The influence of a simple shear deformation on a long wave motion in a pre-stressed incompressible elastic layer

Svetlana R. Amirova

Submitted in Partial Fulfillment of the Requirements
of the Degree of Doctor of Philosophy
in Applied Mathematics
September 2008

Keele University, UK

Abstract

Nowadays rubber-like materials are popular in modern technology including various engineering and biomedical applications. This interest is partially motivated by their important ability to undergo finite primary deformations, leading to a pre-stressed state. Guided by experimental studies, theories of non-linear elasticity are being proposed to model dynamic response of such materials under large external loads. To simplify algebraically complicated analysis, lower-dimensional theories with minimal number of essential parameters can be effectively employed.

It is possible to extend static theories, such as Kirchhoff-Love theory of shells and the refined Timoshenko-Reissner theory to various dynamic problems. However in doing so additional approximations needs to be taken into account. In comparison to only long wave limits of fundamental mode in static case in dynamic case one have to consider long wave low frequency, long wave high frequency and short wave high frequency spectrum.

This thesis has two main purposes each of which aim to elucidate the dynamic response of incompressible elastic layer subject to primary simple shear deformation considering free, fixed, and fixed-free face boundary conditions. The first goal is to provide an asymptotic analysis of associated dispersion relations. The second aim is to construct appropriate asymptotically consistent models. Accordingly, Chapter 1 is devoted to the basic equations which governs small time dependent motion superimposed on large simple shear deformation. The associated dispersion relations are derived and analyzed in Chapters 2-3 considering traction free, zero displacement and mixed boundary value problems. In Chapters 4-6 an appropriate asymptotic dynamic models for long wave motion are obtained and shown to be asymptotically consistent with analysis of exact dispersion relations.

Contents

Introduction **1**

1 Basic equations **8**

1.1 Equations of incremental motion . . . 9

1.2 Features of the simple shear deformation . . . 12

1.2.1 The simple shear deformation . . . 12

1.2.2 The relation between Eulerian axes and the natural axes of the layer . . . 17

1.3 Linearized equations of motion in the natural coordinate system of the layer . . . 19

1.4 Characteristic equation, displacements and incremental pressure in the natural coordinate system of the layer . . . 23

1.5 Components of linearized incremental traction in the natural coordinate system of the layer . . . 26

1.6 Some specific models for rubber-like materials subject to simple shear . . . 33

2 Long wave motion in a layer with free faces **35**

2.1 The dispersion relation . . . 36

2.2 Numerical analysis of the dispersion relation . . . 37

2.3 Analysis of the dispersion relation for long wave low frequency motion . . . 40

2.3.1 Leading order approximation . . . 43

2.3.2 Second order approximation . . . 44

2.3.3 Higher order approximations in the case $\boldsymbol{\delta = 0}$ . . . 44

2.4 Analysis of the dispersion relation for long wave high frequency motion . . . 48

2.4.1 Leading order approximation . . . 50

2.4.2 Higher order approximations in the case $\boldsymbol{\delta = 0}$ . . . 51

2.5 Relative asymptotic orders of displacements and incremental pressure . . . 54

2.5.1 Long wave low frequency regime 54
2.5.2 Long wave high frequency regime 59

3 Long wave motion in a layer subject to some non-classical boundary conditions 62
3.1 The dispersion relations 63
3.1.1 Layer with fixed faces 63
3.1.2 Layer with one free and one fixed faces 64
3.2 Numerical analysis of the dispersion relations 65
3.2.1 Layer with fixed faces 65
3.2.2 Layer with one free and one fixed faces 65
3.3 Analysis of the dispersion relations for long wave high frequency motion 65
3.3.1 Layer with fixed faces 65
3.3.2 Layer with one free and one fixed faces 73
3.4 Relative asymptotic orders of displacements and incremental pressure 77
3.4.1 Layer with fixed faces 77
3.4.2 Layer with one free and one fixed faces 80

4 Asymptotically consistent model for a long wave low frequency motion in a layer with free faces 83
4.1 Asymptotic model in respect of the most general strain-energy function 84
4.1.1 Asymptotic scaling and dimensionless equations 84
4.1.2 Leading order problem 86
4.1.3 Second order problem 87
4.1.4 Third order problem 89
4.2 Asymptotic model in respect of neo-Hookean strain-energy function 95
4.2.1 Non-dimensional equations 95
4.2.2 Leading order problem 96
4.2.3 Second order problem 96
4.2.4 Third order problem 98

5 Asymptotically consistent model for a long wave high frequency motion in a layer with free faces 101
5.1 Asymptotic scaling and dimensionless equations 101
5.2 Leading order problem 103

5.3 Second order problem 105
5.4 Third order problem 108

6 Asymptotically consistent models for a long wave high frequency motion in a layer subject to some non-classical boundary conditions 114

6.1 Asymptotically approximate equations for the layer with fixed faces: first family of cut-off frequencies 115
6.1.1 Asymptotic scaling and dimensionless equations 115
6.1.2 Leading order problem 115
6.1.3 Second order problem 116
6.1.4 Third order problem 118
6.2 Asymptotically approximate equations for the layer with fixed faces: second family of cut-off frequencies 123
6.2.1 Asymptotic scaling and dimensionless equations 123
6.2.2 Leading order problem 124
6.2.3 Second order problem 125
6.2.4 Third order problem 127
6.3 Asymptotically approximate equations for the layer with one free and one fixed faces 132
6.3.1 Asymptotic scaling and dimensionless equations 132
6.3.2 Leading order problem 132
6.3.3 Second order problem 133
6.3.4 Third order problem 134

