QUASI-PERIODIC HÉNON-LIKE ATTRACTORS IN THE LORENZ-84 CLIMATE MODEL WITH SEASONAL FORCING

H. W. BROER AND R. VITOLO

Department of Mathematics and Computing Science,
University of Groningen, PO Box 800,
9700 AV Groningen, The Netherlands
E-mail: broer@math.rug.nl, renato@math.rug.nl

C. SIMÓ

Departament de Matemàtica Aplicada i Anàlisi,
Universitat de Barcelona, Gran Via, 585,
08071 Barcelona, Spain
E-mail: carles@maia.ub.es

A class of strange attractors is described, occurring in a low-dimensional model of general atmospheric circulation. The differential equations of the system are subject to periodic forcing, where the period is one year – as suggested by Lorenz in 1984. The dynamics of the system is described in terms of a Poincaré map, computed by numerical means. It is conjectured that certain strange attractors observed in the Poincaré map are of quasi-periodic Hénon-like type, *i.e.*, they coincide with the closure of the unstable manifold of a quasi-periodic invariant circle of saddle type. A route leading to the formation of such strange attractors is presented. It involves a finite number of quasi-periodic period doubling bifurcations, followed by the destruction of an invariant circle due to homoclinic tangency.

1. Introduction

In this note we examine a class of strange attractors occurring in the model

$$\begin{aligned} \dot{x} &= -ax - y^2 - z^2 + aF(1 + \varepsilon\cos(\omega t)), \\ \dot{y} &= -y + xy - bxz + G(1 + \varepsilon\cos(\omega t)), \\ \dot{z} &= -z + bxy + xz. \end{aligned} \tag{1}$$

This is a variation on an autonomous model proposed by Lorenz in 1984[1] for the long term atmospheric circulation at mid latidute of the northern hemisphere. The autonomous Lorenz-84 model, given by Eq. (1) with $\varepsilon = 0$, is used in climatological research, *e.g.* by coupling it with a low-dimensional model for ocean dynamics.[2] See Ref. 3 for the bifurcation diagram of the autonomous system and Ref. 4 for its derivation from the Navier-Stokes equations.

The variable x in (1) stands for the strength of a symmetric, globally averaged westerly wind current. The variables y and z are the strengths of cosine and sine phases of a chain of superposed waves transporting heat poleward. The terms in F and G are thermal forcings: F is the *symmetric* cross-latitude heating contrast and G accounts for the *asymmetric* heating contrast between oceans and continents. The periodic forcing of frequency $\omega = 2\pi/T$, where the period T is fixed at 73, simulates a seasonal variation of F and G. Indeed, $T = 73$ corresponds to one year in the time-scale unit of Eq. (1), estimated to be five days. As in Refs. 1–4, the coefficients a and b are fixed at $a = 1/4$ and $b = 4$.

In this note we only consider one of the dynamical phenomena observed by numerical simulations in system (1), namely the occurrence of attractors which we conjecture to be of *quasi-periodic Hénon-like* type. Moreover, only G is used here as control parameter, while ε and F are kept fixed at 0.5 and 11 respectively. See Refs. 5–6 for a more detailed study of the bifurcation diagram of Eq. (1) in the three-dimensional parameter space $\{F, G, \varepsilon\}$.

The dynamics of the forced system (1) is described in terms of the one-parameter family of diffeomorphisms given by the Poincaré map $P_G : \mathbb{R}^3 \to \mathbb{R}^3$, also called stroboscopic, first return or period map. The map P_G is computed by numerical integration of Eq. (1) over a period T, see Refs. 5–6 for the methods used.

2. The dynamics on quasi-periodic Hénon-like attractors

Let $H : \mathbb{R}^m \to \mathbb{R}^m$ be a map and $\mathscr{A} \subset \mathbb{R}^m$. Then $\mathscr{A}$ is called an *attractor* if $\mathscr{A}$ is compact and H-invariant, if the stable set (basin of attraction) $W^s(\mathscr{A})$ has nonempty interior and if there exists a point $p \in \mathscr{A}$ such that the orbit $\mathrm{Orb}(p) = \{H^j(p)\}_{j\geq 0}$ is dense in $\mathscr{A}$. The attractor $\mathscr{A}$ is called *Hénon-like*[7,8,9] if there exist a saddle periodic orbit $\mathrm{Orb}(s) = \{s, H(s), \ldots, H^n(s)\}$, a point p in the unstable manifold $W^u(\mathrm{Orb}(s))$, and a tangent vector $v \in T_p\mathbb{R}^m$ such that the orbit of p is dense in $\mathscr{A}$ and

