Dynamic models for a long wave motion in a sheared pre-stressed incompressible elastic layer

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Outline

Motivation

1. Further understand influence of pre-stress on dispersion and stability;
2. Geo-physical applications of shear deformation.

S. Ide, G.C. Beroza, D.R. Shelly,T. Uchide, "A scaling law for slow earthquakes", NATURE, 447 (7140): 76-79 (2007);

P. M. Sheridan, F. O. James and T. S. Miller, Design of components, in A.N. Gent, Engineering with rubber, 209-235, Munich, Hanser (1992).

Small superimposed time-dependent motion

$$\tilde{x}_i(X_A, t) = x_i(X_A) + u_i(x_j, t).$$

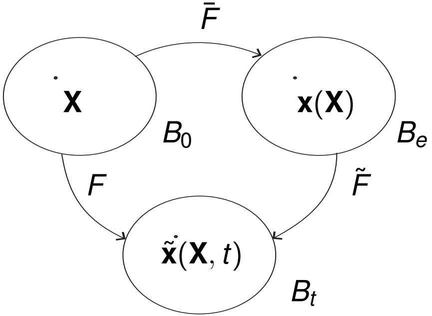


Figure: *Configurations of a pre-stressed body.*

Simple shear primary deformation

$$\mathbf{x_1} = \mathbf{X_1} + \epsilon \mathbf{X_2}, \quad \mathbf{x_2} = \mathbf{X_2}, \quad \mathbf{x_3} = \mathbf{X_3}.$$

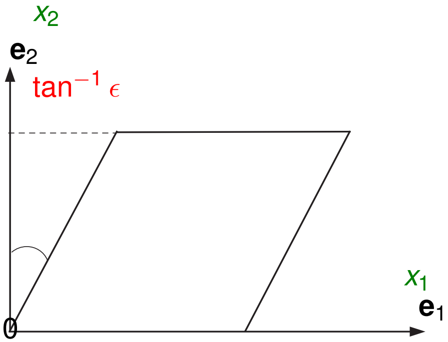


Figure: *The simple shear deformation*

Axes of deformation and natural axes

$$2\theta = tan^{-1}\left(\frac{2}{\epsilon}\right).$$

$$\lambda_1 = \cot\theta \equiv \lambda, \quad \lambda_2 = \tan\theta \equiv \lambda^{-1}, \quad \lambda_3 = 1.$$

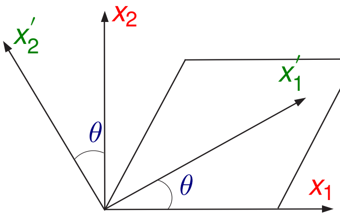


Figure: *The angle θ between Eulerian coordinates (x_1', x_2') and natural coordinate system of the layer (x_1, x_2)*

Main features of models

- Propagation of 2D waves in an incompressible elastic layer subject to **primary simple shear deformation**;
- Wave length considerably exceeds the layer thickness= long wave with small parameter scaled wave number ;
- Long wave low and high frequency regimes ;
  - Long-wave low-frequency $\eta \to 0$ and wave speed is finite $v \to const$.
  - Long-wave high-frequency $\eta \to 0$ and finite frequency $\Omega \to const$.

Equations of motion in layer axes for neo-Hookean material

Neo-Hookean strain energy function

$$W = \frac{\mu}{2}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) - p(\lambda_1\lambda_2\lambda_3 - 1).$$

Equations of motion

$$-\lambda^2 p_{t,1} - \lambda p_{t,2} - \rho\lambda^2\ddot{u}_1 - \rho\lambda\ddot{u}_2 + \mu\left(\lambda^4 + \lambda^2(p-1) + 1\right)(u_{1,11} + u_{2,11})$$
$$+ \mu\lambda\left(\lambda^2(p-2) + 2\right)(u_{1,12} + u_{2,12}) + \mu\lambda^2 u_{1,22} + \mu\lambda\,(1+p)\,u_{2,22} = 0.$$

$$\lambda^2 p_{t,1} - \lambda^3 p_{t,2} + \rho\lambda^2\ddot{u}_1 - \rho\lambda^3\ddot{u}_2 - \mu\left(\lambda^4 + \lambda^2(p-1) + 1\right)(u_{1,11} + u_{2,11})$$
$$+ \mu\lambda\left(\lambda^2(p-2) + 2\right)(u_{1,12} + u_{2,12}) - \mu\lambda^2 u_{1,22} + \mu\lambda^3\,(1+p)\,u_{2,22} = 0.$$