Acknowledgement

I would like to thank my supervisor Prof. Graham Rogerson for introducing me into the field of mechanics of pre-stressed materials, encouraging me, sharing numerous ideas and being very patient. Also I would like to thank Prof. Yibin Fu for providing useful advice and my friends postgraduate students Robert, Simon, William and Steven for keeping me in a good sport shape. Finally I would like to thank my friend Armin and my parents, especially my mother, for help and support. An Overseas Research Student Award and maintenance grant from Keele University are very gratefully appreciated.

Introduction

Elastic materials are extensively used in modern technology due to their important ability to undergo finite strain deformations, see for example Hirst (1969) and Sheridan et al. (1992) in respect of applications in bridges and earthquake protection devices, respectively. In addition, due to recent extensive investigations in biomechanics we mention studies in respect to biological tissues, see for example Beatty (1987), Ogden et al. (2005), Gasser et al. (2006) and Holzapfel and Ogden (2006). Motivated by practical applications, the theory of elastic materials subject to large deformations was first proposed in the early 1940's. The initial studies on this subject were made in respect of incompressible materials which do not change volume during deformation. The main principle in such models is the simplifying incompressibility constraint, see for example Chadwick (1999). As an illustration of the analysis, which became possible with an assumption of the incompressibility constraint, we cite Scott (1986). The initial theoretical investigations of incompressible materials were supported by experimental studies, see for example Rivlin and Saunders (1951) and Treloar (1975). This thesis is focused on incompressible elastic materials subject to primary finite simple shear deformations. The simple shear deformation commonly occurs in geo-mechanics, see for example Gessner et al. (2007) and Ide et al. (2007). In addition, we mention several applications of shear type deformations in bio-mechanics, see Hard and Toll (2008), Furness et al. (1997) and Manoussaki et al. (2006). In respect of theoretical modelling we remark that finite shear in various types of materials was extensively studied by Boulander and Hayes (2000, 2002, 2004, 2007).

We proceed with an explanation of the term *pre-stressed body*, which is used to analyze the behavior of an elastic body under primary finite deformation to indicate the presence of stress for example prior to wave propagation. There is an appropriate non-linear theory, see Green and Adkins (1960) and Green and Zerna (1954), which leads to complicated non-linear equations of elasticity. The approach to simplify the analysis of materials under large initial stress is a decomposition into a large static deformation and superimposed incremental motion. By this means, linearized equations of motion are derived by expanding the stress as a Taylor series about the initial deformed state. The static theory of incremental deformations superimposed on finite strains was apparently first developed by Green et al. (1952). The linearized theory of incremental motion, superimposed on a large elastic deformation, can be applied to the main problem in this thesis, the propagation of waves in a incompressible elastic layer subject to a primary simple shear deformation.

The problem of wave propagation in elastic half spaces and plates goes back to the study of surface waves by Lord Rayleigh (1885). In more recent studies, the influence of pre-stress on surface wave propagation was investigated by Dowaikh and Ogden (1990) for the two-dimensional problem. In addition, surface waves in pre-stressed materials under finite simple shear deformation were investigated by Connor and Ogden (1995) and Destrade and Ogden (2005).

We remark that there is no dispersion associated with surface wave propagation in elastic media. The initial studies on the propagation of dispersive waves were performed by Lord Rayleigh (1889) and Lamb (1917), who derived the dispersion relation for wave propagation in a plate composed of linear isotropic elastic material. Generally, dispersion is a property of certain types of media. It can also arise due to the presence of a length scale and indicates that phase speed depends on wave number. The relationship between phase speed and wave number, referred to as the *dispersion relation* or *frequency equation*, can be obtained by satisfying the equations of motion and boundary conditions. Normally, the dispersion relation is a complicated transcendental equation which has an infinite number of solution branches or modes. For practical applications it is often assumed that only the fundamental modes are significant, see for example Manoussaki et al. (2006). However, studies by Rogerson (1992), Kaplunov and Markushevich (1993) show that for the certain types of problems the higher modes are important. Historically, Lamb (1917) was the first to analyze the higher order modes, identifying the cut-off frequencies, the Lame modes and other properties of the high frequency spectrum. A complete analysis of spectrum of the Rayleigh-Lamb equations was first presented by Mindlin (1960).

A theory has been developed to describe infinitesimal time-dependent wave propagation, superimposed upon a large static primary deformation, see for example Ogden (1984). The initial studies were proposed in respect of dispersion in pre-stressed incompressible materials by Biot (1965) and Willson (1977). There are also several papers investigating the question of stability in pre-stressed elastic structures, see for example Roxburg and Ogden (1994), Fu and Ogden (1999), Triantafyllidis and Abeyarante (1983) and Connor and Ogden (1996). Plane waves in a pre-stressed elastic plate were studied in detail by Ogden and Roxburgh (1993).

