,\varepsilon}$ of $P_{\alpha,\varepsilon}$ can be written as a C^r-graph over $\mathbb{S}^1$:

$$\mathscr{C}_{\alpha,\varepsilon} = \left\{ (x_\varepsilon(\theta), y_\varepsilon(\theta), \theta) \in \mathbb{R}^2 \times \mathbb{S}^1 \;\middle|\; \theta \in \mathbb{S}^1 \right\}, \tag{19}$$

where $x_\varepsilon : \mathbb{S}^1 \to \mathbb{R}$, $x_\varepsilon(\theta) = x_0 + \mathcal{O}(\varepsilon)$, and similarly for $y_\varepsilon(\theta)$. So the restriction of $P_{\alpha,\varepsilon}$ to $\mathscr{C}_{\alpha,\varepsilon}$ has the following form

$$P_{\alpha,\varepsilon}\big|_{\mathscr{C}_{\alpha,\varepsilon}} : \mathscr{C}_{\alpha,\varepsilon} \to \mathscr{C}_{\alpha,\varepsilon}, \qquad P_{\alpha,\varepsilon}(\theta) = \theta + \alpha + \varepsilon g_\varepsilon(x_0, y_0, \theta, \alpha) + \mathcal{O}(\varepsilon^2).$$

By (19), we may consider $P_{\alpha,\varepsilon}$ as a map on $\mathbb{S}^1$. Fix $\gamma > 0$, $\tau > 3$ and take D_γ as in (18). For $\alpha \in D_\gamma$, the map $P_{\alpha,\varepsilon}$ can be averaged repeatedly over the circle, putting the θ-dependency into terms of higher order in ε, compare [8, Proposition

2.7] and [11, Section 4]. After such changes of variables, $P_{\alpha,\varepsilon}$ is brought into the normal form

$$P_{\alpha,\varepsilon}(\theta) = \theta + \alpha + c(\alpha, \varepsilon) + \mathcal{O}(\varepsilon^{r+1}).$$

In fact, it is convenient to consider α as a variable, and to define the cylinder maps

$$\begin{aligned} P_\varepsilon : \mathbb{S}^1 \times [0,1] \to \mathbb{S}^1 \times [0,1], \qquad & P_\varepsilon(\theta, \alpha) = (P_{\alpha,\varepsilon}(\theta), \alpha) \\ R : \mathbb{S}^1 \times [0,1] \to \mathbb{S}^1 \times [0,1], \qquad & R(\theta, \alpha) = (R_\alpha(\theta), \alpha), \end{aligned}$$

where $R_\alpha : \mathbb{S}^1 \to \mathbb{S}^1$ is the rigid rotation of an angle α. We now apply a version of the KAM Theorem, holding for non-conservative, finitely differentiable systems (see *e.g.* [5, Chap. 5] and [6]), to the family of diffeomorphisms P_ε. There exists an integer m with $1 \le m < r$ and a C^m-map

$$\Phi_\varepsilon : \mathbb{S}^1 \times [0,1] \to \mathbb{S}^1 \times [0,1], \quad \Phi_\varepsilon(\theta, \alpha) = (\theta + \varepsilon A(\theta, \alpha, \varepsilon), \alpha + \varepsilon B(\alpha, \varepsilon)), \qquad (20)$$

such that the restriction of Φ_ε to $\mathbb{S}^1 \times D_\gamma$ makes the following diagram commute:

$$\begin{array}{ccc} \mathbb{S}^1 \times D_\gamma & \xrightarrow{R} & \mathbb{S}^1 \times D_\gamma \\ \Phi_\varepsilon \uparrow & & \Phi_\varepsilon \uparrow \\ \mathbb{S}^1 \times D_\gamma & \xrightarrow{P_\varepsilon} & \mathbb{S}^1 \times D_\gamma. \end{array}$$

The differentiability of Φ_ε restricted to $\mathbb{S}^1 \times D_\gamma$ is of Whitney type. Since $P_{\alpha,\varepsilon}\big|_{\mathscr{C}_{\alpha,\varepsilon}}$ is C^m-conjugate to a rigid rotation on $\mathbb{S}^1$, the circle $\mathscr{C}_{\alpha,\varepsilon}$ is in fact C^m. This proves parts 1 and 2 of the Theorem.

Furthermore, the constant γ in (18) can be taken equal to ε^r. This gives that the measure of the complement of D_γ in $[0,1]$ is of order ε^r as $\varepsilon \to 0$. □

4.2. **Hénon-like attractors do exist.** Our proof of Theorem 2.5 is based on a result of Díaz-Rocha-Viana [16]. We begin by stating this result.

4.2.1. *Perturbations of multimodal families.* Two definitions from [16, Section 5.2] are introduced now. For more information about the terminology, we refer to [27, Sections II.5, II.6].

Definition 4.3. Let $J \subset \mathbb{R}$ be a compact interval. Fix $d \ge 1$, $k \ge 3$, $a^* \in \mathbb{R}$, and an interval of parameter values $\mathfrak{U} = [a_-, a_+]$, with $a^* \in \operatorname{Int} \mathfrak{U}$. A C^k-family of maps $M_a : J \to J$, with $a \in \mathfrak{U}$, is called a d-family if it satisfies the following conditions:

1. Invariance:**:** $M_{a^*}(J) \subset \operatorname{Int}(J)$;
2. Nondegenerate critical points:**:** M_{a^*} has d critical points $\{c_1, \dots, c_d\} \stackrel{\text{def}}{=} \operatorname{Cr} M_{a^*}$ that satisfy
$$M''_{a^*}(c_i) \neq 0 \;\text{ for all } i \quad \text{ and } \quad M_{a^*}(c_i) \neq c_j \;\text{ for all }\; i, j;$$
3. Negative Schwarzian derivative:**:** $SM_{a^*} < 0$ for all $x \neq c_i$, where
$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2;$$
4. Topological mixing:**:** There exists an interval $J' \subset \operatorname{Int}(J)$ such that $M_{a^*}(J) = M_{a^*}(J') = J'$ and such that map M_{a^*} is topologically mixing on J' (see Definition 2.4);
5. Preperiodicity:**:** for each $1 \le i \le d$ there exists m_i such that $p_i = M_{a^*}^{m_i}(c_i)$ is a (repelling) periodic point of M_{a^*};

6. Genericity of unfolding:**:** For all $c_i \in \operatorname{Cr} M_{a^*}$, denote by $c_i(a)$ and $p_i(a)$ the continuations of c_i and p_i, respectively, for a close to a^*. Then
$$\frac{\mathrm{d}}{\mathrm{d}a}\big(M_a^{m_i}(c_i(a)) - p_i(a)\big) \neq 0 \qquad \text{at } a = a^*.$$

Next we introduce the notion of *η-perturbation* of a d-family M_a, with $a \in \mathfrak{U}$ and $d \geq 1$ fixed.

Definition 4.4. Fix $\sigma > 0$ and consider the family $\overline{M}_a$ obtained by extending M_a as follows:
$$\overline{M}_a : J \times I_\sigma \to J \times I_\sigma, \quad \overline{M}_a(x,y) \stackrel{\text{def}}{=} (M_a(x), 0). \tag{21}$$
Also denote by M the map
$$M : \mathfrak{U} \times J \times I_\sigma \to J \times I_\sigma, \quad M(a,x,y) \stackrel{\text{def}}{=} \overline{M}_a(x,y) = (M_a(x), 0).$$
Given a C^k-family of diffeomorphisms
$$G_a : J \times I_\sigma \to J \times I_\sigma, \qquad a \in J,$$
for a $k \geq 3$, denote by G its extension
$$G : \mathfrak{U} \times J \times I_\sigma \to J \times I_\sigma, \quad G(a,x,y) \stackrel{\text{def}}{=} G_a(x,y).$$
Then G is called a *η-perturbation* of the d-family $\{M_a\}_a$ if
$$\|M - G\|_{C^k} \leq \eta,$$
where $\|\cdot\|_{C^k}$ denotes the C^k-norm over $\mathfrak{U} \times J \times I_\sigma$.

