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Long wave low frequency motion in a sheared pre-stressed layer composed of Neo-Hookean material

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Abstract

We consider two-dimensional long wave low frequency motion in a pre-stressed layer composed of Neo-Hookean material. Specifically, the pre-stress is a simple shear deformation. Derivation of the dispersion relation associated with traction-free boundary conditions is briefly reviewed. Appropriate approximations are established for the two associated long wave modes. From these approximations it is clear that there may be either two, one or no real long wave limiting phase speeds. These approximations are also used to establish the relative asymptotic orders of the displacement components and pressure increment. Using these relative orders to motivate the introduction of appropriate a scales, an asymptotically consistent model long wave low frequency motion is established. It is shown that in the presence of shear there is neither bending nor extension, or analogues of their previously established pre-stressed counterparts. In fact, both the in-plane and normal displacement components have the same asymptotic orders and the derived governing equation is of vector form.

1 Introduction

The problem of long wave low frequency motion in a pre-stressed elastic layer composed of incompressible neo-Hookean material and subject to a simple shear primary deformation is considered. Shear-type deformations may occur in a number of geo-physical systems, for example in the Earth's crust, see Gessner et al. (2007). In addition to that movements over long time scales may cause shear-type deformations, see Ide et al. (2007). Wave propagation in homogeneously pre-stressed plates has been discussed in detail by a number of authors, see for example Ogden and Roxburg (1991) and Rogerson (1997) in respect of incompressible elastic plates, Nolde et al. (2004) for compressible elastic plates, Rogerson (1998) and Y.Fu (2005) for anisotropic plates. Connor and Ogden (1996) seemingly first derived the dispersion relation for a layer with an initial simple shear deformation, showing that additional complexity arises because no principal axis is normal to layer.

In the present study we derive long wave low frequency approximation of the dispersion relation and use these to establish the asymptotic orders of displacements and pressure, enabling the derivation of an asymptotically consistent model for long wave low frequency motion. This is done through the use of asymptotic integration techniques established by Kaplunov et al. (1998), in the linear isotropic elastic context and by Kaplunov et al. (2002) in the pre-stressed case. A novel feature of the model derived is the matrix representation of governing equation. This arises because a consequence of the simple shear deformation is that both the in-plane and normal displacement components are the same asymptotic order. No analogues of classical bending or extension, such as were derived in Kaplunov et al. (1998), are therefore possible for this type of primary deformation.

2 Governing equations

In this section we present the basic equations associated with small amplitude time dependent motions superimposed upon a primary simple shear deformation in the incompressible layer of Neo-Hookean material. The position vector of a representative particle is denoted by X_A in the initial unstressed isotropic configuration B_u, by $x_i(X_A)$ in a pre-stressed equilibrium state B_e and by $\tilde{x}_i(X_A, t)$ in the final time-dependent configuration B_t, these being related through

$$\tilde{x}_i(X_A, t) = x_i(X_A) + u_i(x_j, t), \tag{2.1}$$

with $u_i(x_j, t)$ the small time-dependent displacement. A strain-energy is assumed in the form

$$W = \frac{\mu}{2}\left({\lambda_1}^2 + {\lambda_2}^2 + {\lambda_3}^2 - 3\right) - p(\lambda_1\lambda_2\lambda_3 - 1), \tag{2.2}$$

where λ_i are principal stretches, p a hydro-static pressure and μ a material constant. We consider two-dimensional motion, with $u_3 = 0$, u_1 and u_2 independent of x_3, and $\lambda_3 = 1$. The simple shear primary deformation we are concerned with may be defined by

$$x_1 = X_1 + \epsilon X_2, \quad x_2 = X_2, \quad x_3 = X_3. \tag{2.3}$$

The equations of motion may be represented in the component form

$$B_{jilk}u_{k,jl} - p_{t,i} = \rho\ddot{u}_i, \tag{2.4}$$

with $\mathbf{B}$ the elasticity tensor, p_t the pressure increment and ρ the material density. A complicating feature of this primary deformation is that no principal axis is normal to the faces of the plate. The natural coordinates of the layer (x_1, x_2), parallel and normal to layer faces, and Eulerian (x_1', x_2') coordinate, the principal axes of deformation, are connected via the following relations

$$\mathbf{x} = \mathbf{R}\mathbf{x}', \;\; \mathbf{x}' = \mathbf{R}^T\mathbf{x}, \;\; \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \;\; \mathbf{x}' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}, \;\; \mathbf{R} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{2.5}$$

With origin on its upper face, the layer occupies the domain $-\infty \le x_1 \le \infty, -h \le x_2 \le 0$. In the incompressible two dimensional case, simple shear is one parameter deformation, with $\lambda_1 = \lambda$ and $\lambda_2 = \lambda^{-1}$. It may further be shown that

$$\cot\theta = \lambda, \quad \sin\theta = \frac{1}{\sqrt{\lambda^2 + 1}}, \quad \cos\theta = \frac{\lambda}{\sqrt{\lambda^2 + 1}}. \tag{2.6}$$