Depending on the type of large static deformation, the complexity of the equations required to describe wave propagation in pre-stressed materials is different. The asymptotic analysis of the two-dimensional dispersion relation for wave propagation in an incompressible and slightly compressible plate subject to primary plain strain was performed in the studies by Rogerson and Fu (1995), Rogerson (1997), Sandiford and Rogerson (2000). These investigations were generalized by Pichugin and Rogerson (2001, 2002) to the three dimensional problem. We remark that in these studies the dispersion relation allows decomposition into symmetric and anti-symmetric components. This feature allows simplification of the analysis of the appropriate problems. However this is not always the case. For example in the studies by Connor and Ogden (1995, 1996) the equations to describe wave propagation in an incompressible elastic layer subject to simple shear deformation were derived. These equations lead to the corresponding dispersion relation which cannot be decoupled into symmetric and anti-symmetric parts. The long wave asymptotic analysis of the associated dispersion relation has been performed by Amirova and Rogerson (2008) and forms the basis of Chapter 2 of this thesis. We remark that in the paper by Destrade and Ogden (2005) some additional information about the characteristic equation and the propagation of surface waves in a half space subject to simple shear deformation have been obtained.

The dispersion relation is dependent on the type of boundary conditions. In addition to classical traction free boundary condition, wave propagation in an incompressible and a nearly incompressible layer with fixed faces was investigated by Nolde and Rogerson (2002) and Kaplunov and Nolde (2002). This problem was motivated by practical applications in geo-mechanics, see Liang et al. (1993). The various effects of dispersion in a pre-stressed compressible elastic plate has been studies by Nolde et al. (2004) in respect to traction free boundary conditions, and Prikazchikova (2004) for some non-classical boundary conditions. The case of mixed boundary conditions was investigated by Fu (2007) and Connor and Ogden (1996). The above papers indicate that in case of the layer with one fixed and one free face boundary condition the corresponding dispersion relation cannot be decomposed into symmetric and anti-symmetric parts. Additionally we remark that a layer with fixed faces can be used to model coal seams, these usually being surrounded by much stiffer rocks, see for example Liang et al. (1993).

To elucidate the dynamic response of thin structures, lower dimensional theories, with minimal essential parameters have been shown to provide significant simplification. In the static case, there are widely used lower-dimensional structural theories, such as the Kirchhoff theory of plates, the

Kirchhoff-Love theory of shells and the refined Timoshenko-Reissner theories, see Lord Rayleigh (1877), Love (1927), Timoshenko (1938) and Reissner (1944). Such theories are based on physical hypothesis and do not readily provide an approach for refining to higher order.

An analysis of equations of elasticity, and the associated method of asymptotic integration utilized to derive an asymptotically consistent dynamic theories are presented in a detailed monograph by Kaplunov et al. (1998). It was shown that long wave low frequency, long wave high frequency and short wave high frequency theories are needed to fully describe the dynamic behavior of an elastic body, in comparison to only the long wave low frequency limit in the static case. Some lower dimensional asymptotically consistent models for the two-dimensional motion were derived for an incompressible elastic plate subject to primary homogeneous strain with traction free faces, see Kaplunov (2000, 2002). These models were generalized to the three dimensional case by Pichugin and Rogerson (2001, 2002). Asymptotically consistent models were also derived for an incompressible transversely isotropic elastic plate, see Kossovitch (2002, 2006). The purpose of this thesis is to extend such asymptotic models in order to elucidate the dynamic behavior of an incompressible elastic layer subject to a primary simple shear deformation.

This thesis has the following structure. In Chapter 1 we summarize the basic equations of nonlinear elasticity. The body is assumed to possess a natural isotropic state and has been subjected to a large primary static deformation, leading to a finitely deformed equilibrium state. An infinitesimal time-dependent motion is then superimposed upon the pre-stressed state. In order to elucidate the dynamic response we obtain the linearized three dimensional equation of motion by expanding the stress tensor as a Taylor series about the equilibrium state and introducing a measure of incremental surface traction. In this thesis we restrict our attention to the two dimensional problem and specify a simple shear as the primary static deformation. An important feature of this deformation is that no principal axis is normal to the faces of the layer, with the Lagrangean and Eulerian axes non-coincident. We remark that in respect of wave propagation in a layer subject to such a primary simple shear deformation, lack of symmetry exists and there is to no analogue of symmetric or anti-symmetric motion. In addition, we impose the strong ellipticity condition in order to ensure that the phase speeds of all associated body waves are real in every direction.

In this thesis we consider three types of boundary conditions for an incompressible elastic layer: free faces, fixed faces, and one fixed and one free face. The free faces boundary conditions are widely used while the latter two may be regarded as non-classical. We seek solutions of all three boundary value problems in the form of a travelling wave which results in the corresponding dispersion relation. This relation is then analyzed both numerically and asymptotically. Where possible, we perform the analysis in respect of the most general strain-energy function. However, we remark that in the case of a neo-Hookean strain-energy function the linearized equations of motion, together with the incremental traction components, can be significantly simplified. We use this illustrative simplified model to highlight the main features and perform a highly transparent analysis. In addition, we employ a Varga strain energy function to model incompressible rubber-like elastic materials subjected to a simple shear deformation. In this case we produce illustrative numerical results which demonstrate good agreement between analytical approximations and numerical results.