The following proposition is used in the sequel to prove existence of Hénon-like attractors for the map (2). See [2, 3, 29, 35, 41, 43, 46] for similar results.

Proposition 2. [16, Theorem 5.2] *Let $\{M_a\}_a$ be a d-family and p a periodic point of M_{a^*}. Then there exist $\eta > 0$, $\bar{a}$ and $\chi > 0$ such that, given any η-perturbation $\{G_a\}_a$ of $\{M_a\}_a$ the following holds.*

1. *For all a with $|a - a^*| < \bar{a}$ the map G_a has a periodic point p_a which is the continuation of the periodic point $(p, 0)$ of the map $\overline{M}_a$ in (21).*
2. *There exists a set $\mathfrak{S}$, contained in the interval $[a^* - \bar{a}, a^* + \bar{a}] \subset \mathfrak{U}$, with* $\operatorname{meas}(\mathfrak{S}) > \chi$*, such that for all $a \in \mathfrak{S}$ there exists $z \in W^u(p_a)$ satisfying:*
   (a) the orbit $\{G_a^n(z) \mid n \geq 0\}$ is dense in $\operatorname{Cl}(W^u(\operatorname{Orb}(p_a)))$;
   (b) G_a has a positive Lyapunov exponent at z, i.e.*, there exist $k > 0$, $\lambda > 1$ and $v \neq 0$ such that $\|DG_a^n(z)v\| \geq k\lambda^n$ for all $n \geq 0$;*
   (c) there exist $w \neq 0$ such that $\|DG_a^n(z)w\| \to 0$ as $n \to \pm\infty$.

4.2.2. *Multimodal families arising from powers of the Logistic map.* The proof of Theorem 2.5, which we present in this section, is based on three facts. First, suppose that $a^* \in [0,2]$ is such that the quadratic family $Q_a(x) = 1 - ax^2$ in (12) is a d-family in the sense of Definition 4.3, with $d = 1$. Then for all $n \geq 1$ the family $M_a \stackrel{\text{def}}{=} Q_a^n$ given by the n-th iterate of Q_a is a d-family for some $d \leq 2^n$. Second, for all $\eta_1 > 0$, the composition of an η_1-perturbation of Q_a with an η_1-perturbation of Q_a^n is an η_2-perturbation of Q_a^{n+1}, where $\eta_2 = C(n)\eta_1$ and $C(n)$ is a positive constant depending on n. Third, for each $n > q \geq 1$ and for each $(\alpha, \delta) \in \mathfrak{A}^{q/n}$, the asymptotic dynamics of $T_{\alpha,\delta,a,\varepsilon}$ is described by a map that turns out to be an η-perturbation of the d-family M_a, with $\eta = \mathcal{O}(\varepsilon)$. Moreover, M_a has a periodic point p such that its analytic continuation in the family $T_{\alpha,\delta,a,\varepsilon}$ possesses a transversal homoclinic intersection. Application of Proposition 2 to the point p concludes the proof.

In the next lemma we show that M_a is a d-family. For each $\tilde{a} \in [0, 2)$ there exists a $\beta > 0$ such that for all a with $a \in [0, \tilde{a}]$ the interval $J = [-1-\beta, 1+\beta] \subset \mathbb{R}$ satisfies $Q_a(J) \subset \mathrm{Int}(J)$. In the sequel, it is always assumed that the family Q_a is defined on such an interval J, and that the values of a we consider are such that $Q_a(J) \subset \mathrm{Int}(J)$.

Lemma 4.5. *Suppose $a^* \in [0, 2) \stackrel{\text{def}}{=} \mathfrak{U}$ is such that the quadratic family*

$$Q_a : J \to J, \quad Q_a(x) = 1 - ax^2$$

satisfies hypotheses 4 and 5 of Definition 4.3. Then for all $n \geq 1$ there exists $d \geq 1$ such that the family

$$M_a : J \to J, \quad M_a \stackrel{\text{def}}{=} Q_a^n$$

is a d-family with $d \leq 2^n - 1$ critical points.

Proof. Take a^* as above. We first prove the case $n = 1$, that is, $Q_a : J \to J$ is a 1-family. Conditions 1, 2, 3 of Definition 4.3 are obviously satisfied by Q_a. Condition 6 will now be proved. By conditions 4 and 5 (assumed by hypothesis), Q_{a^*} is a Misiurewicz map [28], *i.e.*, it has no periodic attractor and $c \notin \omega(c)$, where $c = 0$ is the critical point of Q_{a^*}. Moreover, by [27, Theorem III.6.3] the map Q_{a^*} is Collet-Eckmann (see *e.g.* [27, Section V.4]), that is, there exist constants $\kappa > 0$ and $\lambda > 1$ such that

$$\left| \frac{\mathrm{d}}{\mathrm{d}x} Q_{a^*}^n(Q_{a^*}(c)) \right| \geq \kappa \lambda^n \quad \text{for all } n \geq 0. \tag{22}$$

Therefore, by combining [42, Theorem 3] with the Collet-Eckmann condition (22) we get

$$\lim_{n \to \infty} \frac{\frac{\mathrm{d}}{\mathrm{d}a} Q_a^n(c) \mid_{a=a^*}}{\frac{\mathrm{d}}{\mathrm{d}x} Q_{a^*}^{n-1}(Q_{a^*}(c))} > 0. \tag{23}$$

Assume $Q_{a^*}^k(c) = p$, with p periodic (and repelling) under Q_{a^*}. By $p(a)$ denote the continuation of p for a close to a^*. Then, for all n sufficiently large,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}a} Q_a^n(c) \mid_{a=a^*} \\
=&\frac{\partial Q_a^{n-k}}{\partial a}(Q_{a^*}^k(c)) \mid_{a=a^*} + \frac{\partial Q_a^{n-k}}{\partial x}(Q_{a^*}^k(c)) \mid_{a=a^*} \frac{\mathrm{d}}{\mathrm{d}a} Q_a^k(c) \mid_{a=a^*} = \\
=&\frac{\partial}{\partial a} Q_a^{n-k}(p) \mid_{a=a^*} + \frac{\partial}{\partial x} Q_a^{n-k}(p) \mid_{a=a^*} \frac{\mathrm{d}}{\mathrm{d}a} \left[p(a) + Q_a^k(c) - p(a) \right] \mid_{a=a^*} = \\
=&\frac{\mathrm{d}}{\mathrm{d}a} \left(Q_a^{n-k}(p(a)) \right) \mid_{a=a^*} + \frac{\partial}{\partial x} Q_{a^*}^{n-k}(p) \frac{\mathrm{d}}{\mathrm{d}a} \left[Q_a^k(c) - p(a) \right] \mid_{a=a^*} .
\end{aligned} \tag{24}$$

The point $Q_a^{n-k}(p(a))$ belongs to a hyperbolic periodic orbit, that varies smoothly with the parameter a. Therefore, its derivative with respect to a (which is the first term in the last equality) is uniformly bounded in n. On the other hand,

$$\frac{\mathrm{d}}{\mathrm{d}x} Q_{a^*}^{n-1}(Q_{a^*}(c)) = \frac{\partial}{\partial x} Q_{a^*}^{n-k}(p) \frac{\mathrm{d}}{\mathrm{d}x} Q_{a^*}^{k-1}(Q_{a^*}(c)).$$

Therefore, by (22), (23), and (24) we conclude that

$$0 < \lim_{n \to \infty} \frac{\frac{\mathrm{d}}{\mathrm{d}a} Q_a^n(c) \mid_{a=a^*}}{\frac{\mathrm{d}}{\mathrm{d}x} Q_{a^*}^{n-1}(Q_{a^*}(c))} = \frac{\frac{\mathrm{d}}{\mathrm{d}a} \left[Q_a^k(c) - p(a) \right]_{a=a^*}}{\frac{\mathrm{d}}{\mathrm{d}x} Q_{a^*}^{k-1}(Q_{a^*}(c))}. \tag{25}$$

This proves that Q_a satisfies condition 6 of Definition 4.3.