The equations of motion, relative to the natural coordinate system of the layer, are given by

$$\begin{aligned}
&-\lambda^2 p_{t,1} - \lambda p_{t,2} - \rho\lambda^2 \ddot{u}_1 - \rho\lambda\ddot{u}_2 + \mu\left(\lambda^4 + \lambda^2(p-1)+1\right)(u_{1,11}+u_{2,11}) \\
&\qquad + \mu\lambda\left(\lambda^2(p-2)+2\right)(u_{1,12}+u_{2,12}) + \mu\lambda^2 u_{1,22} + \mu\lambda\,(1+p)\,u_{2,22} = 0, \\
&\lambda^2 p_{t,1} - \lambda^3 p_{t,2} + \rho\lambda^2\ddot{u}_1 - \rho\lambda^3\ddot{u}_2 - \mu\left(\lambda^4 + \lambda^2(p-1)+1\right)(u_{1,11}+u_{2,11}) \\
&\qquad + \mu\lambda\left(\lambda^2(p-2)+2\right)(u_{1,12}+u_{2,12}) - \mu\lambda^2 u_{1,22} + \mu\lambda^3\,(1+p)\,u_{2,22} = 0,
\end{aligned} \tag{2.7}$$

where the connection $\mu\lambda^2 p = \mu - \sigma_2\lambda^2$, with σ_2 the normal principal Cauchy stress, is noted. The two dimensional incompressibility condition takes the following form

$$u_{1,1} + u_{2,2} = 0. \tag{2.8}$$

Solutions of the three governing equations are then sought in the form

$$(u_1, u_2, p_t) = (U_1, U_2, kP)e^{ikqx_2}e^{ik(vt-x_1)}, \tag{2.9}$$

withe k the wave number, v the phase speed and q a parameter determined from the governing equations. From (2.7), (2.8) and (2.9) we obtain an equation for q, previously derived by Connor and Ogden (1996), given by

$$q^4 - 2\,\epsilon\,q^3 + \left(2+\epsilon^2-\hat{v}\right)q^2 - 2\,\epsilon\,q + 1 + \epsilon^2 - \hat{v} = 0, \quad \hat{v} = \frac{\rho v^2}{\mu}. \tag{2.10}$$

Equation (2.10) may be factorized, yielding the four roots

$$q_1 = i, \quad q_2 = -i, \quad q_3 = \epsilon + i\kappa, \quad q_4 = \epsilon - i\kappa, \quad \kappa^2 = 1 - \hat{v}. \tag{2.11}$$

Also from (2.7) and (2.8), we are able to establish that $U_1 = qU_2$ and $P = \mathcal{P}(q)U_2$, where

$$P(q) = -\frac{\mu}{kv\lambda^2}(\lambda^2 q^3 - 2\lambda(\lambda^2-1)^2 q^2 + (\lambda^4 - \lambda - \hat{v} + 1)q). \tag{2.12}$$

The incremental traction components, associated with the layer faces, are given by

$$\tau_i = B_{milk}u_{k,l}n_m + p_0 u_{m,i}n_m - p_t n_i. \tag{2.13}$$

It is possible to derive from (2.13) forms for the two traction components presented in the natural coordinate system, namely

$$\begin{aligned}
\tau_1 &= \frac{(\lambda^2+1)}{\lambda}(\mu(\lambda^2(\lambda^2-1)u_{1,1} + \lambda^2 u_{1,2} + (\lambda^2(p+1)-1)u_{2,1} + \lambda(p+1)u_{2,2}) - \lambda p_t), \\
\tau_2 &= \frac{(\lambda^2+1)}{\lambda}(\mu((\lambda^2-1)u_{1,1} - \lambda u_{1,2} + (\lambda^2-(p+1))u_{2,1} + \lambda^2(p+1)u_{2,2}) - \lambda p_t).
\end{aligned} \tag{2.14}$$