Chapter 2 is devoted to the analysis of two dimensional long wave motion in an incompressible elastic layer subject to a primary simple shear deformation in respect of the free face boundary value problem. Considering traction free boundary conditions, the dispersion relation is first derived for the most general incompressible strain energy function and then simplified to a specific class of strain energy functions. A numerical analysis of the dispersion relation reveals that dependent on the amount of shear and pre-stress there may be non, one or two long wave limits of the fundamental modes. Motivated by the numerical investigation, an asymptotic analysis of the dispersion relation is performed for both long wave low and long wave high frequency motion. The resulting analytical approximations give phase speed and frequency as explicit function of wave and mode number. Good agreement between numerical and asymptotic solutions over a relatively large wave number regime is illustrated. The relative asymptotic orders of the displacement components and incremental pressure are established and used to provide appropriate asymptotic scalings for the long wave asymptotic models to be derived in the later chapters of the thesis. In respect of the high frequency motion, it is shown that the incremental pressure is asymptotically leading and the in-plane displacement component asymptotically larger than its normal counterpart. For the low frequency motion, the incremental pressure, in-plane and normal displacement components all have the same asymptotic orders. The paper by Amirova and Rogerson (2008) is based on the material in Chapter 2.

In Chapter 3 we consider wave propagation in a layer with some non-classical boundary conditions. Specifically, the fixed faces boundary value problem and one fixed one free face boundary value problem are examined. A numerical analysis demonstrates that in the long wave limit there is no low frequency motion for both types of boundary condition. After a numerical investigation we proceed to an asymptotic analysis of the corresponding dispersion relations. It is shown that for the two considered boundary value problems there is no analogue of symmetric or anti-symmetric motion. The dispersion relations are first derived in respect of the most general incompressible strain-energy function and then represented in simplified forms for a specific class of materials. Taking the scaled wave number as a small parameter, asymptotic expansions are derived for motion in the vicinity of cut-off frequencies and provide frequency as an explicit function of wave and mode number for both problems. We conclude Chapter 3 by establishing the relative asymptotic orders of the displacement components and incremental pressure. It is shown that for both boundary value problems the incremental pressure is asymptotically leading, with the in-plane displacement component larger than its normal counterpart.

In Chapter 4 we derive a simplified asymptotic model for two-dimensional long wave low frequency motion in an sheared pre-stressed incompressible elastic layer with traction free faces. The cases of the most general strain-energy function and a neo-Hookean strain-energy function are considered. Non-dimensional equations of motion, incompressibility condition and incremental traction components lead to the hierarchial systems of traction free boundary value problems at various orders. We take into consideration the first three asymptotic orders both for the general and neo-Hookean material and employ asymptotic integration to derive analytical solutions of the asymptotic boundary value problems. The presence of pre-stress in the form of simple shear yields no analogue of symmetric and anti-symmetric motion and leads to a novel type of one dimensional vector governing equation, which is obtained only at third order. We remark that the governing equations may be used to confirm the existence of two fundamental modes and illustrate the asymptotic consistency of the derived simplified dynamic model for both the general and neo-Hookean material models. Based on work outlined in Chapter 4, is the paper by Rogerson and Amirova (2008).

In Chapters 5 and 6 we derive a one-dimensional asymptotically consistent model for the two-dimensional long wave high frequency motion, i.e. motion in the vicinity of the cut-off frequencies. To simplify the analysis and highlight the main features of the model we focus attention on the motion in a layer composed of the neo-Hookean material. In Chapter 5 we consider free faces boundary conditions whereas in Chapter 6 we examine some non-classical boundary conditions. The asymptotic methodology is similar to the long wave low frequency analysis. However, direct asymptotic integration is performed in the vicinity of the cut-off frequencies. Again, three orders of the problem are set up and investigated. The one-dimensional governing equation is derived only at the stage of the third order problem in all the considered boundary values problems: free faces, fixed faces, one free one fixed faces. In addition, in each of these cases the governing equation was shown to be hyperbolic or elliptic depending on the amount of shear and mode number. To conclude, we demonstrate that the obtained governing equations can be employed to illustrate consistency of the derived simplified dynamic models with the asymptotic approximations of the corresponding dispersion relations.

Chapter 1

Basic equations

In this chapter we summarize the basic equations of nonlinear elasticity, relevant to this thesis. We examine the problem of harmonic wave propagation in an incompressible elastic layer of finite thickness, infinite lateral extent and which is subjected to a primary large static deformation. An infinitesimal time-dependent motion is then superimposed on this large static deformation. To begin with we derive the equations of infinitesimal time-dependent motion, superimposed upon a large static primary deformation and consider their linearized form in the general three dimensional case. Then we restrict our attention to the case in which static deformation is a simple shear, analyze its features and emphasize the fact that no principal axis is normal to the incrementally traction free faces of the plate. After that the connection is established between what we will refer to as the natural coordinate system of the layer (coincident with its natural axes) and Eulerian coordinate system formed by the principal axes of the simple shear deformation. In this thesis we restrict our attention to a two dimensional problem. For the propagation of harmonic planes wave the strong ellipticity condition is imposed to ensure physically realistic response. Linearized equations of two dimensional motion superimposed on the primary simple shear deformation and associated linearized incompressibility condition are initially established within the Eulerian coordinates and then transferred into the natural coordinates of the layer. Then the travelling harmonic wave propagating along the longitudinal direction of the layer is considered as the solution of homogeneous system of two linearized equations of motion. In addition, we take into account the linearized incompressibility condition to derive the characteristic equation and obtain a representation for the incremental pressure component. An appropriate linearized measure of incremental surface traction is introduced and representations for the components of incremental traction associated with the upper and lower faces of the layer are obtained in the natural coordinates of the layer. Finally, we consider neo-Hookean and Varga strain-energy functions to model incompressible rubber-like elastic materials subjected to a simple shear deformation. These material models will be used for numerical calculations and comparison of numerical and asymptotic results thought the thesis. We remark that in case of the neo-Hookean material model linearized equations of motion together with incremental traction components can be significantly simplified.