We now show that the n-th iterate M_a of the quadratic map is a d-family for all $n > 1$ and for some $d \le 2^n$. For simplicity, we denote Q_{a^*} by Q for the rest of this proof. Condition 1 holds for M_{a^*} since it holds for Q_{a^*}. Condition 3 follows from the fact that the composition of maps with negative Schwarzian derivative also has negative Schwarzian derivative, see *e.g.* [27, II.6]. Condition 4 is obviously satisfied.

Condition 2 is now proved by induction on n, where the case $n = 1$ is obvious. Obviously, the set $\operatorname{Cr} M_{a^*}$ of critical points of M_{a^*} has cardinality $d \le 2^n - 1$. Moreover,

$$\operatorname{Cr} M_{a^*} = Q^{-1}\left(\operatorname{Cr} Q^{n-1}\right) \cup \operatorname{Cr} Q = \bigcup_{j=0}^{n-1} (Q^{-j})(\operatorname{Cr} Q). \tag{26}$$

Suppose that condition 2 holds for a given $n \ge 1$. We first show that

$$(Q^{n+1})''(x) \neq 0 \qquad \text{for all} \;\; x \in \operatorname{Cr} Q^{n+1}. \tag{27}$$

By (26), if $x \in \operatorname{Cr} Q^{n+1}$ then either $x = c$, or $Q(x) \in \operatorname{Cr} Q^n$. If $x = c$ then

$$(Q^{n+1})''(x) = (Q^n)'\left(Q(c)\right) \cdot (Q)''(c). \tag{28}$$

The second factor is nonzero. If the first factor is zero, then

$$0 = (Q^n)'\left(Q(c)\right) = Q'(Q^n(c)) \dots Q'(Q(c)).$$

Therefore there exists j such that $Q^j(c) = c$, so that c is an attracting periodic point of Q. But this contradicts the fact that Q is Misiurewicz, so that (28) is nonzero. The other possibility is that $c \neq x$ and $Q(x) \in \operatorname{Cr} Q^n$. In this case,

$$(Q^{n+1})''(x) = (Q^n)''\left(Q(x)\right) \cdot Q'(x)^2,$$

which is nonzero. Indeed, $Q'(x) \neq 0$, otherwise $x = c$. Moreover $(Q^n)''\left(Q(x)\right) \neq 0$ by the induction hypotheses since the critical points of Q^n are nondegenerate. This proves (27), from which the first part of condition 2 follows.

We now prove, again arguing by contradiction, that

$$Q^{n+1}(x) \neq y \qquad \text{for all} \;\; x, y \in \operatorname{Cr} Q^{n+1}.$$

Suppose that there exist $x, y \in \operatorname{Cr} Q^{n+1}$ such that $Q^{n+1}(x) = y$. By (26) there exist i and j such that $Q^i(x) = Q^j(y) = c$, where $0 \le i, j \le n$. This would imply that

$$Q^{n+1+j-i}(c) = Q^j(Q^{n+1}(x)) = Q^j(y) = c,$$

with $n + 1 + j - i \ge 1$ and, therefore, c would be an attracting periodic point of Q, which is impossible since Q is Misiurewicz. Condition 2 is proved.

To prove condition 5, fix $y \in \operatorname{Cr} M_{a^*}$ and $j \ge 0$ such that $Q^j(y) = c$. Since c is preperiodic for Q by hypothesis, there exists $k \ge 1$ such that $Q^{j+k}(y) = p$, where p is periodic under Q with period $u \ge 1$. The orbit of y under M_{a^*} is, except for a finite number of initial iterates, a subset of the orbit of p under Q. This shows that y is preperiodic for M_{a^*}.

To prove condition 6, take $y \in \operatorname{Cr} M_{a^*}$, j, u, k and $p \in J$ as in the proof of condition 5. Then there exist integers l and m, with $0 \le l < u$ and $m \ge 1$, such that

$$M_{a^*}^m(y) = Q^{k+l}(c) = Q^l(p) \in \operatorname{Orb}_Q(p). \tag{29}$$

By condition 5 (assumed by hypothesis) and by (29), the point $z = Q^l(p)$ is periodic (and repelling) under M_{a^*}. Denote by $y(a)$, $z(a)$, and $p(a)$ the continuations of y, z, and p, respectively, for a close to a^*. In particular,

$$Q_a^j(y(a)) = c \quad \text{and} \quad Q_a^l(p(a)) = z(a).$$

We have to show that

$$\frac{\mathrm{d}}{\mathrm{d}a}\left[M_a^m(y(a)) - z(a)\right]\big|_{a=a^*} \neq 0. \tag{30}$$

By the chain rule we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}a}\,Q_a^{l+k}(c)\big|_{a=a^*} &= \frac{\partial Q_a^l}{\partial a}\,(Q_a^k(c))\big|_{a=a^*} + \frac{\partial Q_a^l}{\partial x}\,(Q_a^k(c))\big|_{a=a^*}\,\frac{\mathrm{d}Q_a^k}{\mathrm{d}a}\,(c)\big|_{a=a^*} = \\ &= \frac{\partial Q_a^l}{\partial a}(p(a))\bigg|_{a=a^*} + \frac{\partial Q_{a^*}^l}{\partial x}(p)\;\frac{\mathrm{d}Q_a^k}{\mathrm{d}a}(c)\bigg|_{a=a^*}, \\ \frac{\mathrm{d}}{\mathrm{d}a}\,Q_a^l(p(a))\big|_{a=a^*} &= \frac{\partial Q_a^l}{\partial a}(p(a))\bigg|_{a=a^*} + \frac{\partial Q_{a^*}^l}{\partial x}(p)\,\frac{\mathrm{d}}{\mathrm{d}a}p(a^*),\end{aligned}$$

where $p = p(a^*) = Q_{a^*}^k(c)$. Therefore,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}a}\left[M_a^m(y(a)) - z(a)\right]\big|_{a=a^*} &= \frac{\mathrm{d}}{\mathrm{d}a}\left[Q_a^{k+l}(c) - Q_a^l(p(a))\right]\big|_{a=a^*} = \\ &= \frac{\partial Q_{a^*}^l}{\partial x}(p)\,\frac{\mathrm{d}}{\mathrm{d}a}\left[Q_a^k(c) - p(a)\right]\big|_{a=a^*}.\end{aligned}$$

The factor $\frac{\mathrm{d}}{\mathrm{d}a}\left[Q_a^k(c(a)) - p(a)\right]\big|_{a=a^*}$ is nonzero by (25). The same holds for the other factor, otherwise p would be an attracting periodic point of Q_{a^*}. This proves inequality (30). □

Proposition 2 does not provide a nontrivial basin of attraction for the closure $\mathrm{Cl}\,(W^u(p_a))$. We now show that, under the hypotheses of Theorem 2.5, there exists a periodic point p_a for which a nontrivial basin of attraction of $\mathrm{Cl}\,(W^u(p_a))$ can be constructed. Therefore, in this case $\mathrm{Cl}\,(W^u(p_a))$ is a Hénon-like attractor.