From (2.9), solutions for two displacement components, incremental pressure and traction components may now be expressed as the following linear combinations

$$u_1 = \sum_{i=1}^{4} q_i\mathcal{E}^{(i)}, \;\; u_2 = \sum_{i=1}^{4}\mathcal{E}^{(i)}, \;\; p_t = k\sum_{i=1}^{4}P(q_i)\mathcal{E}^{(i)}, \;\; \tau_1 = \mathcal{C}\sum_{i=1}^{4}f(q_i)\mathcal{E}^{(i)}, \;\; \tau_2 = \mathcal{C}\sum_{i=1}^{4}g(q_i)\mathcal{E}^{(i)}, \tag{2.15}$$

with $\mathcal{E}_i = A_i e^{ikq_ix_2}$, $A_1, A_2, A_3, A_4, \mathcal{C}$ constants, $P(q)$ given by (2.12) and where use of (2.9), (2.12) and (2.14) enables $f(q)$ and $g(q)$ to be represented as

$$\begin{aligned}
f(q) &= q^3 + \left(-\lambda + 2\,\lambda^{-1}\right)q^2 + \left(1 + \lambda^{-2} + p - \hat{v}\right)q + \lambda^{-1} - \lambda - p\lambda, \\
g(q) &= q^3 + \left(-2\,\lambda + \lambda^{-1}\right)q^2 + \left(1 + \lambda^2 + p - \hat{v}\right)q - \lambda + \lambda^{-1} + \frac{p}{\lambda}.
\end{aligned} \tag{2.16}$$

3 The dispersion relation

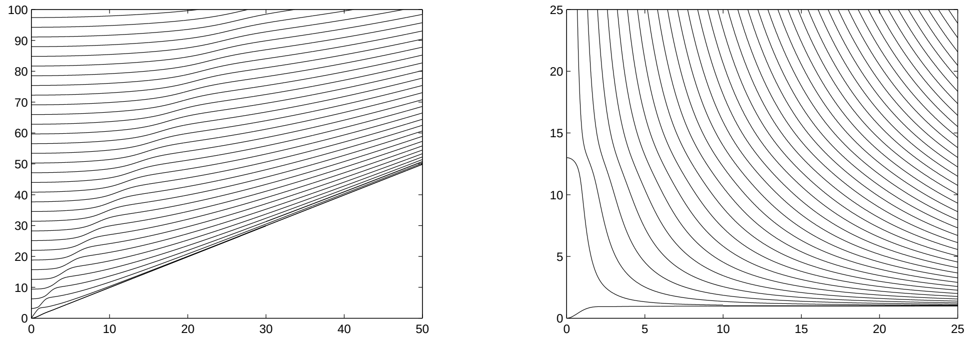
In order to derive a dispersion relation we specify the traction free boundary conditions

$$\tau_1 = \tau_2 = 0 \quad \text{on} \quad x_2 = 0, -h. \tag{3.1}$$

We now introduce the small non-dimensional parameter η, the ratio of a layer thickness h to typical wave length l, so $\eta \equiv h/l \equiv kh$. Using (2.16, (2.11) and (2.10), the dispersion relation associated with the traction free boundary conditions (3.1) takes the form

$$\begin{aligned}&\Big(q_0\left(p^2-\kappa^2\right)^2+q_0\kappa^2\left(q_0+2\,p\right)^2+4\,\kappa^2\left(p^2-\kappa^2\right)\left(q_0+2\,p\right)\Big)\sinh\left(\eta\right)\sinh\left(\eta\,\kappa\right)\\&+2\,\kappa\,\left(pq_0+p^2+\kappa^2\right)^2\left(\cos\left(\eta\,\epsilon\right)-\cosh\left(\eta\right)\cosh\left(\eta\,\kappa\right)\right)=0,\\&q_0=1+\epsilon^2+\kappa^2.\end{aligned} \tag{3.2}$$

Shown in Figure (1) is the numerical solution of dispersion relation (3.2). This problem was first investigated by Connor and Ogden (1996), who presented a number of numerical solutions.



(a) (b)

Figure 1: *Numerical solution of dispersion relation for a Neo-Hookean material , showing dimensionless frequency ω (vertical axis) against scaled wave number η (horizontal axis) cases (a), and dimensional squared wave speed $\hat{v}$ (vertical axis) against scaled wave number η (horizontal axis) cases (b) for parameters ϵ=3 p=1*

4 Long-wave low-frequency approximations of the dispersion relation

To derive long-wave low frequency approximations for the dimensionless squared wave speed $\hat{v}$, we expand all the functions involving the small parameter η in (3.2) as series up to and including $O(\eta^4)$, resulting in a dispersion relation in the following form

$$\mathcal{D} = \mathcal{D}^{(0)} + \eta^2\mathcal{D}^{(2)} + \eta^4\mathcal{D}^{(4)} + O(\eta^6). \tag{4.1}$$

The leading order approximation, $\mathcal{D}^{(0)} = 0$, provides the following two limits

$$\hat{v}_1^{(0)} = 1 - p^2, \quad \hat{v}_2^{(0)} = \epsilon^2 + 2p + 2. \tag{4.2}$$