Incompressibility condition

$$u_{1,1} + u_{2,2} = 0.$$

Harmonic wave solutions

Seek solutions of the form

$$(u_1, u_2, p_t) = (U_1, U_2, kP)e^{ikqx_2}e^{ik(vt-x_1)}.$$

Equation for q

$$q^4 - 2\,\epsilon\,q^3 + \left(2 + \epsilon^2 - \hat{v}\right)q^2 - 2\,\epsilon\,q + 1 + \epsilon^2 - \hat{v} = 0, \quad \hat{v} = \frac{\rho v^2}{\mu}.$$

Solutions for q

$$q_1 = i, \quad q_2 = -i, \quad q_3 = \epsilon + i\kappa, \quad q_4 = \epsilon - i\kappa, \quad \kappa^2 = 1 - \hat{v}.$$

Solutions for u_1, u_2, p_t, τ_1 and τ_2

$$u_1 = \sum_{i=1}^{4} q_i A_i e^{ikq_i x_2}, \quad u_2 = \sum_{i=1}^{4} A_i e^{ikq_i x_2}, \quad p_t = k\sum_{i=1}^{4} P(q_i) A_i e^{ikq_i x_2},$$

$$\tau_1 = \mathcal{C}\sum_{i=1}^{4} f(q_i) A_i e^{ikq_i x_2}, \quad \tau_2 = \mathcal{C}\sum_{i=1}^{4} g(q_i) A_i e^{ikq_i x_2}.$$

Benefits of asymptotically consistent dynamic models

Motivation to construct dynamic models

- Analytical solutions of displacement and incremental pressure components considering several asymptotic orders of the problem;
- One equation can be used instead of system of PDE to describe 2D motion.

Asymptotic methodology

J.D.Kaplunov, L.Yu.Kossovich and E.V.Nolde, *Dynamics of thin walled elastic bodies*, Academic Press, (1998);

Long wave low frequency motion in a layer with free faces

Dispersion relation

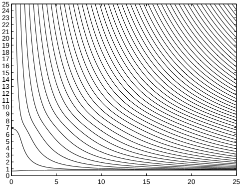
Dispersion relation

$$\left(q_0 \left(p^2 - \kappa^2\right)^2 + q_0 \kappa^2 \left(q_0 + 2\,p\right)^2 + 4\,\kappa^2 \left(p^2 - \kappa^2\right) \left(q_0 + 2\,p\right) \right) \sinh\left(\eta\right) \sinh\left(\eta\,\kappa\right)$$

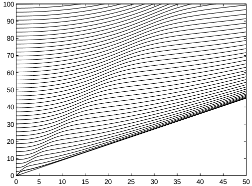
$$+ 2\kappa \left(p q_0 + p^2 + \kappa^2\right)^2 \left(\cos\left(\eta\,\epsilon\right) - \cosh\left(\eta\right) \cosh\left(\eta\,\kappa\right)\right) = 0,$$

$$q_0 = 1 + \epsilon^2 + \kappa^2, \quad \eta = kh.$$

Numerical solution with $\epsilon = 2, p = 0.5$



(a)



(b)