1.1 Equations of incremental motion

In this section, we summarize relevant equations from the theory of infinitesimal time-dependent motion, superimposed upon a large static primary deformation, more details can be found in books on continuum mechanics, see for example Ogden (1984).

We shall consider a homogeneous body B, composed of isotropic elastic material which possesses an initial unstressed configuration B_0. A purely homogeneous static deformation is imposed upon B_0, to produce a finitely deformed equilibrium configuration B_e, which we will use as our reference configuration. Finally, an infinitesimal time-dependent motion is superimposed on B_e, with the resulting configuration termed the current and denoted by B_t. The deformation gradients associated with the deformations $B_0 \to B_e$, $B_e \to B_t$ and $B_0 \to B_t$ are given in component form by

$$\bar{F}_{mA} = \frac{\partial x_m}{\partial X_A}, \qquad \tilde{F}_{im} = \frac{\partial \tilde{x}_i}{\partial x_m}, \qquad F_{iA} = \frac{\partial \tilde{x}_i}{\partial X_A} = \tilde{F}_{im}\bar{F}_{mA}, \tag{1.1}$$

where the tensor $\bar{F}_{mA}$ is constant. We use an over-bar and an over-tilde to denote quantities associated with the equilibrium and the current configurations.

The position vector of a representative particle is denoted by X_A in the initial unstressed isotropic configuration B_0, by $x_i(X_A)$ in a pre-stressed equilibrium state B_e and by $\tilde{x}_i(X_A, t)$ in the final time-dependent configuration B_t, see Figure 1.1. These position vectors are related through

$$\tilde{x}_i(X_A, t) = x_i(X_A) + u_i(x_j, t), \tag{1.2}$$

where $u_i(x_j, t)$ denotes the infinitesimal time-dependent super-imposed motion associated with the secondary deformation $B_e \to B_t$. Such motions are usually referred to as infinitesimal, with all the second and higher terms in the displacement gradient neglected within the expansions used to establish the governing equations. In this thesis, we will specifically consider two-dimensional infinitesimal motions, with $u_3 = 0$, u_1 and u_2 independent of x_3. However, initially we will

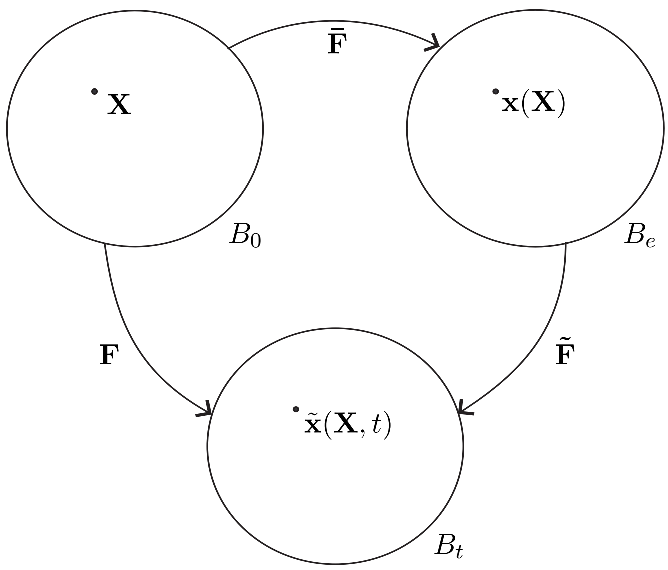


Figure 1.1: *Configurations of a pre-stressed body.*

establish the full three-dimensional equations. With the help of equations (1.1) and (1.2), it is possible to show that components of deformation gradients $\boldsymbol{F}$ and $\bar{\boldsymbol{F}}$ are related by

$$F_{iA} = (\delta_{im} + u_{i,m})\bar{F}_{mA}, \tag{1.3}$$

where the comma indicates differentiation with respect to the implied spatial coordinate component in B_e and δ_{ij} is the Kronecker delta.