Proposition 3. *Consider the map $\{M_a^*\}_a = Q_{a^*}^n$, where Q_{a^*} satisfies the hypotheses of Lemma 4.5. There exist a periodic point p of Q_{a^*}, and positive constants η, $\bar{a}$ and χ such that for any η-perturbation $\{G_a\}_a$ of $\{M_a\}_a = Q_{a^*}^n$ the following holds.*

1. *For all a with $|a - a^*| < \bar{a}$ the map G_a has a periodic point p_a which is the continuation of the periodic point $(p, 0)$ of the map $\overline{M}_a$ in (21). Moreover, p_a has a transversal homoclinic intersection.*
2. *There exists a set $\mathfrak{S}$, contained in the interval $[a^* - \bar{a}, a^* + \bar{a}] \subset \mathfrak{U}$, with $\mathrm{meas}(\mathfrak{S}) > \chi$, such that for all $a \in \mathfrak{S}$ the set $\mathrm{Cl}\,(W^u(\mathrm{Orb}(p_a)))$ is a Hénon-like attractor of the map G_a.*

Proof. To construct a non-trivial basin of attraction for $\mathrm{Cl}\,(W^u(\mathrm{Orb}(p_a)))$, it is sufficient to find a periodic point p_a of $\{G_a\}_a$ that has a transversal homoclinic intersection. Then the basin is provided by the Tangerman-Szewc Theorem (see Theorem 2.1 and subsequent remark). Indeed, for all η sufficiently small, all η-perturbations of the map Q_a^* are uniformly dissipative. Moreover, the unstable manifold of p_a is bounded, since it is bounded for Q_{a^*} and since the invariant manifolds of a map depend continuously on the map [29, Prop. 7.1]. Therefore, the second part of the proposition follows from the first part, together with the Tangerman-Szewc argument and Proposition 2.

To prove the first part, we claim that the map Q_{a^*} has a periodic point p belonging to a non-degenerate homoclinic orbit. Indeed, if the claim is true, then for η sufficiently small and for a close to a^*, any η-perturbation of M_a possesses a periodic point p_a which is the analytic continuation of p and such that p_a has a transverse homoclinic intersection. The latter property again follows from continuous dependence of the invariant manifolds on the map [29, Prop. 7.1].

To prove the claim that Q_{a^*} has a periodic point p belonging to a non-degenerate homoclinic orbit we first show that there exists a point y_0 belonging to a degenerate homoclinic orbit of Q_{a^*}. Since the critical point c of Q_{a^*} is preperiodic, there exist positive integers k, h such that $Q^k_{a^*}(c) = y_0$ and y_0 is periodic with period h. The unstable manifold of any periodic point of Q_{a^*} is the whole core $[1-a^*, 1]$, since Q_{a^*} is topologically mixing. Therefore, since the critical point c belongs to $W^u(y_0)$, by taking preimages of c, a point q can be found such that $q \in W^u_{loc}(y_0)$, $Q^l_{a^*}(q) = c$ and $Q^{l+k}_{a^*}(q) = y_0$ for some integer $l > 0$. This means that y_0 belongs to a degenerate homoclinic orbit of Q_{a^*}.

We now prove that there exists a periodic point p of Q_{a^*} having a non-degenerate homoclinic orbit. This is achieved by examining a power of Q_{a^*} for which all points of the orbit of y_0 are fixed. Denote by $\mathrm{Orb}_{Q_{a^*}}(y_0) = \{y_j \mid j = 0, \dots, h-1\}$ the orbit of y_0, under Q_{a^*}, where $y_j = Q^j_{a^*}(y_0)$. Let m be the smallest multiple of h which is larger than k, and denote $f \overset{\text{def}}{=} Q^m_{a^*}$. Then, $f(c)$ belongs to $\mathrm{Orb}_{Q_{a^*}}(y_0)$ and all points y_j are fixed for f. We can assume that

$$f'(y_j) > 1 \quad \text{for all} \quad y_j \in \mathrm{Orb}_{Q_{a^*}}(y_0) \tag{31}$$

by taking f^2 instead of f if necessary.

Brouwer's fixed point Theorem and continuity arguments ensure the existence of a fixed point p of f, a critical point c' of f, and an interval $I = (c' - \delta, c')$ such that:

1. $f'(p) < -1$;
2. c' lies in the interval (y, p);
3. f is monotonically increasing in I;
4. p falls in the interval $f(I)$.

The configuration of c, c' and $y = f(c)$ within the graph of f looks like the sketch in Figure 10, in the case $y < c$ and $f''(c) > 0$ (the other combinations of the sign of $y - c$ and $f''(c)$ are treated similarly). Since f is topologically mixing, the interval $f(I)$ is contained in the unstable manifold of p. Therefore p belongs to a homoclinic orbit $\mathcal{O}$.

Moreover, the homoclinic orbit $\mathcal{O}$ is non-degenerate. Indeed, if this was not the case, then there would exist a critical point c'' of f belonging to $\mathcal{O}$, so that $f^j(c'') = p$ for some $j \in \mathbb{N}$. However, according to (26), and since c is preperiodic, the orbit of c'' under Q_{a^*} eventually lands inside $\mathrm{Orb}_{Q_{a^*}}(y_0)$. It follows that $p \in \mathrm{Orb}_{Q_{a^*}}(y_0)$, which is absurd, since $f'(p) < -1$ whereas (31) holds. ∎

In the next lemma we show that the composition of a small perturbation of the map $\overline{Q}_a(x, y) = (Q_a(x), 0)$ (we use here the notation of Definition 4.4) with a small perturbation of $\overline{Q^n_a}(x, y) = (Q^n_a(x), 0)$ yields a small perturbation of $\overline{Q^{n+1}_a}(x, y)$. As in Definition 4.4, denote by $Q, Q^n : [0, 2] \times J \times I \to J \times I$ the functions $Q(a, x, y) = (Q_a(x), 0)$ and $Q^n(a, x, y) = (Q^n_a(x), 0)$, respectively.

Lemma 4.6. *For each $\eta > 0$ there exists a $\zeta > 0$ such that for all $F, G : [0, 2] \times J \times I \to J \times I$ such that*

$$\|G - Q\|_{C^3} < \zeta \quad \textit{and} \quad \|F - Q^n\|_{C^3} < \zeta, \tag{32}$$

we have

$$\left\|G \circ F - Q^{n+1}\right\|_{C^3} < \eta. \tag{33}$$

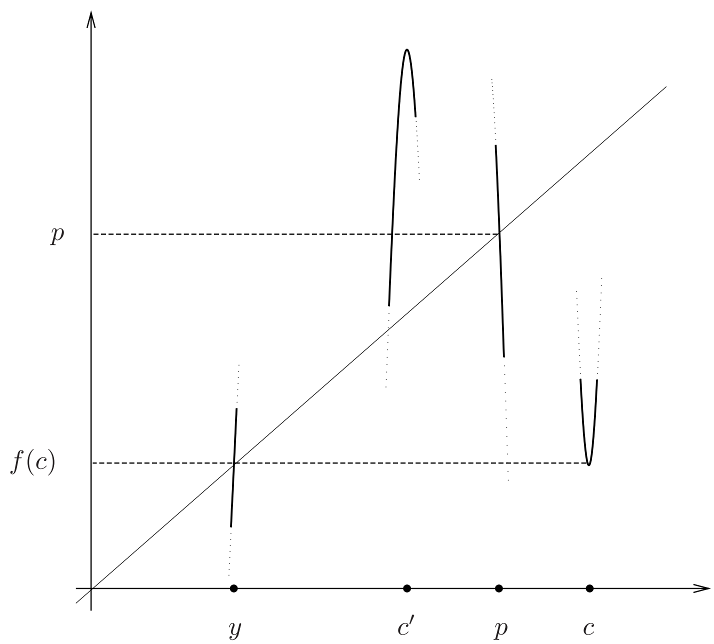


FIGURE 10. Graph of the map f from the proof of Proposition 3. Only the relevant branches of the graph are plotted, in relation to the fixed points y, p and the critical points c and c'. See the text for details.