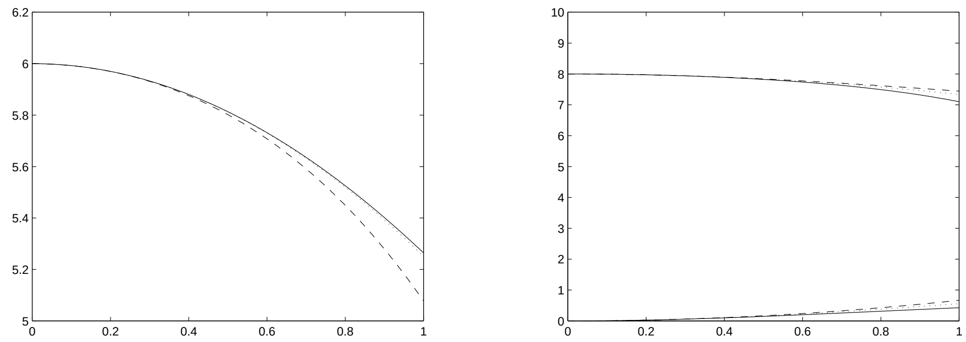
It is now possible to establish that the second order term $\mathcal{D}^{(2)}$ provides

$$\hat{v}_1^{(2)} = \frac{1}{12}(\epsilon^2 + (p+1)^2)p^2, \quad \hat{v}_2^{(2)} = -\frac{1}{12}(\epsilon^2 + (p+1)^2). \tag{4.3}$$

Approximations for the two fundamental modes may be expressed as

$$\hat{v}_m = \hat{v}_m^{(0)} + \eta^2 \hat{v}_m^{(2)} + \eta^4 \hat{v}_m^{(4)} + O(\eta^6), \tag{4.4}$$

within which $\hat{v}_m^{(n)}$, $m = 1, 2, n = 0, 2$ are given in equations (4.2) and (4.3), respectively and for brevity the result for $n = 4$ is omitted. Comparison of approximate and numerical solutions is shown to provide excellent agreement over a surprisingly large wave number range, see Figure 2.



(a) (b)

Figure 2: *Comparison of numerical and asymptotic results for the fundamental modes, showing $\hat{v}$ (vertical axis) against η (horizontal axis) for Neo-Hookean material: (a) ϵ=0 p=2 (one fundamental mode), (b) ϵ=2 p=1 (two fundamental modes). Dotted curves show third order approximations, dashed curves second order and unbroken curves numerical results.*

5 Relative asymptotic orders of displacements and pressure

We now wish to establish the relative orders of the displacement components and incremental pressure within the long wave low frequency regime. We first introduce the notation $\zeta = x_2/h$, and then use (2.15), together with the boundary condition (3.1), to establish connections between the constants A_1, A_2, A_3 and A_4. Then it is possible to establish the relative asymptotic orders of A_1, A_2, A_3 and A_4 in terms of one parameter, $\tilde{U}$ and obtain the relative orders of u_1, u_2 and p_t, yielding

Case 1: $\hat{v}^{(0)} = \hat{v}_1^{(0)} = 1 - p^2$,

$u_1 = \epsilon \mathcal{M}_1^1(\epsilon, p)\tilde{U}\eta + O(\eta^2), \quad u_2 = \mathcal{M}_2^1(\epsilon, p)\tilde{U}\eta + O(\eta^2), \quad \mathcal{M}_1^1(0, p) \neq 0, \quad \mathcal{M}_2^1(0, p) \neq 0,$

Case 2: $\hat{v}^{(0)} = \hat{v}_2^{(0)} = \epsilon^2 + 2p + 2$,

$$u_1 = \mathcal{M}_1^2(\epsilon, p)\tilde{U}\eta + O(\eta^2), \quad u_2 = \epsilon \mathcal{M}_2^2(\epsilon, p)\tilde{U}\eta + O(\eta^2), \quad \mathcal{M}_1^2(0, p) \neq 0, \quad \mathcal{M}_2^2(0, p) \neq 0. \tag{5.1}$$

We remark that the exponential function $e^{ik(x_1-vt)}$ has been incorporated into the definition of $\tilde{U}$. We therefore conclude that u_1 and u_2 have the same asymptotic orders. The order of hydrostatic pressure may similarly be established and shown to be of the same form for both two cases

$$p_t = k\mathcal{Q}(\epsilon, p)\tilde{U}\eta + O(\eta^2), \quad \mathcal{Q}(0,p) \neq 0. \tag{5.2}$$

For both associated fundamental modes, the coefficients $\mathcal{M}_1^1, \mathcal{M}_2^1, \mathcal{M}_1^2, \mathcal{M}_2^2, \mathcal{Q}$ are algebraically complex and is therefore not written explicitly here. Therefore we conclude that both displacements components u_1, u_2 and hydrostatic pressure p_t have the same asymptotic orders. Also we note that if $\epsilon = 0$, the relative asymptotic orders of displacements and pressure components agree with previously obtained classical results by Kaplunov et al. (2002).