Long wave low frequency approximations and asymptotic orders

Wave speed for two fundamental modes

$$\text{Mode 1:}\quad v_1 = v_1^{(0)} + \eta^2 v_1^{(2)} + O(\eta^4) =$$

$$1 - p^2 + \frac{\eta^2}{12}(\epsilon^2 + (p+1)^2)p^2 + O(\eta^4),$$

$$\text{Mode 2:}\quad v_2 = v_2^{(0)} + \eta^2 v_2^{(2)} + O(\eta^4) =$$

$$\epsilon^2 + 2p + 2 - \frac{\eta^2}{12}(\epsilon^2 + (p+1)^2) + O(\eta^4).$$

Scalings for long wave low frequency non-dimensional governing equations

$$u_1 \approx u_2 \approx p_t,$$

$$u_1 = lu_1^*, \quad u_2 = lu_2^*, \quad p_t = \mu p_t^*, \quad x_1 = l\xi, \quad x_2 = l\eta\zeta, \quad t = l\sqrt{\frac{\rho}{\mu}}\tau.$$

Analytical solutions for displacements and incremental pressure

Form of solution

$$(u_1^*, u_2^*, p_t^*) = \sum_{l=0}^{m} (u_1^{(l)}, u_2^{(l)}, p_t^{(l)})\eta^l + O(\eta^{m+1}).$$

Leading order problem

$$u_1^{(0)} = U^{(0,0)}(\xi, \tau), \quad u_2^{(0)} = V^{(0,0)}(\xi, \tau).$$

Second order problem

$$u_1^{(1)} = \{\left(-\lambda + \lambda^{-1}\right) U_{,\xi}^{(0,0)} - V_{,\xi}^{(0,0)} p\}\zeta + U_1^{(0,1)}, \quad u_2^{(1)} = -U_{,\xi}^{(0,0)}\zeta + U_2^{(0,1)},$$
$$p_t^{(0)} = -(p+1)\, U_{,\xi}^{(0,0)} + \left(\lambda - \lambda^{-1}\right) V_{,\xi}^{(0,0)}.$$

Third order problem

$$u_1^{(2)} = \left\{\left(p + \lambda^2 - 2 + \lambda^{-2}\right) U_{\xi\xi}^{(0,0)} + \left(\lambda - \lambda^{-1}\right) pV_{\xi\xi}^{(0,0)}\right\}\zeta^2 + \left\{\left(-\lambda + \lambda^{-1}\right) U_{1,\xi}^{(0,1)} - U_{2,\xi}^{(0,1)} p\right\}\zeta + U_1^{(0,2)},$$
$$u_2^{(2)} = \left\{\left(\lambda - \lambda^{-1}\right) U_{\xi\xi}^{(0,0)} + pV_{\xi\xi}^{(0,0)}\right\}\zeta^2 - \zeta U_{1,\xi}^{(0,1)} + U_2^{(0,2)},$$
$$p_t^{(1)} = \left\{\left(\lambda - \lambda^{-1}\right) pU_{\xi\xi}^{(0,0)} + p\,(p+1)\, V_{\xi\xi}^{(0,0)}\right\}\zeta - (p+1)\, U_{1,\xi}^{(0,1)} + \left(\lambda - \lambda^{-1}\right) U_{2,\xi}^{(0,1)}.$$

Governing equation

to define $U^{(0,0)}$ and $V^{(0,0)}$ and hence determine asymptotic solutions for u_1, u_2, p_t

$$\frac{\partial^2}{\partial \tau^2}\begin{pmatrix} U^{(0,0)} \\ V^{(0,0)} \end{pmatrix} + \boldsymbol{D}\frac{\partial^2}{\partial \zeta^2}\begin{pmatrix} U^{(0,0)} \\ V^{(0,0)} \end{pmatrix} = 0.$$

where the matrix **D** given by

$$\boldsymbol{D} = \begin{pmatrix} -2(p+1) & \frac{\left(\lambda^2-1\right)(1+p)}{\lambda} \\ \frac{\left(\lambda^2-1\right)(1+p)}{\lambda} & \frac{\lambda^2(1+p)-\lambda^4-1}{\lambda} \end{pmatrix}.$$