$$\mathscr{A} = \overline{W^u(\mathrm{Orb}(s))}, \tag{2}$$

$$\|DH^n(p)v\| \geq \kappa\lambda^n \qquad \text{for } n \geq 0, \tag{3}$$

where overline denotes topological closure. Condition (3) means that the dense orbit $\mathrm{Orb}(p)$ has a positive Lyapunov exponent. We say that the attractor $\mathscr{A}$ is *quasi-periodic Hénon-like* if there exist a quasi-periodic invariant circle $\mathscr{C}$ of saddle type, a point $p \in W^u(\mathscr{C})$, and a vector $v \in T_p\mathbb{R}^m$ such that condition (3) holds while

$$\mathscr{A} = \overline{W^u(\mathscr{C})}.$$

The conjectural occurrence of this type of attractors in the family P_G is now illustrated by numerical results. An attractor $\mathscr{D}$ of the map P_G is plotted in Fig. 1 (A), where $\mathscr{D} = \mathscr{D}_1 \cup \mathscr{D}_2$ is the union of two disjoint circles $\mathscr{D}_1$ and $\mathscr{D}_2$ such that $P_G(\mathscr{D}_1) = \mathscr{D}_2$ and $P_G^2(\mathscr{D}_j) = \mathscr{D}_j$ for $j = 1, 2$. Upon a slight variation of the parameters, this period two circle becomes resonant (*i.e.*, phase-locked to a periodic attractor) and it gets eventually destroyed by homoclinic tangency[10,11] of a periodic saddle point. For nearby parameter values the attractor $\mathscr{A}$ in Fig. 1 (B) is detected. The attractor $\mathscr{A}$

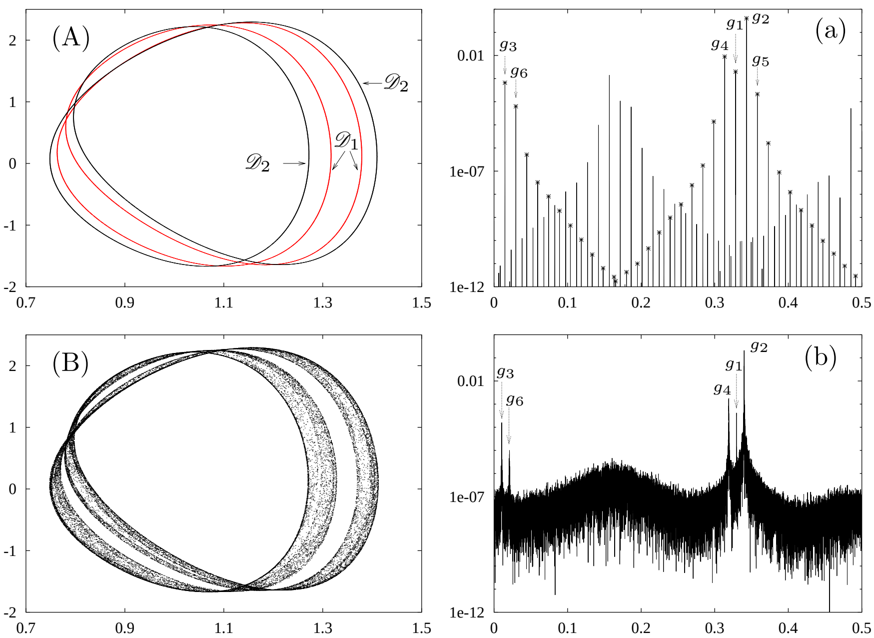

Figure 1. (A) Projection on (x, z) of the attracting period two circle $\mathscr{D} = \mathscr{D}_1 \cup \mathscr{D}_2$ occurring at $G = 0.4969$. (B) Same as (A) for the strange attractor $\mathscr{A}$, at $G = 0.4972$. (a) Power spectrum of the attractor in (A). The square modulus of the Fourier coefficients (vertical axis) is plotted against the Fourier frequencies $f_k = k/N$ for $k = 1, \ldots, N/2$ (horizontal axis). Here $N = 2^{16}$ is the sample length of the time series given by the y-coordinate along an orbit on the attractor. The first six harmonics $g_k = kg_1$ of the internal frequency g_1 are labelled, and up to order $k = 35$ the harmonics are marked by asterisks. (b) Power spectrum of the attractor in (B).