6 An asymptotically consistent model for long wave low frequency motion

Motivated by the asymptotic orders established in (5.1) and (5.2), we introduce non-dimensional displacement components and incremental pressure defined through

$$u_1 = lu_1^*, \quad u_2 = lu_2^*, \quad p_t = \mu p_t^*, \tag{6.1}$$

where $*$ indicates quantities of the same asymptotic order. Appropriate scales for spatial and time variables are similarly introduced, taking the forms

$$x_1 = l\xi, \quad x_2 = l\eta\zeta, \quad t = l\sqrt{\frac{\rho}{\mu}}\tau, \tag{6.2}$$

with l a typical wavelength. We now substitute (6.1) and (6.2) into (2.7), re-casting the governing equations in terms of non-dimensional variables, leading to two equations of the motion in the form

$$c_2^1\eta^2 + c_1^1\eta + c_0^1 = 0, \quad c_2^2\eta^2 + c_1^2\eta + c_0^2 = 0, \tag{6.3}$$

within which

$$\begin{aligned}
c_0^1 &= -\lambda^3 u_{1,\zeta\zeta}^* - (p+1)\,\lambda^2 u_{2,\zeta\zeta}^*, \quad c_0^2 = \lambda^2 u_{1,\zeta\zeta}^* - (p+1)\,\lambda^3 u_{2,\zeta\zeta}^*,\\
c_1^1 &= \lambda\left(2-(2+p)\,\lambda^2\right)u_{2,\xi\zeta}^* - \lambda^2\left(2\lambda^2-(2-p)\right)u_{1,\xi\zeta}^* + p_{t,\zeta}^*\lambda^2,\\
c_1^2 &= -\lambda^2\left(2\lambda^2-(2+p)\right)u_{2,\xi\zeta}^* - \lambda\left(2-(2-p)\lambda^2\right)u_{1,\xi\zeta}^* + p_{t,\zeta}^*\lambda^3,\\
c_2^1 &= -\left(\lambda^4-\lambda^2+1\right)u_{2,\xi\xi}^* + p_{t,\xi}^*\lambda^3 + \lambda^3 u_{1,\tau\tau}^* + \lambda^2 u_{2,\tau\tau}^* - \lambda\left(\lambda^2(p-1)+1+\lambda^4\right)u_{1,\xi\xi}^*,\\
c_2^2 &= -\lambda\left(\lambda^4-\lambda^2+1\right)u_{2,\xi\xi}^* - p_{t,\xi}^*\lambda^2 - \lambda^2 u_{1,\tau\tau}^* + \lambda^3 u_{2,\tau\tau}^* + \left(\lambda^2(p-1)+1+\lambda^4\right)u_{1,\xi\xi}^*,
\end{aligned} \tag{6.4}$$

with the scaled form of the incompressibility condition being

$$u_{1,\xi}^*\eta + u_{2,\zeta}^* = 0. \tag{6.5}$$

The corresponding forms of the traction free boundary conditions are obtained through a combination of (2.14, (6.1) and (6.2), leading to

$$t_1 = t_1^1\eta + t_0^1 = 0, \quad t_2 = t_1^2\eta + t_0^2 = 0, \quad \zeta = 0, -1, \tag{6.6}$$

within which

$$
\begin{aligned}
t_0^1 &= -\lambda\,(1+p)\,u^*_{2,\zeta} - \lambda^2 u^*_{1,\zeta}, \quad t_0^2 = \lambda\,u^*_{1,\zeta} - (p+1)\,\lambda^2 u^*_{2,\zeta},\\
t_1^1 &= -\lambda\left(\lambda^2-1\right)u^*_{1,\xi} - \left(\lambda^2(p+1)-1\right)u^*_{2,\xi} + \lambda p^*_t,\\
t_1^2 &= \left(\lambda^2-1\right)u^*_{1,\xi} - \lambda\left(\lambda^2-(1+p)\right)u^*_{2,\xi} + \lambda^2 p^*_t.
\end{aligned}
\tag{6.7}
$$

The solutions of equations (6.3) subject to boundary conditions (6.6) are now to be sought in the series form

$$
(u_1^*, u_2^*, p_t^*) = \sum_{l=0}^{m} (u_1^{(l)}, u_2^{(l)}, p_t^{(l)})\eta^l + O(\eta^{m+1}),
\tag{6.8}
$$

leading to a hierarchical system of equations at various orders.