Eigenvalues of **D** give leading order long wave phase speed limits

Long wave high frequency motion in a layer with fixed faces

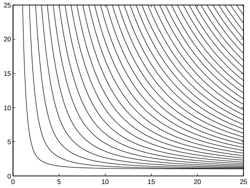
The dispersion relation

The dispersion relation

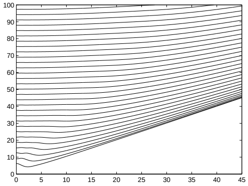
$$2\,\kappa\,\left(\cosh(\eta)\cosh(\eta\,\kappa) - \cos(\eta\,\epsilon)\right) - \left(1 + \epsilon^2 + \kappa^2\right)\sinh(\eta)\sinh(\eta\,\kappa) = 0,$$

$$\kappa = \sqrt{1 - \hat{v}}.$$

Numerical solution of the dispersion relation with $\epsilon = 3, p = 2$ indicates no fundamental modes



(c)



(d)

Long wave high frequency approximations and asymptotic orders

Two families of cut-off frequencies

$$\Omega_1 = 2k\pi, \quad \tan(A) = A \quad \text{where} \quad A = \frac{\Omega_2}{2},$$

Long wave high frequency third order approximations

$$\Omega = \Omega_i^{(0)} + \Omega_i^{(2)}\eta^2 + \Omega_i^{(4)}\eta^4 + O(\eta^6), \quad i = 1, 2.$$

Relative asymptotic orders of displacements and pressure
first family of cut-off frequencies

$$u_1 = u_1^*, \quad u_2 = \eta u_2^*, \quad p = \eta^{-1} p^*,$$

scalings $u_1 = l u_1^*, \quad u_2 = l\eta u_2^*, \quad p_t = \mu p_t^* \eta^{-1} = \mu p_t^* \eta^{-1}, \quad x_1 = l\xi, \quad x_2 = l\eta\zeta, \quad t = l\eta\sqrt{\frac{\rho}{\mu}}\tau = l\eta\sqrt{\frac{\rho}{\mu}}\tau,$

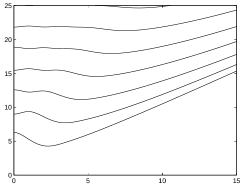
second family of cut-off frequencies

$$u_1 = u_1^*, \quad u_2 = \eta u_2^*, \quad p = \eta^{-2} p^*.$$

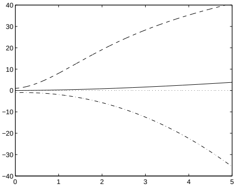
scalings $u_1 = l u_1^*, \quad u_2 = l\eta u_2^*, \quad p_t = \mu p_t^* \eta^{-2} = \mu p_t^* \eta^{-2}, \quad x_1 = l\xi, \quad x_2 = l\eta\zeta, \quad t = l\eta\sqrt{\frac{\rho}{\mu}}\tau = l\eta\sqrt{\frac{\rho}{\mu}}\tau.$ (1)

Phenomenon of negative group velocity in dispersive curves

Numerical analysis: negative group velocity in the vicinity of cut-off frequencies for harmonics with even index



(e)



(f)

Analytical explanation: gradients in the vicinity of cut-off frequencies

$$\Omega_1^{(2)} = -\frac{1+2\,\epsilon^2}{4\pi\, n}, \quad \Omega_2^{(2)} = \frac{1+6\,\epsilon^2}{6\Omega^{(0)}},$$

Analytical solutions for displacement components and incremental pressure- part 1

Cut-off frequencies $2k\pi$, main parameter in model long wave amplitude $U_{1s}^{(0,0)}$, general solution of order m

$$(u_1^*, u_2^*, p_t^*) = \sum_{m=0}^{r} (u_1^{(m)}, u_2^{(m)}, p_t^{(m)})\eta^m + O(\eta^{r+1}),$$

Leading order problem: $u_1^{(0)} = U_{1s}^{(0,0)} \sin(\Omega\zeta), \quad u_2^{(0)} = \dfrac{U_{1s,\xi}^{(0,0)}}{\Omega} \cos(\Omega\zeta) - \dfrac{U_{1s,\xi}^{(0,0)}}{\Omega},$

Second order problem: $u_1^{(1)} = U_{1c}^{(0,1)} \cos(\Omega\zeta) + U_{1s}^{(0,1)} \sin(\Omega\zeta) + U_{1s}^{(1,1)} \zeta \sin(\Omega\zeta) + v_1^{(0,1)},$