Table 1. Numerical values of the Lyapunov dimension $\mathrm{D_L}$ and Lyapunov exponents $\lambda_1 \geq \lambda_2 \geq \lambda_3$ of the attractors in Fig. 1.

Fig. 1	attractor	$\mathrm{D_L}$	λ_1	λ_2	λ_3
(A)	$\mathscr{D}$	1	0	-0.18	-14.5
(B)	$\mathscr{A}$	2.016	0.24	0	-14.9

is contained inside a Möbius strip which is slightly fattened in the normal direction. Indeed, the Lyapunov dimension $\mathrm{D_L}$ of $\mathscr{A}$ is quite close to 2 (Table 1). This is due to the large absolute value of the negative Lyapunov exponent λ_3, corresponding to strong normal contraction, and to the fact that $\lambda_2 \simeq 0$. Since λ_1 is positive, the dynamics on $\mathscr{A}$ is sensitive to initial conditions. However, the property $\lambda_2 \simeq 0$ suggests that the dynamics on $\mathscr{A}$ still contains a quasi-periodic direction.

This idea is also supported by examination of power spectra, displayed for $\mathscr{D}$ and $\mathscr{A}$ in Fig. 1 (a) and (b) respectively. The period two circle $\mathscr{D}$ has two internal frequencies, $g_1 = 0.328$ and $h_1 = \frac{1}{2}$. The second harmonic $g_2 = 2g_1$ is the internal frequency of P_G^2 on $\mathscr{D}_1$ and $\mathscr{D}_2$. Only a few harmonics of g_1 persist in the spectrum of $\mathscr{A}$ (Fig. 1 (b)), all others having turned into broad band. Power spectra like in Fig. 1 (b) are of mixed type:[12] they contain marked peaks (atoms of the spectral density) but also have a broad band component (locally continuous density).

The process leading to the formation of attractors like $\mathscr{A}$ (Fig. 1 (B)) passes through a finite number of quasi-periodic period doubling bifurcations.[13] A whole quasi-periodic period doubling cascade does not take place, since the attracting periodic circles are eventually destroyed due to homoclinic tangencies of a saddle periodic point.[10,11] An attracting invariant circle $\mathscr{C}$ of P_G occurs at $G = 0.4872$ (Fig. 2 (A)). As G increases, $\mathscr{C}$ loses stability through a quasi-periodic period doubling, and a circle attractor $\mathscr{C}^2$ is created (Fig. 2 (B)). The circle $\mathscr{C}$ still exists, is of saddle type and its two-dimensional unstable manifold is a Möbius strip with $\mathscr{C}^2$ as its boundary. Through another quasi-periodic doubling, the attracting period two circle $\mathscr{D}$ is born (Fig. 1 (A)), and $\mathscr{C}^2$ also becomes of saddle type. We stress that the latter bifurcation is different from the previous "length-doubling", since two disjoint circles $\mathscr{D}_1$ and $\mathscr{D}_2$ are now created.

A strange attractor $\mathscr{B}$, consisting of a single fattened Möbius strip, is plotted in Fig. 3 (A). This attractor is formed as the two "belts" of $\mathscr{A}$ (Fig. 1 (B)) melt together. Sections of $\mathscr{B}$ and $\mathscr{A}$ have a planar Hénon-like structure,[7,8,9] compare the slice Σ in Fig. 3 (B) and condition (2). To