Proof. Write

$$G(a,x,y) = \begin{pmatrix} Q_a(x) + g_1(a,x,y) \\ g_2(a,x,y) \end{pmatrix} \quad \text{and} \quad F(a,x,y) = \begin{pmatrix} Q_a^n(x) + f_1(a,x,y) \\ f_2(a,x,y) \end{pmatrix}.$$

Then

$$\begin{aligned} & G \circ F(a,x,y) - \begin{pmatrix} Q_a^{n+1}(x) \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -2a(f_1(a,x,y))^2 - 2af_1(a,x,y)Q_a^n(x) + g_1\big(a, \tilde{f}_1(a,x,y), f_2(a,x,y)\big) \\ g_2\big(a, \tilde{f}_1(a,x,y), f_2(a,x,y)\big) \end{pmatrix}, \end{aligned}$$

where $\tilde{f}_1(a,x,y) = Q_a^n(x) + f_1(a,x,y)$. The C^3-norm of the terms $-2a(f_1(a,x,y))^2$ and $-2af_1(a,x,y)Q_a^n(x)$ is bounded by a constant times the C^3-norm of f_1. We now estimate the norm of $\tilde{g}_1$, defined by

$$\tilde{g}_1(x_0, x_1, x_2) = g_1(a, \tilde{f}_1(a,x,y), f_2(a,x,y)).$$

Denote $x_0 = a$, $x_1 = x$, and $x_2 = y$. Then any second order derivative of $\tilde{g}_1$ is a sum of terms of the following type:

$$\frac{\partial^2 g_1}{\partial x_j x_k} \frac{\partial \tilde{f}_k}{\partial x_l}, \quad \frac{\partial g_1}{\partial x_k} \frac{\partial^2 \tilde{f}_k}{\partial x_j x_l},$$

where we put $\tilde{f}_2 = f_2$ to simplify the notation. For the third order derivatives a similar property holds. Since the C^3-norm of $\tilde{f}_k$ is bounded, we get that each term in the third order derivative of $\tilde{g}_1$ is bounded by a constant times the C^3-norm of the g_j. This concludes the proof. □

Proof of Theorem* 2.5*. The Theorem will be first proved for $a^* < 2$. The case $a^* = 2$ follows by choosing another value $\bar{a}^* < 2$ sufficiently close to 2. Fix $a^* \in (1, 2)$ verifying the hypotheses of Lemma 4.5. To begin with, we consider the case $(\alpha, \delta) \in \operatorname{Int} \mathfrak{A}^1$, the interior of the tongue of period one. Then the Arnol′d family $A_{\alpha,\delta}$ on $\mathbb{S}^1$ has two hyperbolic fixed points θ_1^s (attracting) and θ_1^r (repelling), see [15, Section 1.14]. The θ-coordinate of both points depends on the choice of $(\alpha, \delta) \in \operatorname{Int} \mathfrak{A}^1$. So for all $\theta \in \mathbb{S}^1$ with $\theta \neq \theta_1^r$, the orbit of θ under $A_{\alpha,\delta}$ converges to θ_1^s. This means that the manifold

$$\Theta_1 = \left\{(x, y, \theta) \in \mathbb{R}^2 \times \mathbb{S}^1 \ \mid \ \theta = \theta_1^s\right\} \subset \mathbb{R}^2 \times \mathbb{S}^1$$

is invariant and attracting under $T_{\alpha,\delta,a,\varepsilon}$. Denote by $G_{a,1}$ the restriction of $T_{\alpha,\delta,a,\varepsilon}$ to Θ_1:

$$G_{a,1} : \Theta_1 \to \Theta_1, \qquad (x, y, \theta_1^s) \mapsto (1 - ax^2 + \varepsilon f_1,\ \varepsilon g_1,\ \theta_1^s),$$

where $f_1 = f(a, x, y, \theta_1^s, \alpha, \delta)$ and similarly for g_1. Since $Q_{a^*}(J) \subset \operatorname{Int}(J)$, there exists a constant $\sigma > 0$ such that for all ε sufficiently small and all a close enough to a^*,

$$\begin{aligned} G_{a,1}(J \times I_\sigma \times \{\theta_1^s\}) &\subset \operatorname{Int}(J \times I_\sigma \times \{\theta_1^s\}) \quad \text{and} \\ T_{\alpha,\delta,a,\varepsilon}\left(J \times I_\sigma \times (\mathbb{S}^1 \setminus \{\theta_1^r\})\right) &\subset \operatorname{Int}\left(J \times I_\sigma \times (\mathbb{S}^1 \setminus \{\theta_1^r\})\right). \end{aligned} \tag{34}$$

Since Θ_1 is diffeomorphic to $\mathbb{R}^2$, we consider $G_{a,1}$ as a map of $\mathbb{R}^2$. Then $G_{a,1}$, is an η-perturbation of the quadratic family $Q_a(x)$, where $\eta = \mathcal{O}(\varepsilon)$. We now apply Proposition 3 to the family $G_{a,1}$. Let p_0 be the periodic point of M_{a^*} as given by Proposition 3. For all ε sufficiently small there exists a constant $\bar{a} > 0$ and a set $\mathfrak{S}$ of positive Lebesgue measure, contained in the interval $[a^* - \bar{a}, a^* + \bar{a}]$, such that the following holds. For all $a \in [a^* - \bar{a}, a^* + \bar{a}]$, $G_{a,1}$ has a saddle periodic point $\bar{p}$ which is the continuation of the point p_0. Furthermore, for all $a \in \mathfrak{S}$ the closure $\widetilde{\mathscr{A}} = \operatorname{Cl}\left(W^u(\operatorname{Orb}_{G_{a,1}}(\bar{p}))\right)$ is a Hénon-like attractor of $G_{a,1}$ contained inside Θ_1. The point $p = (\bar{p}, \theta_1^s)$ is a saddle periodic point of the map $T_{\alpha,\delta,a,\varepsilon}$, and $W^u(\operatorname{Orb}_{T_{\alpha,\delta,a,\varepsilon}}(p)) = W^u(\operatorname{Orb}_{G_{a,1}}(\bar{p})) \times \{\theta_1^s\}$. Therefore $\mathscr{A} = \operatorname{Cl}(W^u(p)) = \widetilde{\mathscr{A}} \times \{\theta_1^s\}$ is a Hénon-like attractor of $T_{\alpha,\delta,a,\varepsilon}$. Moreover, the basin of attraction of $\operatorname{Cl}(W^u(p))$ has nonempty interior in $\mathbb{R}^2 \times \mathbb{S}^1$ because of (34). This proves the claim for $(\alpha, \delta) \in \operatorname{Int} \mathfrak{A}^1$.