$u_2^{(1)} = U_{2c}^{(0,1)} \cos(\Omega\zeta) + U_{2c}^{(1,1)} \zeta \cos(\Omega\zeta) + v_2^{(1,1)} \zeta + v_2^{(0,1)}, \quad p_t^{(1)} = P_t^{(0,1)} + P_t^{(1,1)} \zeta,$

Third order problem:

$$u_1^{(2)} = U_{1c}^{(0,2)} \cos(\Omega\zeta) + U_{1s}^{(0,2)} \sin(\Omega\zeta) + U_{1c}^{(1,2)} \zeta \cos(\Omega\zeta) + U_{1s}^{(1,2)} \zeta \sin(\Omega\zeta) + U_{1s}^{(2,2)} \zeta^2 \sin(\Omega\zeta) + v_1^{(0,2)} + v_1^{(1,2)} \zeta,$$

$$u_2^{(2)} = U_{2c}^{(0,2)} \cos(\Omega\zeta) + U_{2s}^{(0,2)} \sin(\Omega\zeta) + U_{2c}^{(1,2)} \zeta \cos(\Omega\zeta) + U_{2s}^{(1,2)} \zeta \sin(\Omega\zeta) + U_{2c}^{(2,2)} \zeta^2 \cos(\Omega\zeta)$$

$$+ v_2^{(0,2)} + v_2^{(1,2)} \zeta + v_2^{(2,2)} \zeta^2, \quad p_t^{(2)} = P_t^{(0,2)} + P_t^{(1,2)} \zeta + P_t^{(2,2)} \zeta^2.$$

Analytical solutions for displacement components and incremental pressure- part 2

Cut-off frequencies $\tan(A) = A \quad A = \frac{\Omega_2}{2}$, main parameter in model pressure increment $P_t^{(0,0)}$

Leading order problem: $u_1^{(0)} = -\frac{P_{t,\xi}^{(0,0)} \cos(\Omega\,\zeta)}{\Omega^2} + \frac{P_{t,\xi}^{(0,0)} \sin(\Omega\,\zeta)}{2\Omega} + \frac{P_{t,\xi}^{(0,0)}}{\Omega^2}$,

$$u_2^{(0)} = \frac{P_{t,\xi,\xi}^{(0,0)} \cos(\Omega\,\zeta)}{2\Omega^2} + \frac{P_{t,\xi,\xi}^{(0,0)} \sin(\Omega\,\zeta)}{\Omega^3} - \frac{P_{t,\xi,\xi}^{(0,0)}}{2\Omega^2} - \frac{P_{t,\xi,\xi}^{(0,0)}\zeta}{\Omega^2}, \quad p_t^{(0)} = P_t^{(0,0)}(\xi,\tau).$$

Second order problem: $u_1^{(1)} = U_{1c}^{(0,1)}\cos(\Omega\zeta) + U_{1s}^{(0,1)}\sin(\Omega\zeta) + U_{1c}^{(1,1)}\zeta\cos(\Omega\zeta) + U_{1s}^{(1,1)}\zeta\sin(\Omega\zeta) + v_1^{(0,1)}$,

$$u_2^{(1)} = U_{2c}^{(0,1)}\cos(\Omega\zeta) + U_{2s}^{(0,1)}\sin(\Omega\zeta) + U_{2c}^{(1,1)}\zeta\cos(\Omega\zeta) + U_{2s}^{(1,1)}\zeta\sin(\Omega\zeta) + v_2^{(0,1)} + v_2^{(1,1)}\zeta, \quad p_t^{(1)} = P_t^{(0,1)}(\xi,\tau),$$