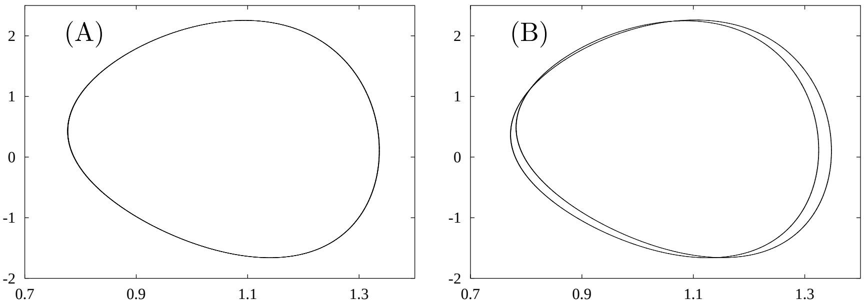

Figure 2. (A) Projection on (x, z) of the circle attractor $\mathscr{C}$ of P_G, occurring at $G = 0.4872$. (B) Same as (A), for the circle $\mathscr{C}^2$ at $G = 0.4874$.

illustrate the dynamics inside $\mathscr{B}$, we computed the image under P_G of all points in the slice Σ. The image $P_G(\Sigma)$ is stretched and folded (Fig. 3 (A)), and again has a planar Hénon-like stucture.

The main point of this note is the conjecture that the strange attractors $\mathscr{A}$ Fig. 1 (B) and $\mathscr{B}$ Fig. 3 (A) are indeed quasi-periodic Hénon-like. To be more precise, there exists a positive measure subset of the parameter space for which $\mathscr{A}$ (resp. $\mathscr{B}$) occurs. For such parameter values:

(1) the circle $\mathscr{C}^2$ coexists with $\mathscr{A}$ (resp. $\mathscr{C}$ coexists with $\mathscr{B}$).
(2) $\mathscr{C}^2$ is quasi-periodic and of saddle type (resp. $\mathscr{C}$ is);
(3) $\mathscr{A} = \overline{W^u(\mathscr{C}^2)}$ (resp. $\mathscr{B} = \overline{W^u(\mathscr{C})}$).

3. Concluding remarks

Quasi-periodic Hénon-like attractors are also numerically observed in a model map for the Hopf-saddle-node bifurcation of fixed point of diffeomorphisms.[14] This bifurcation is one of the organizing centers of the Poincaré map $P_{F,G,\varepsilon}$ for ε not too large.[5,6]

However, for the above models a rigorous proof for the existence of quasi-periodic Hénon-like attractors is out of reach, though a computer-assisted proof may be possible. So in Ref. 15 we turn to the setting of product maps on $\mathbb{R}^2 \times \mathbb{S}^1$, which is easier to deal with. In particular, a new result on Hénon-like attractors is obtained for maps given by the skew-product of a planar Hénon map[7,9] with the Arnol′d map on $\mathbb{S}^1$. We also consider perturbations of the product of certain dissipative planar maps with a rigid rotation on $\mathbb{S}^1$. In this case, it is proved that there exists a quasi-periodic saddle-like invariant circle $\mathscr{C}$ such that the closure $\overline{W^u(\mathscr{C})}$ attracts an open set of points. However, the characterization of quasi-periodic Hénon-like

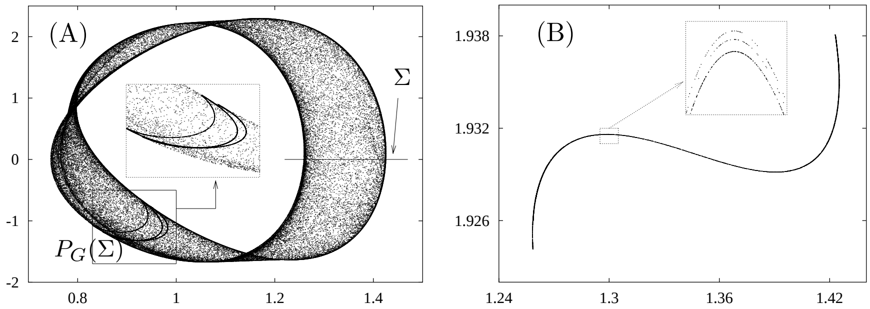

Figure 3. (A) Projection on (x, z) of the attractor $\mathscr{B}$ of the map P_G, occurring at $G = 0.5$. A slice contained in the layer centered at $z = 0$ with thickness 0.0001 is labelled by Σ. The image of Σ under P_G is labelled by $P_G(\Sigma)$ and magnified in the central box. (B) Projection on $(x, y - 0.133 * z)$ of the slice Σ.

attractors largely remains open even in this simplified setting.

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