6.1 Leading order problem

At leading order, the equations of motion and incompressibility condition are given by

$$
-\lambda u^{(0)}_{1,\zeta\zeta} - (p+1)\,u^{(0)}_{2,\zeta\zeta} = 0, \quad u^{(0)}_{1,\zeta\zeta} - (p+1)\,\lambda u^{(0)}_{2,\zeta\zeta} = 0, \quad u^{(0)}_{2,\zeta} = 0,
\tag{6.9}
$$

with the associated leading order boundary conditions given by

$$
-(1+p)\,u^{(0)}_{2,\zeta} - \lambda u^{(0)}_{1,\zeta} = 0, \quad u^{(0)}_{1,\zeta} - (p+1)\,\lambda u^{(0)}_{2,\zeta} = 0, \ \text{ at } \ \zeta = 0, -1.
\tag{6.10}
$$

Equations (6.9), in conjunction with the boundary conditions (6.10), dictate that the solutions $u_1^{(0)}$ and $u_2^{(0)}$ take the following forms

$$
u_1^{(0)} = U^{(0,0)}(\xi,\tau), \quad u_2^{(0)} = V^{(0,0)}(\xi,\tau),
\tag{6.11}
$$

indicating that they are both independent of the normal variable ζ. At this stage, it is not possible to derive governing equations for $U^{(0,0)}(\xi,\tau)$ and $V^{(0,0)}(\xi,\tau)$. This will require considering higher order problems. We remark that throughout functions denoted by capitals, and involving the double superscripts, are independent of ζ. Moreover, within the two indices of superscript the first indicates the asymptotic order, with the second denoting the degree of zeta.

6.2 Second order problem

To establish second order problem we substitute general solution (6.8) with $m = 1$ into equations (6.3) subject to boundary conditions (6.6) and consider orders $O(\eta)$ in expansion on small η. We remark that at the stage of second order problem the solution of leading order problem (6.11) provides the orders $O(1)$ being identically zero. Therefore at the second order problem the equations of motion are the following

$$
\begin{aligned}
&\lambda^2 u^{(1)}_{1,\zeta\zeta} + \lambda\,(p+1)\,u^{(1)}_{2,\zeta\zeta} = \lambda((2-p)+2\lambda^2)u^{(0)}_{1,\xi\zeta} + (2-\lambda^2(p+2))u^{(0)}_{2,\xi\zeta} + \lambda p^{(0)}_{t,\zeta},\\
&\lambda u^{(1)}_{1,\zeta\zeta} - \lambda^2\,(1+p)\,u^{(1)}_{2,\zeta\zeta} = (2+\lambda^2(p-2))u^{(0)}_{1,\xi\zeta} + \lambda(2\lambda^2-(p+2))u^{(0)}_{2,\xi\zeta} - \lambda^2 p^{(0)}_{t,\zeta},\\
&u^{(0)}_{1,\xi} + u^{(1)}_{2,\zeta} = 0.
\end{aligned}
\tag{6.12}
$$

Finally at second order problem the traction zero boundary conditions are the following

$$\lambda^2 u_{1,\zeta}^{(1)} + \lambda\,(p+1)\,u_{2,\zeta,1} = \lambda p_t^{(0)} + \lambda\left(1-\lambda^2\right) u_{1,\xi}^{(0)} + \left(1-\lambda^2(p+1)\right) u_{2,\xi}^{(0)}, \quad \zeta = 0,-1;$$
$$\lambda\, u_{1,\zeta}^{(1)} - \lambda^2\,(p+1)\,u_{2,\zeta}^{(1)} = -\lambda^2 p_t^{(0)} + \left(1-\lambda^2\right) u_{1,\xi}^{(0)} + \lambda\left(\lambda^2-(p+1)\right) u_{2,\xi}^{(0)}, \quad \zeta = 0,-1. \tag{6.13}$$

From the incompressibility condition $(6.12)_3$ taking into account solution of the leading order problem (6.11) we obtain the following representation of second displacement component

$$u_2^{(1)} = -\zeta U_{,\xi}^{(0,0)}(\xi,\tau) + U_2^{(0,1)}(\xi,\tau). \tag{6.14}$$

If we substitute (6.14) into $(6.12)_1$ and $(6.12)_2$ we get the homogeneous system of two equation and two unknowns $u_{1,\zeta\zeta}^{(1)}$, $p_{t,\zeta}^{(0)}$. As determinant of this system is not equal to zero the only possible solutions for two variables are the trivial ones, therefore

$$u_1^{(1)} = U_1^{(1,1)}(\xi,\tau)\zeta + U_1^{(0,1)}(\xi,\tau), \quad p_t^{(0)} = P_t^{(0)}(\xi,\tau). \tag{6.15}$$

The unknown functions $U_1^{(1,1)}(\xi,\tau)$ and $P_t^{(0)}(\xi,\tau)$ can are determined from boundary conditions (6.13) taking into account leading order solution (6.11). Therefore solutions of second order problem are the following

$$u_1^{(1)} = \{\left(-\lambda+\lambda^{-1}\right) U_{,\xi}^{(0,0)} - V_{,\xi}^{(0,0)} p\}\zeta + U_1^{(0,1)}, \quad u_2^{(1)} = -U_{,\xi}^{(0,0)}\zeta + U_2^{(0,1)},$$
$$p_t^{(0)} = -\,(p+1)\,U_{,\xi}^{(0,0)} + \left(\lambda-\lambda^{-1}\right) V_{,\xi}^{(0,0)}. \tag{6.16}$$

We note that the new functions $U_2^{(0,1)}(\xi,\tau), U_1^{(0,1)}(\xi,\tau)$ and also leading order functions $U^{(0,0)}, V^{(0,0)}$ in (6.16) remain undefined at the second order problem, to obtain them we have to switch to the higher order problems.