Third order problem:

$$u_1^{(2)} = U_{1c}^{(0,2)}\cos(\Omega\zeta) + U_{1s}^{(0,2)}\sin(\Omega\zeta) + U_{1c}^{(1,2)}\zeta\cos(\Omega\zeta) + U_{1s}^{(1,2)}\zeta\sin(\Omega\zeta)$$
$$+ U_{1c}^{(2,2)}\zeta^2\cos(\Omega\zeta) + U_{1s}^{(2,2)}\zeta^2\sin(\Omega\zeta) + v_1^{(0,2)} + v_1^{(1,2)}\zeta + v_1^{(2,2)}\zeta^2,$$
$$u_2^{(2)} = U_{2c}^{(0,2)}\cos(\Omega\zeta) + U_{2s}^{(0,2)}\sin(\Omega\zeta) + U_{2c}^{(1,2)}\zeta\cos(\Omega\zeta) + U_{2s}^{(1,2)}\zeta\sin(\Omega\zeta)$$
$$+ U_{2c}^{(2,2)}\zeta^2\cos(\Omega\zeta) + U_{2s}^{(2,2)}\zeta^2\sin(\Omega\zeta) + v_2^{(0,2)} + v_2^{(1,2)}\zeta + v_2^{(2,2)}\zeta^2 + v_2^{(3,2)}\zeta^3,$$
$$p_t^{(2)} = P_t^{(0,2)} + P_t^{(1,2)}\zeta + P_t^{(2,2)}\zeta^2.$$

Governing equations

1D governing equations aim to describe 2D motion

Cut-off frequencies $2k\pi$:

$$\frac{\rho h^2}{\lambda^2\gamma}\frac{\partial^2 U_{1s}^{(0,0)}}{\partial^2 t^2}+4k^2\pi^2 U_{1s}^{(0,0)}+h^2\left(\frac{\left(2+2\,\lambda^4-3\,\lambda^2\right)}{\lambda^2}\right)\frac{\partial^2 U_{1s}^{(0,0)}}{\partial^2 x_1^2}=0,$$

elliptic type:

$$\mathcal{B}_e^{(1)}=\frac{\left(2+2\,\lambda^4-3\,\lambda^2\right)}{\lambda^2}>0.$$

Cut-off frequencies $\tan(A)=A \quad A=\frac{\Omega_2}{2}$:

$$\frac{\rho h^2}{\lambda^2\gamma}\frac{\partial^2 P_t^{(0,0)}}{\partial^2 t^2}+\Omega^2 P_t^{(0,0)}-h^2\frac{\left(6\,\lambda^4+6-11\,\lambda^2\right)}{3\lambda^2}\frac{\partial^2 P_t^{(0,0)}}{\partial^2 x_1^2}=0,$$

hyperbolic type:

$$\mathcal{B}_e^{(2)}=\frac{\left(6\,\lambda^4+6-11\,\lambda^2\right)}{3\lambda^2}>0.$$

both coincide with previous results provided: $\lambda=1, \alpha=\beta=\gamma.$

Summary

- Equations of motions were obtained;
- Dispersion relations was derived and analyzed for different boundary value problems;
- Relative asymptotic orders of displacements and incremental pressure was established to construct dynamic model;
- Simplified dynamic model was presented for long wave low frequency 2D motion in a layer with free faces;
- 1D dynamic model can was presented for long wave high frequency 2D motion ;
- Analytical solutions for displacement components and incremental pressure were obtained till third order;
- Phenomenon of negative group velocity was investigated.

Conferences and summer school

conferences

- CanCNSM (2008) Canada: 3rd Canadian Conference on Nonlinear Solid Mechanics;
- BAMC (2008) Manchester, UK : British applied mathematics colloquium;
- Euromech Colloquium 481 (2007) UK : Edge and surface waves ;
- BAMC (2006) Keele, UK : British applied mathematics colloquium;
- CISM advanced course (2006): Waves in non-linear pre-stressed materials.

papers

S. R. Amirova and G. A. Rogerson, The influence of simple shear deformation on long-wave motion in a pre-stressed incompressible elastic layer, *JoMMS*, **3**, 831-851 (2008).

G. A. Rogerson and S. R. Amirova, Long wave dispersion in a neo-Hookean layer subject to simple shear, *3rd Canadian Conference on Nonlinear Solid Mechanics, CanCNSM*, 85-92, University of Toronto, Ontario, Canada (2008).