We now consider an elastic body composed of incompressible material. This constraint indicates that such materials are only able to undergo isochoric deformations. The requirement of incompressibility may be shown to imply

$$J - 1 = 0\,, \qquad J = \det \mathbf{F}, \tag{1.4}$$

throughout every possible material deformation, for details see for example Chadwick (1999, p. 90) or Ogden (1984, p. 199). In order to accommodate this constraint, we introduce a strain-energy function in the form

$$W(\mathbf{F}, p\,) = W_0(\mathbf{F}) - p\,(J-1)\,. \tag{1.5}$$

The first term on the right hand side of (1.5) generates the constitutive part of the stress, which is determined by the deformation. The second term generates a workless reaction stress and is constrained to be zero throughout every possible material deformation. The scalar p, usually interpreted as a hydro-static pressure, plays the role of a Lagrange multiplier and must ultimately be chosen to satisfy the equations of motion and boundary conditions. We note that the pressure p may be decomposed into a static pressure in B_e, denoted by $\bar{p}$, and a small time dependent increment, denoted by p_t, thus

$$p = \bar{p} + p_t. \tag{1.6}$$

We shall outline the derivation of the equations of time-dependent infinitesimal incremental motion by taking into account the conservation of the linear momentum, see for example Spencer (1980, p. 97). In the absence of a body forces, these equations may be represented as

$$\frac{\partial S_{Ai}}{\partial X_A} = \rho \ddot{u}_i, \tag{1.7}$$

within which $\mathbf{S} = \mathbf{S}(\mathbf{F}, p\,)$ is the nominal stress tensor, ρ is the material density and a dot denotes differentiation with respect to time. Equations (1.7) are evaluated in the natural configuration B_0. If we consider B_e as the reference configuration the equations of motion are better represented in the form

$$\frac{\partial}{\partial x_m}\left(\bar{F}_{mA} S_{Ai}\right) = \rho \ddot{u}_i, \tag{1.8}$$

where $\bar{F}_{mA}$ is the constant deformation gradient tensor associated with the deformation $B_0 \rightarrow B_e$, see equation (1.1).

We expand the nominal stress $\mathbf{S}$ as a Taylor series around the finitely deformed state B_e and then obtain the following expansion

$$S_{Ai} = \bar{S}_{Ai} + (F_{kB} - \bar{F}_{kB}) \frac{\partial S_{Ai}}{\partial F_{kB}}\bigg|_{B_e} + (p - \bar{p}\,) \frac{\partial S_{Ai}}{\partial p}\bigg|_{B_e} + \dots . \tag{1.9}$$

Taking into account the relations (1.3) and (1.6), the formal definition of the concept of small amplitude motion around B_e may be given by

$$\left|F_{kB} - \bar{F}_{kB}\right| = \left|u_{k,l}\bar{F}_{lB}\right| \ll 1\,, \qquad |p - \bar{p}\,| = |p_t| \ll 1\,, \tag{1.10}$$

see for example Ogden (1984, p. 328). The constitutive equations for a finitely deformed incompressible elastic body may be written in terms of the nominal stress and strain-energy function as

$$S_{Ai} = \frac{\partial W}{\partial F_{iA}} = \frac{\partial W_0}{\partial F_{iA}} - pJF^{-1}_{Ai}. \tag{1.11}$$

Equations (1.11) may now be employed to calculate the required derivatives of the components of nominal stress in the expansion (1.9), yielding

$$\frac{\partial S_{Ai}}{\partial p} = -JF^{-1}_{Ai}\,, \qquad \frac{\partial S_{Ai}}{\partial F_{kB}} = \frac{\partial^2 W_0}{\partial F_{kB}\,\partial F_{iA}} - pJ\left(F^{-1}_{Bk}F^{-1}_{Ai} - F^{-1}_{Bi}F^{-1}_{Ak}\right)\,. \tag{1.12}$$

Now the expansion (1.9) may be substituted into the equations of motion (1.8), which together with (1.12) gives, in component form, the *linearized equations of motion* as

$$B_{milk} u_{k,lm} - p_{t,i} = \rho \ddot{u}_i \,, \tag{1.13}$$

where $\mathbf{B}$ is the fourth order elasticity tensor, for which the components are defined by

$$B_{milk} = \bar{F}_{mA} \bar{F}_{lB} \left. \frac{\partial^2 W_0}{\partial F_{kB}\, \partial F_{iA}} \right|_{B_e} . \tag{1.14}$$

The equations of motion (1.13) must be considered in conjunction with the linearized incompressibility condition, which may be shown to be

$$u_{i,i} = 0, \tag{1.15}$$

see for example Spencer (1980, p. 93).

For an isotropic elastic body B, the components of the elasticity tensor $\mathbf{B}$ allow especially simple representation relative to axes coincident with the principal axes of the primary deformation. The only non-zero components of $\mathbf{B}$ then have the form B_{iijj}, B_{ijij} or B_{ijji}, $i, j = 1, 2, 3$, and these may be given in terms of the principal stretches λ_m, $m = 1, 2, 3$, of the static deformation as

$$\begin{aligned}
B_{iijj} &= \lambda_i \lambda_j \frac{\partial^2 W_0}{\partial \lambda_i \partial \lambda_j} \,, \\
B_{ijij} &= \begin{cases} \dfrac{\lambda_i^2}{\lambda_i^2 - \lambda_j^2} \left(\lambda_i \dfrac{\partial W_0}{\partial \lambda_i} - \lambda_j \dfrac{\partial W_0}{\partial \lambda_j} \right), & i \neq j \,, \quad \lambda_i \neq \lambda_j \,, \\ \dfrac{1}{2} \left(B_{iiii} - B_{iijj} + \lambda_i \dfrac{\partial W_0}{\partial \lambda_i} \right), & i \neq j \,, \quad \lambda_i = \lambda_j \,, \end{cases} \\
B_{ijji} &= B_{ijij} - \lambda_i \frac{\partial W_0}{\partial \lambda_i} \qquad i \neq j \,,
\end{aligned} \tag{1.16}$$

within which no summation over repeated suffices is assumed, see Ogden (1984, p. 412).

