The Hopf-saddle-node bifurcation for fixed points of 3D-diffeomorphisms

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The goal of the present investigation is to understand the typical bifurcation patterns organized around a Hopf-saddle-node (HSN) bifurcation of fixed points, defined as follows: a C^∞-family of diffeomorphisms $F_\alpha : \mathbb{R}^3 \to \mathbb{R}^3$, where $\alpha \in \mathbb{R}^p$ is a multi-parameter, is an *HSN-family* if

$$F_0(0) = 0, \quad \text{and} \quad \operatorname{spec} DF_0(0) = \{e^{i\omega_0}, e^{-i\omega_0}, 1\}, \tag{1}$$

where the complex eigenvalues satisfy the non-resonance conditions

$$e^{in\omega_0} \neq 1 \qquad \text{for} \quad n = 1, 2, 3, 4. \tag{2}$$

We observe that this bifurcation is one of the organizing centers of the bifurcation diagram of a model of the atmospheric jet.[2] We carry out a case study of the family G of 3D maps, given by

$$G\begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} e^{i(\omega_0+\gamma\delta)} w[1 - \gamma(\gamma\mu + az + \gamma z^2)] \\ z + \gamma(1 - |w|^2 - z^2) \end{pmatrix} + \begin{pmatrix} \gamma^3(\varepsilon_1 \overline{w}^4 + \varepsilon_2 z^4) \\ 0 \end{pmatrix}. \tag{3}$$

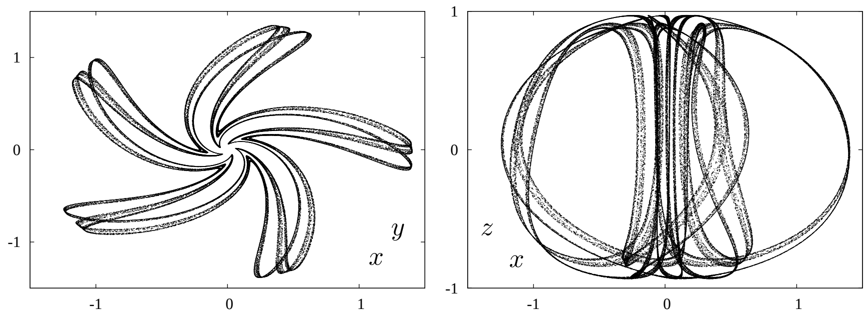

Fig. 1. Strange attractor of map G (3), projections on (x, y) (left) and (x, z) (right).

The family G depends on the parameters (γ, μ, δ) and is given in the coordinates (w, z), where $w = x + iy \in \mathbb{C}$ and $z \in \mathbb{R}$. The coefficients $a = a_1 + ia_2 \in \mathbb{C}$, $\varepsilon_j \in \mathbb{R}$, $j = 1, 2$ are constants belonging to a fixed compact set. Specifically, we investigate the effect on the dynamics of resonances in the complex eigenvalues: therefore ω_0 is fixed at $2\pi/5$ and map G is constructed to be 'as generic as possible' in the class of diffeomorphisms unfolding a HSN bifurcation in the neighbournood of a 1:5 resonance (this is the strongest among the weak resonances).

The family G has two secondary Hopf-saddle-node bifurcations of period five points which, near the origin of parameter space, organize a cone-like structure formed by two surfaces of saddle-node and a surface of Hopf bifurcations. Among the detected phenomena there is an intricate bifurcation structure near a 1:5 resonance gap which occurs along a frayed boundary of quasi-periodic Hopf bifurcations[1] of an invariant circle: a sort of cascade of quasi-periodic bifurcations of invariant circles and two-tori takes place nearby. Another scenario involves the creation of $\mathbb{Z}_5$-symmetric strange attractors (as in Fig. 1) through a sequence of quasi-periodic period doublings of an invariant circle. A full account is in preparation.[3]

References

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