6.3 Third order problem

To establish third order problem we substitute general solution (6.8) with $m=2$ into equations (6.3) subject to boundary conditions (6.6) and consider orders $O(\eta^2)$ in expansion on small η. We remark that at the stage of third order problem the solutions of leading (6.11) and second order problems (6.16) provide the orders $O(1), O(\eta)$ being identically zero respectively. Therefore at the third order problem the equations of motion are the following

$$\lambda^3 u_{1,\zeta\zeta}^{(2)} + \lambda^2\,(p+1)\,u_{2,\zeta\zeta}^{(2)} = \lambda(\lambda^2(p+1)-(\lambda^4+1))u_{1,\xi\xi}^{(0)} + (-\lambda^4+\lambda^2-1)u_{2,\xi\xi}^{(0)}$$
$$+\,\lambda^2((2-p)-2\lambda^2)u_{1,\xi\zeta}^{(1)} + \lambda(2-\lambda^2(p-2))u_{2,\xi\zeta}^{(1)} + \lambda^2 p_{t,\zeta}^{(1)} + \lambda^2 u_{2,\tau\tau}^{(0)} + \lambda^3 u_{1,\tau\tau}^{(0)} + \lambda^3 p_{t,\xi}^{(0)},$$
$$\lambda^3\,(1+p)\,u_{2,\zeta\zeta}^{(2)} - \lambda^2 u_{1,\zeta\zeta}^{(2)} = (\lambda^2(p-1)+\lambda^4+1)u_{1,\xi\xi}^{(0)} + \lambda(-\lambda^4+\lambda^2-1)u_{2,\xi\xi}^{(0)}$$
$$+\,\lambda(\lambda^2(2-p)-2)u_{1,\xi\zeta}^{(1)} + \lambda^2(p+2-2\lambda^2)u_{2,\xi\zeta}^{(1)} - \lambda^2 u_{1,\tau\tau}^{(0)} + \lambda^3 u_{2,\tau\tau}^{(0)} - \lambda^2 p_{t,\xi}^{(0)} + \lambda^3 p_{t,\zeta}^{(1)},$$
$$u_{1,\xi}^{(1)} + u_{2,\zeta}^{(2)} = 0. \tag{6.17}$$

Finally at third order problem the traction zero boundary conditions are the following

$$\lambda^2 u_{1,\zeta}^{(2)} + \lambda\,(p+1)\,u_{2,\zeta}^{(2)} = \lambda p_t^{(1)} + \lambda\left(1-\lambda^2\right) u_{1,\xi}^{(1)} + \left(1-\lambda^2(p+1)\right) u_{2,\xi}^{(1)}, \quad \zeta = 0,-1;$$
$$-\,\lambda\, u_{1,\zeta}^{(2)} + \lambda^2\,(p+1)\,u_{2,\zeta}^{(2)} = \lambda^2 p_t^{(1)} + \left(\lambda^2-1\right) u_{1,\xi}^{(1)} + \lambda\left(p+1-\lambda^2\right) u_{2,\xi}^{(1)}, \quad \zeta = 0,-1. \tag{6.18}$$

From third order incompressibility condition $(6.17)_3$, taking into account solutions of leading order problem (6.11) and second order problem (6.16) we obtain the following representation of second displacement component

$$u_2^{(2)} = ((\lambda - \lambda^{-1}) U_{\xi\xi}^{(0,0)} + pV_{\xi\xi}^{(0,0)})\zeta^2 - \zeta U_{1,\xi}^{(0,1)} + U_2^{(0,2)}. \tag{6.19}$$

Based on (6.11), (6.16) and (6.19) we represent in third order problem the first displacement component and incremental pressure component in the following form

$$\begin{aligned} u_1^{(2)} &= U_1^{(2,2)}\zeta^2 + U_1^{(1,2)}\zeta + U_1^{(0,2)} \\ p_t^{(1)} &= P_t^{(1,1)}\zeta + P_t^{(0,1)}. \end{aligned} \tag{6.20}$$