1.2 Features of the simple shear deformation

1.2.1 The simple shear deformation

In this section we analyze the simple shear deformation. Let (X_1, X_2, X_3) and (x_1, x_2, x_3) be Cartesian coordinates corresponding to the position vector $\mathbf{X}$ in the reference configuration and the position vector $\mathbf{x}$ in a pre-stressed equilibrium configuration. The simple shear deformation is defined by

$$x_1 = X_1 + \epsilon X_2, \quad x_2 = X_2, \quad x_3 = X_3, \tag{1.17}$$

where ϵ is called the amount of shear, see Figure 1.2.

Relative to the basis vectors $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$, where the deformation gradient tensor $\mathbf{F}$ may be written as

$$\mathbf{F} = \mathbf{I} + \epsilon \mathbf{e}_1 \otimes \mathbf{e}_2. \tag{1.18}$$

Here the notation $\mathbf{e}_1 \otimes \mathbf{e}_2$ denotes the *dyadic product* of $\mathbf{e}_1$ and $\mathbf{e}_2$, the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ are orthogonal unit vectors, which together with unit vector $\mathbf{e}_3$ forms an orthonormal basis. The vector $\mathbf{e}_1$ is said to define the *direction of shear* and the planes orthogonal to $\mathbf{e}_2$ and $\mathbf{e}_3$ are called the *glide plane* and *the plane of the shear*, respectively.

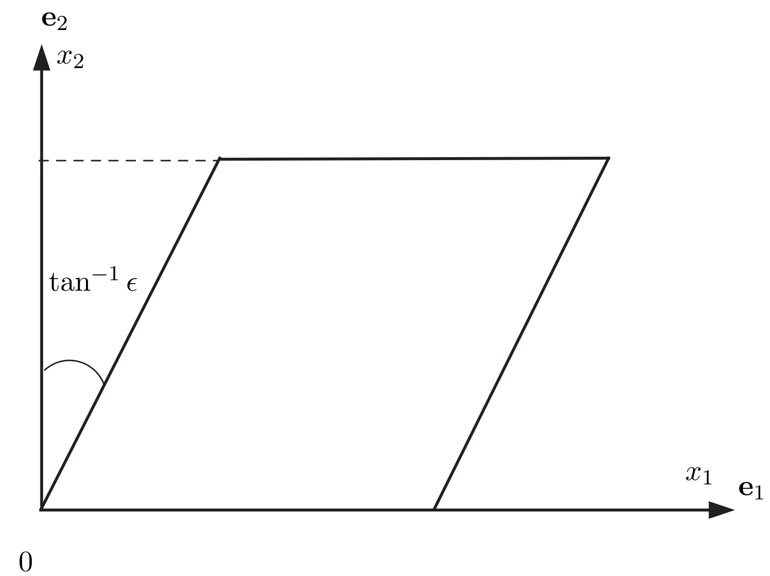


Figure 1.2: *The simple shear deformation.*

According to the polar decomposition theorem, see Ogden (1984, p. 92), there exist unique, symmetric, positive definite right and left stretch tensors $\mathbf{U}$ and $\mathbf{V}$, and unique proper orthogonal rotation tensor $\mathbf{R}$, such that the deformation gradient tensor $\mathbf{F}$ may be expressed as

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR}. \tag{1.19}$$

For the deformation gradient tensor associated with the simple shear deformation (1.17), the matrix of components of $\mathbf{F}$ is given by

$$\boldsymbol{F} = \begin{bmatrix} 1 & \epsilon & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{1.20}$$

Using the polar decomposition theorem (1.19) for the deformation gradient $\mathbf{F}$ we are able to form the following tensor measures of deformation

$$\mathbf{C} = \mathbf{F^T F}, \quad \mathbf{B} = \mathbf{F F^T}, \tag{1.21}$$

where $\mathbf{C}$ and $\mathbf{B}$ referred to the right and left Cauchy-Green deformation tensors, respectively. The tensors $\mathbf{C}$ and $\mathbf{B}$ can be represented in terms of the basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, yielding

$$\begin{aligned} \mathbf{C} &= \mathbf{e}_1 \otimes \mathbf{e}_1 + (1+\epsilon^2)\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3 + \epsilon(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1), \\ \mathbf{B} &= (1+\epsilon^2)\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3 + \epsilon(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1). \end{aligned} \tag{1.22}$$

Taking into account (1.22), the components of the tensors $\mathbf{C}$ and $\mathbf{B}$ may be represented in matrix form through

$$\mathbf{C} = \begin{bmatrix} 1 & \epsilon & 0 \\ \epsilon & 1+\epsilon^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1+\epsilon^2 & \epsilon & 0 \\ \epsilon & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{1.23}$$