We pass to the case of higher period tongues. Suppose that $(\alpha, \delta) \in \operatorname{Int} \mathfrak{A}^{q/n}$, with $n > q \geq 1$. Then $A_{\alpha,\delta}$ has (at least) two hyperbolic periodic orbits

$$\begin{aligned} \operatorname{Orb}(\theta_1^s) &= \{\theta_1^s, \theta_2^s, \ldots, \theta_n^s\} \quad \text{attracting, and} \\ \operatorname{Orb}(\theta_1^r) &= \{\theta_1^r, \theta_2^r, \ldots, \theta_n^r\} \quad \text{repelling.} \end{aligned}$$

For $j = 1, \ldots, n$, denote by Θ_j the manifold

$$\Theta_j = \left\{(x, y, \theta) \in \mathbb{R}^2 \times \mathbb{S}^1) \ \mid \ \theta = \theta_j^s\right\},$$

and define maps G_j as the restriction of $T_{\alpha,\delta,a,\varepsilon}$ to Θ_j:

$$\begin{aligned} &G_j : \Theta_j \to \Theta_{j+1} \qquad \text{for } j = 1, \ldots, n-1 \\ &G_n : \Theta_n \to \Theta_1, \qquad \text{where} \end{aligned}$$

$$\begin{aligned} (x,\ y,\ \theta_1^s) &\overset{G_j}{\mapsto} (Q_a(x) + \varepsilon f_j,\ \varepsilon g_j,\ \theta_{j+1}^s), \qquad \text{for } j = 1, \ldots, n-1 \\ (x,\ y,\ \theta_n^s) &\overset{G_n}{\mapsto} (Q_a(x) + \varepsilon f_n,\ \varepsilon g_n,\ \theta_1^s). \end{aligned}$$

Here, $f_j = f(a, x, y, \theta_j^s, \alpha, \delta)$. The manifold Θ_1 is invariant and attracting under the n-th iterate of the map $T_{\alpha,\delta,a,\varepsilon}$. For all (x, y, θ) in the complement of the set

$$\{(x, y, \theta) \mid \theta \in \mathrm{Orb}(\theta_1^r)\},$$

the asymptotic dynamics is given by the map

$$G_{a,1,\ldots n} \stackrel{\mathrm{def}}{=} G_n \circ G_{n-1} \circ \cdots \circ G_1.$$

Notice that each of the G_j's is an η_j-perturbation of the family Q_a in the sense of Definition 4.4, where $\eta_j = B\varepsilon$ and B can be chosen uniform on θ_j^s (and, therefore, on (α, δ)).

Let p_0 be the periodic point of M_{a^*} as given by Proposition 3. Then $(p_0, 0)$ is a saddle periodic point for the map $\overline{M}_a$ defined as in (21). Take η, $\bar{a}$, and χ as in Proposition 3. By inductive application of Lemma 4.6 there exists an $\bar{\varepsilon} > 0$ depending on η and n such that

$$\|G_{a,1,\ldots n} - Q^n\|_{C^3} < \eta,$$

for all $(\alpha, \delta) \in \mathrm{Int}\, \mathfrak{A}^{q/n}$ and all $|\varepsilon| < \bar{\varepsilon}$. That is, $G_{a,1,\ldots n}$ is an η-perturbation of M_a for all q with $1 \le q < n$ and all $(\alpha, \delta, a, \varepsilon)$ with

$$(\alpha, \delta) \in \mathfrak{A}^{q/n}, \quad \varepsilon \in [-\bar{\varepsilon}, \bar{\varepsilon}].$$

By Proposition 3 there exist an $\bar{a} > 0$ and a set $\mathfrak{S}$ contained in the interval $[a^* - \bar{a}, a^* + \bar{a}]$ such that $\mathrm{meas}(\mathfrak{S}) \ge \chi$ and the following holds. For all $a \in [a^* - \bar{a}, a^* + \bar{a}]$ the map $G_{a,1,\ldots,n}$ has a periodic point $\bar{p}_a$ which is the continuation of the periodic point $(p_0, 0)$ of $\overline{M}_a$. Moreover, for all $a \in \mathfrak{S}$ the closure $\widetilde{\mathscr{A}} = \mathrm{Cl}\left(W^u(\mathrm{Orb}_{G_{a,1,\ldots,n}}(\bar{p}_a))\right)$ is a Hénon-like attractor of $G_{a,1,\ldots,n}$, contained inside Θ_1.

To finish the proof, observe that $p_a = (\bar{p}_a, \theta_1^s)$ is a saddle periodic point of $T_{\alpha,\delta,a,\varepsilon}$. The set $\mathscr{A} = \mathrm{Cl}\left(W^u(\mathrm{Orb}_{T_{\alpha,\delta,a,\varepsilon}}(p_a))\right)$ is compact and invariant under $T_{\alpha,\delta,a,\varepsilon}$. To show that $\mathscr{A}$ has a dense orbit, fix parameter values as provided by Proposition 3 applied to $G_{a,1,\ldots n}$. Let $z \in \Theta_1$ a point having a dense orbit in $\widetilde{\mathscr{A}}$ and satisfying properties (a)–(c) of Proposition 2. Then given $\eta > 0$ and a point

$$q = T^j_{\alpha,\delta,a,\varepsilon}(q') \in T^j_{\alpha,\delta,a,\varepsilon}(\widetilde{\mathscr{A}} \times \{\theta_1^s\}), \quad \text{with } \ 1 \le j \le n-1,$$

there exists $m > 0$ such that $\mathrm{dist}(G^m_{a,1,\ldots n}(z), q') < \eta$. By continuity of $T^j_{\alpha,\delta,a,\varepsilon}$, for all $\varrho > 0$ there exists $\eta > 0$ such that

$$\mathrm{dist}(T^j_{\alpha,\delta,a,\varepsilon}(q''), T^j_{\alpha,\delta,a,\varepsilon}(q')) < \varrho \quad \text{for all } q'' \text{ with} \quad \mathrm{dist}(q'', q') < \eta.$$

We conclude that for all $\varrho > 0$ there exists $m > 0$ such that

$$\mathrm{dist}(T^j_{\alpha,\delta,a,\varepsilon}(G^m_{a,1,\ldots n}(z)), T^j_{\alpha,\delta,a,\varepsilon}(q')) = \mathrm{dist}(T^{j+mn}_{\alpha,\delta,a,\varepsilon}(z), q) < \varrho.$$

This proves that the orbit of z under $T_{\alpha,\delta,a,\varepsilon}$ is dense in $\mathscr{A}$. Properties (9) and (10) will now be proved. Since $G_{a,1,\ldots,n} = T^n_{\alpha,\delta,a,\varepsilon}$ on Θ_1, for any $m \in \mathbb{N}$ and any $z \in \mathscr{A}$ we have

$$DT^m_{\alpha,\delta,a,\varepsilon}(z) = DT^r_{\alpha,\delta,a,\varepsilon}(G^s_{a,1,\ldots,n}(z)) DG^s_{a,1,\ldots,n}(z),$$

where $s = m \mod n$ and $r = m - s$. Take z as above and a vector $v = (v_x, v_y, 0) \in T_z\mathscr{A}$ such that $\left\|DG^s_{a,1,\ldots,n}(z)v\right\| \ge \kappa\lambda^s$ for all s, where $\kappa > 0$ and $\lambda > 1$ are constants. Since $T^r_{\alpha,\delta,a,\varepsilon}$ is a diffeomorphism for all $r = 1, \ldots, s-1$ and $G^s_{a,1,\ldots,n}(z)$

belongs to the compact set $\mathscr{A}$ for all $s \in \mathbb{N}$, then there exists a constant $c > 0$ such that

$$\left\| DT^m_{\alpha,\delta,a,\varepsilon}(z)v \right\| = \left\| DT^r_{\alpha,\delta,a,\varepsilon}(G^s_{a,1,\dots,n}(z))DG^s_{a,1,\dots,n}(z)v \right\| \geq c \left\| DG^s_{a,1,\dots,n}(z)v \right\|,$$

where c is uniform in r. This proves property (9). Property (10) is proved similarly. This shows that the closure $\mathrm{Cl}\,(W^u(p_a))$ is a Hénon-like attractor of $T_{\alpha,\delta,a,\varepsilon}$. □

Remark 2. At the boundary of a tongue $\mathfrak{A}^{q/n}$ the Arnol′d family $A_{\alpha,\delta}$ has a saddle-node periodic point θ_1. However, the basin of attraction of $\mathrm{Orb}\,\theta_1$ still has nonempty interior, so that the above conclusions hold for all (α, δ) in the closure $\mathrm{Cl}\left(\mathfrak{A}^{q/n}\right)$.