Substituting representations (6.19)-(6.20) into traction components (6.18) we obtain system of four equations. From the two equations corresponding to boundary conditions $\zeta = 0$ we determine the following unknown functions $P_t^{(0,1)}, U_1^{(1,2)}$

$$\begin{aligned} P_t^{(0,1)} &= -(p+1)\, U_{1,\xi}^{(0,1)} + (\lambda - \lambda^{-1})\, U_{2,\xi}^{(0,1)}, \\ U_1^{(1,2)} &= (-\lambda + \lambda^{-1})\, U_{1,\xi}^{(0,1)} - U_{2,\xi}^{(0,1)} p, \end{aligned} \tag{6.21}$$

and from the other two equations corresponding to boundary conditions $\zeta = -1$ we determine the following unknown functions $P_t^{(1,1)}, U_1^{(2,2)}$

$$\begin{aligned} P_t^{(1,1)} &= (\lambda - \lambda^{-1})\, pU_{\xi\xi}^{(0,0)} + p\,(p+1)\, V_{\xi\xi}^{(0,0)} \\ U_1^{(2,2)} &= (p + \lambda^2 - 2 + \lambda^{-2})\, U_{\xi\xi}^{(0,0)} + (\lambda - \lambda^{-1})\, pV_{\xi\xi}^{(0,0)}. \end{aligned} \tag{6.22}$$

we remark that to determine the unknown functions $U_1^{(0,2)}$, $U_2^{(0,2)}$ in (6.20) and (6.19) we have to consider higher order problems.

Hence the solutions of third order problem are the following

$$\begin{aligned} u_1^{(2)} &= \{(p + \lambda^2 - 2 + \lambda^{-2})\, U_{\xi\xi}^{(0,0)} + (\lambda - \lambda^{-1})\, pV_{\xi\xi}^{(0,0)}\}\zeta^2 + \{(-\lambda + \lambda^{-1})\, U_{1,\xi}^{(0,1)} - U_{2,\xi}^{(0,1)} p\}\zeta + U_1^{(0,2)}, \\ u_2^{(2)} &= \{(\lambda - \lambda^{-1})\, U_{\xi\xi}^{(0,0)} + pV_{\xi\xi}^{(0,0)}\}\zeta^2 - \zeta U_{1,\xi}^{(0,1)} + U_2^{(0,2)}, \\ p_t^{(1)} &= \{(\lambda - \lambda^{-1})\, pU_{\xi\xi}^{(0,0)} + p\,(p+1)\, V_{\xi\xi}^{(0,0)}\}\zeta - (p+1)\, U_{1,\xi}^{(0,1)} + (\lambda - \lambda^{-1})\, U_{2,\xi}^{(0,1)} \end{aligned} \tag{6.23}$$

We note that the unknown function $U_1^{(0,1)}, U_2^{(0,1)}$ can be determined at fourth order problem and the unknown functions $U_1^{(0,2)}, U_2^{(0,2)}$ can be determined at firth order problem.

If we substitute solution of third order problem (6.23) into $(6.17)_1$ and $(6.17)_2$ we obtain the governing equations to define $U^{(0,0)}$ and $V^{(0,0)}$

$$\begin{aligned} &\lambda^3 U_{\tau\tau}^{(0,0)} + \lambda^2 V_{\tau\tau}^{(0,0)} - \lambda\,(\lambda^2+1)\,(p+1) U_{\xi\xi}^{(0,0)} + (p^2\lambda^2 + \lambda^2(\lambda^2-1)p - 1)\, V_{\xi\xi}^{(0,0)} = 0, \\ &-\lambda^2 U_{\tau\tau}^{(0,0)} + \lambda^3 V_{\tau\tau}^{(0,0)} + \lambda^2\,(\lambda^2+1)\,(p+1) U_{\xi\xi}^{(0,0)} + \lambda\,(p^2\lambda^2 + (1-\lambda^2)p - \lambda^4)\, V_{\xi\xi}^{(0,0)} = 0. \end{aligned} \tag{6.24}$$

The above equations $(6.24)_1$ and $(6.24)_2$ can be written in a matrix representation as the following

$$\frac{\partial^2}{\partial\tau^2}\begin{pmatrix} U^{(0,0)} \\ V^{(0,0)} \end{pmatrix} + \boldsymbol{D}\frac{\partial^2}{\partial\zeta^2}\begin{pmatrix} U^{(0,0)} \\ V^{(0,0)} \end{pmatrix} = 0, \boldsymbol{D} = \begin{pmatrix} -2(p+1) & \frac{(\lambda^2-1)(1+p)}{\lambda} \\ \frac{(\lambda^2-1)(1+p)}{\lambda} & \frac{\lambda^2(1+p)-\lambda^4-1}{\lambda} \end{pmatrix}. \tag{6.25}$$

We remark that the eigenvalues of $\mathbf{D}$ may be shown to be the squared phase speed limit (4.2), confirming asymptotic consistency of the established model.

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