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Appendix A. **Numerical methods.** Computation of Lyapunov exponents is an important tool in the numerical exploration of dynamical systems. Consider an orbit $\left\{x_j = T^j(x_0), j = 0, 1, 2, 3, \dots\right\}$ of a three-dimensional map T. Following [39], three linearly independent vectors are selected. Consecutive iterates of these vectors are computed under the derivative DT, after a transient along the orbit. The vectors are orthonormalised at each step (or every given number of steps to speed up the process). We now define the *Lyapunov sums*, for simplicity just considering the iterates of one initial vector v_0. Let $v_n = DT^n_{x_0}(v_0)/\|DT^n_{x_0}(v_0)\|$ denote the normalised tangent vector obtained after n iterations. Let $\hat{v}_{n+1} = DT_{x_n}(v_n)$. Then $v_{n+1} = \hat{v}_{n+1}/f_{n+1}$, where $f_{n+1} = \|\hat{v}_{n+1}\|$. The Lyapunov sum is then defined as

$$LS_n = \sum_{j=1}^{n} \log(f_j). \tag{35}$$

The maximal Lyapunov exponent is the average slope of the Lyapunov sum (35) as $n \to \infty$, that is, the average logarithmic rate of increase of the length $\log(f_j)$. The other Lyapunov exponents are estimated as averages of Lyapunov sums in which the coefficients f_j are given by the Gram-Schmidt orthonormalisation, see [39] for details.

In the numerical implementation, estimates are produced of the average slope of LS_n for different values of n up to a maximal number N of iterates. The computations are stopped before N iterates in case of escape, or if a periodic orbit is detected, or if different estimates of the average coincide within a prescribed tolerance ρ. Typical values for N and ρ in the computations for this paper are 10^7 and 10^{-6}, respectively.

One of the major problems is to detect values of the Lyapunov exponents which are very close to zero. Several procedures have been proposed to this end. Taking into account that the driving behaviour is quasi-periodic or periodic in the skew product case, a method of successive filtering and fitting, similar to [7] can be suitable. Another method, the MEGNO [13], is based of weighted averages and is very useful to detect small values of the Lyapunov exponent, see [26] for a problem which requires substantial usage of the MEGNO. However, we presently use the Lyapunov sum (35), because this actually helps understand the behaviour of the system in certain elementary cases, see Appendix C. Typically we considered a Lyapunov exponent as equal to zero whenever $|\Lambda_j| < 10^{-5}$, $j = 1, 2$.

Let $\Lambda_1 \geq \Lambda_2 \geq \Lambda_3$ be the three Lyapunov exponents. Since the family (17) is dissipative, the value of Λ_3 is used to decide whether the normal behaviour is of nodal type ($\Lambda_2 > \Lambda_3$) or of focal type ($\Lambda_2 = \Lambda_3$) for periodic or quasi-periodic attractors. The values of Λ_1 and Λ_2 and their position with respect to zero play a major role in classifying the detected attractor.

An alternative criterion is used, mainly for small ε, to decide whether the attractor is quasi-periodic or has some 'strange' character. For our choice of the parameter values (see the next Section) one may expect a period 7 invariant circle if (α, δ) is in the quasi-periodic domain and ε is sufficiently small, in the skew product case $\mu = 0$. Similar behaviour can be expected for $\mu > 0$ small, for a subset of (α, δ) having large relative measure. Iterates of $\mathcal{T}^7$ (where $\mathcal{T}$ is given in (17)), are sorted, after a transient, by the values of θ. If the attractor is an invariant circle $(x(\theta), y(\theta))$, then the variation of the coordinates (x, y) has to remain bounded and tend to the true variation when the number of iterates increases. We refer to this as *the variation criterion for invariant circles*. A similar idea is used to recognise Hénon-like attractors for the fully coupled case $\mu > 0$: the values of the angular coordinate θ along the iterates should cluster around the periodic orbit obtained for $\mu = 0$. We refer to this as *the clustering criterion*.

Appendix B. **On the selection of coefficients for the model map.** Several coefficients and parameters have been fixed in the family $\mathcal{T}$ given in (17), to perform the numerical exploration. Our choice of $\delta = 0.6$ yields a satisfactory compromise between periodicity and quasi-periodicity in the driving Arnol′d family: resonant zones are not too narrow, while quasi-periodic dynamics still occurs for most values of α. Concerning the Hénon family, the parameter b has been fixed to 0.3 for historical reasons: indeed, it is the value used by Hénon [21], and provides a good compromise between dissipation and visibility of the folds in the unstable manifold. It was also used in [38], which studied the types of attractors and the role of homoclinic and heteroclinic tangencies as a function of a. The value $a = 1.25$ corresponds to a periodic attractor of period 7 and allows for moderate values of the forcing ε before escape occurs. The values $\mu = 0$ and $\mu = 0.01$ are used for the skew product and the fully coupled case, respectively.

Appendix C. **On the numerical identification of strange nonchaotic attractors.** We now discuss two distinct phenomena which illustrate the difficulty of identifying strange nonchaotic attractors in a reliable way.

Arithmetic effects. The numerically computed orbit of a given system can depend very strongly on the arithmetics used for the computations. Indeed, consider a quasi-periodic attracting invariant circle for a map: then $\Lambda_1 = 0$ and $\Lambda_2 < 0$. Therefore, the Lyapunov sums (see (35) in Appendix A) of Λ_2 are decreasing *on average*. However, the Lyapunov sums can display *arbitrarily wide* oscillations around a the average straight line whose slope is Λ_2. Roundoff errors are locally amplified by a large factor, if the oscillations are sufficiently wide. This can result in numerically observed behaviour which is *entirely different* from the actual dynamics.

We illustrate this effect with the Logistic family driven by a rigid rotation:

$$\mathbb{R} \times \mathbb{S}^1 \to \mathbb{R} \times \mathbb{S}^1, \quad (x, \theta) \mapsto (1 - (a + \varepsilon \sin(2\pi\theta))x^2, \theta + \alpha). \tag{36}$$

This is a particular case of family (2) for $b = \delta = 0$ (though it is not a diffeomorphism). Denoting by γ the golden mean, we fix $\alpha = \gamma/1000$: this small value ensures that the attractor of (36) varies 'adiabatically' through the the sequence of attractors corresponding to the 'frozen' values of θ. We also fix $(a, \varepsilon) = (1, 30, 0.30)$, thereby ensuring that the frozen values of $(a + \varepsilon \sin(2\pi\theta))$ range over the interval $[1, 1.6]$: here the attractors of the Logistic family range from a period 2 sink to a 'one-piece' chaotic attractor (meaning that the support of the invariant measure has a single connected component). The Lyapunov sum $LS_n = \sum_{j=1}^{n} \log(2a|x_j|)$ decreases during the first 600 iterates along an orbit (Figure 11), reaching a minimum of about -665 (small oscillations of period 2 occur, but are neglected). Then LS_n increases for the next 800 iterates, reaching a maximum of about -460. Therefore, local errors increase by about $\exp(-460 + 665) \approx 10^{89}$ during the first 1400 iterates.

Clearly, roundoff errors produced by standard double precision accuracy are quickly amplified, inducing a departure of the computed orbit from the true orbit. The consequences are illustrated in Figure 12. If sufficient arithmetic accuracy is used (e.g. 150 digits), the computed orbit is indeed an invariant circle. With insufficient accuracy (standard double precision or 'only' 60 decimal digits), the computed orbit no longer is a circle and bears resemblance to a 'strange nonchaotic attractor'. Also the Lyapunov sums change with the number of digits. With sufficient accuracy, the first minimum of LS_n is about -646, attained at $n = 562$. A maximum of -376 is then attained for $n = 1459$. The error amplification factor is therefore $\exp(-376 + 646) \approx 10^{117}$ showing the minimum number of digits required.

For 'frozen' θ the effective value of the parameter in the Logistic family is $\hat{a} = a + \varepsilon \sin(2\pi\theta)$, see (36). Period two points of the frozen system are created through a period doubling bifurcation at $\hat{a} = 3/4$: on this orbit one has $|D^2T(x)| = 4|(\hat{a} - 1)|$. Hence, the Lyapunov sums (35) are Riemann sums of the integral

$$\frac{1}{2\alpha} \int_{\theta_0}^{\theta_0 + n\alpha} \log\left(|4(a - 1 + \varepsilon \sin(2\pi\theta))|\right) d\theta. \tag{37}$$

The value of this integral closely matches the curve of the Lyapunov sums obtained with 150 digits in Figure 12.

Invariant circles with large oscillations. Normally hyperbolic invariant circles might have geometric shapes which *look* fractal at a given scale of visualisation, although they look smooth under sufficient magnification (possibly requiring larger numerical accuracy than standard double precision). In particular, this can happen when the invariant circle has very large local slope, when the phase-space coordinates along the invariant circle are viewed as functions of the angular variable. Since the Lyapunov exponent in the x variable is negative, continuation of the invariant circle with respect to ε is still locally possible.

We illustrate this phenomenon in a quasi-periodically forced Logistic family:

$$(x, \theta) \mapsto (3x(1 - x) + \varepsilon \sin(2\pi\theta)), \theta + \gamma (\text{mod } 1)), \tag{38}$$

in the form studied in [30], where γ is the golden mean. The authors of [30] claim that there exists a parameter range starting at $\varepsilon \approx 0.1553$, characterised by strange

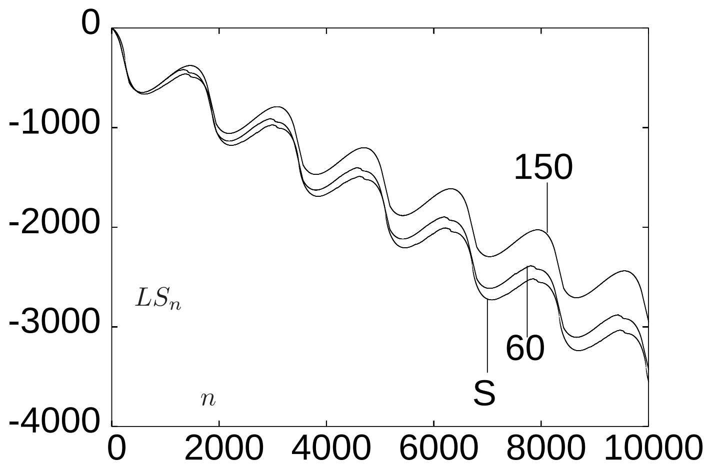

FIGURE 11. Lyapunov sums (35) of Λ_2 as a function of the iterate n for an orbit of (36) starting from $(x_0, \theta_0) = (0.123456789, 0.6)$, for arithmetic in standard double precision (S), 60 and 150 decimal digits. Increasing accuracy results in a larger (though still negative) estimate for Λ_2.

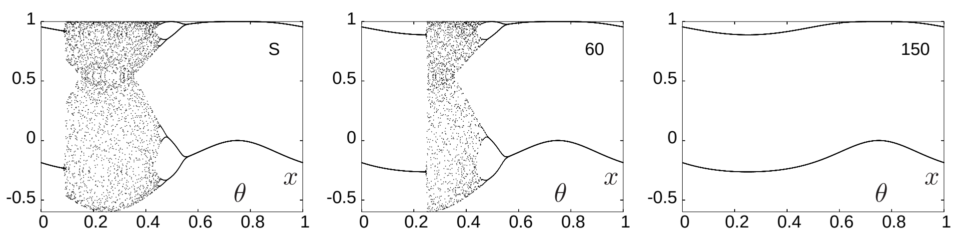

FIGURE 12. Attractors of (36) with standard double precision (S), 60 and 150 decimal digits, for $(a, \varepsilon, \alpha) = (1.30, 0.30, \gamma/1000)$, where γ denotes the golden mean.

nonchaotic attractors. The criterion used to identify the strange nonchaotic attractors is that the Lyapunov exponent in the x direction is negative, but the attractors display a fractal structure. We now show that the actual dynamics takes place on a smooth attracting invariant circle: the misinterpretation as a strange nonchaotic attractor is due to a too coarse visualisation scale.

We compute N iterates of (38), after some transient. These points are sorted with respect to θ and oscillations of the x variable are computed in the intervals $[0.0, 0.1], \ldots, [0.9, 1.0]$. Let $I^1_{j_1} := [j_1/10, (j_1 + 1)/10]$ be the interval with largest oscillation. The process is repeated for the points in $I^1_{j_1}$: $10N$ iterates are recomputed and sorted with respect to θ, and so on. This process is repeated until the maximal slope, based on the computed points, is no longer changing in a significant way. Figure 13(left) shows the results for $\varepsilon = 0.1554$: this attractor is a smooth curve. The curve *looks like a fractal* when using a coarser resolution: this is due to oscillations with very large slope. A similar conclusion is obtained for $\varepsilon = 0.1555$ (Figure 13 right), using a variety of methods: the largest slope seems to exceed 10^{18} in this case. We have used arithmetics with 30 to 40 decimal digits to overcome

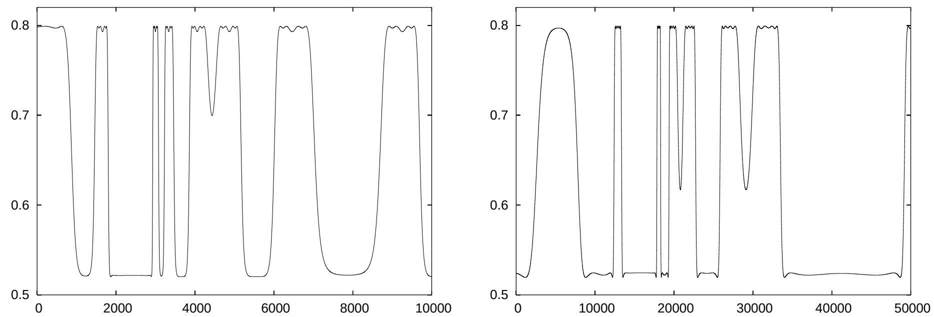

FIGURE 13. Invariant circles for map (38). Left: $\varepsilon = 0.1554$, with $(\theta - 0.0070944247) \times 10^{14}$ on the horizontal axis and x on the vertical. Right: same as left for $\varepsilon = 0.1555$ and $(\theta - 0.007235958375) \times 10^{16}$ on the horizontal axis. Maximum slopes are 1.5×10^{12} (left) and 4.0×10^{14} (right).

accumulation of errors due to the large number of iterations performed. For further examples and theoretical discussion we refer to [20, 24].

The phenomena discussed above do not imply that the 'apparent' results obtained with standard double precision or with too coarse visualisation scale are not important. Indeed, most mathematical models used for concrete applications are approximations and, furthermore, 'real life' problems always contain noise. Round-off errors play the role of noise in the above toy models. Therefore, the observed behaviour of a real system can be closer to the left panel of Figure 12 rather than its right panel.

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