The spectral decomposition theorem, see Ogden (1984, p. 27), establishes that $\mathbf{V}$ and $\mathbf{U}$ may be written as

$$\mathbf{U} = \sum_{i=1}^{3} \lambda_i \mathbf{u}_i \otimes \mathbf{u}_i, \quad \mathbf{V} = \sum_{i=1}^{3} \lambda_i \mathbf{v}_i \otimes \mathbf{v}_i, \tag{1.24}$$

where λ_i are the principal stretches of deformation, $\mathbf{u}_i$ are the principal axes of $\mathbf{U}$ and $\mathbf{v}_i$ are the principal axes of $\mathbf{V}$. The simple shear deformation rotates the principal axes of $\mathbf{U}$ into those of $\mathbf{V}$ through a certain angle ψ, such a rotation can be described by a rotation tensor $\mathbf{R}$. The principal axes of $\mathbf{U}$ and $\mathbf{V}$ are usually referred to as the *Lagrangean and Eulerian principal axes*, respectively.

From the polar decomposition theorem (1.19) it follows that

$$\mathbf{C} = \mathbf{F^T F} = \mathbf{U^2}, \quad \mathbf{B} = \mathbf{F F^T} = \mathbf{V^2}. \tag{1.25}$$

The relations (1.24) indicate that the principal stretches λ_1, λ_2 and λ_3 are eigenvalues of the tensors $\mathbf{U}$ and $\mathbf{V}$. It can be shown using (1.25) and the square root theorem, Ogden (2002, p. 16), that λ_1^2, λ_2^2 and λ_3^2 are the eigenvalues of tensors $\mathbf{C}$ and $\mathbf{B}$. Moreover, the eigenvectors of $\mathbf{B}$ and $\mathbf{V}$ coincide and are termed as *Eulerian axes*, similarly the coincident eigenvectors of $\mathbf{C}$ and $\mathbf{U}$ can be termed as *Lagrangean axes.*

In view of incompressibility, we deduce that

$$det\mathbf{F} \equiv \lambda_1\lambda_2\lambda_3 = 1. \tag{1.26}$$

From (1.26) it follows that in the two dimensional case we can represent the principal stretches in terms of one parameter λ, revealing that

$$\lambda_3 = 1, \quad \lambda_1 = \lambda \geq 1, \quad \lambda_2 = \lambda^{-1}. \tag{1.27}$$

Applying the spectral decomposition theorem to $\mathbf{C}$ and $\mathbf{B}$, and using (1.25) together with the square root theorem, Ogden (2002, p. 16), we obtain

$$\mathbf{C} = \sum_{i=1}^{3} \lambda_i^2 \mathbf{u}_i \otimes \mathbf{u}_i, \quad \mathbf{B} = \sum_{i=1}^{3} \lambda_i^2 \mathbf{v}_i \otimes \mathbf{v}_i, \tag{1.28}$$

where the orthonormal triplets $\mathbf{u}_i$ and $\mathbf{v}_i$ specify the Lagrangean and Eulerian axes, respectively. Taking into account (1.27), for our case the representation (1.28) becomes

$$\begin{aligned} \mathbf{C} &= \lambda^2 \mathbf{u}_1 \otimes \mathbf{u}_1 + \lambda^{-2} \mathbf{u}_2 \otimes \mathbf{u}_2 + \mathbf{u}_3 \otimes \mathbf{u}_3, \\ \mathbf{B} &= \lambda^2 \mathbf{v}_1 \otimes \mathbf{v}_1 + \lambda^{-2} \mathbf{v}_2 \otimes \mathbf{v}_2 + \mathbf{v}_3 \otimes \mathbf{v}_3. \end{aligned} \tag{1.29}$$

Since $\mathbf{u}_1, \mathbf{u}_2$ and $\mathbf{v}_1, \mathbf{v}_2$ are orthogonal pairs of unit vectors all orthogonal to $\mathbf{e}_3$, they can be expressed as the linear combination of $\mathbf{e}_1$ and $\mathbf{e}_2$, thus

$$\begin{aligned} \mathbf{u}_1 &= \cos\phi_1 \mathbf{e}_1 + \sin\phi_1 \mathbf{e}_2, \quad \mathbf{u}_2 = -\sin\phi_1 \mathbf{e}_1 + \cos\phi_1 \mathbf{e}_2, \quad (0 < \phi_1 < \pi/2), \\ \mathbf{v}_1 &= \cos\phi_2 \mathbf{e}_1 + \sin\phi_2 \mathbf{e}_2, \quad \mathbf{v}_2 = -\sin\phi_2 \mathbf{e}_1 + \cos\phi_2 \mathbf{e}_2, \quad (0 < \phi_2 < \pi/2). \end{aligned} \tag{1.30}$$

To determine ϕ_1 and ϕ_2 we substitute expressions (1.30) into (1.29) and use (1.22) together with (1.23) to obtain the following relations

$$\begin{aligned} &\lambda^2 \cos^2\phi_1 + \lambda^{-2} \sin^2\phi_1 = 1, \\ &\lambda^2 \sin^2\phi_1 + \lambda^{-2} \cos^2\phi_1 = 1 + \epsilon^2, \\ &(\lambda^2 - \lambda^{-2}) \sin\phi_1 \cos\phi_1 = \epsilon, \end{aligned} \tag{1.31}$$

$$\begin{aligned} &\lambda^2 \cos^2\phi_2 + \lambda^{-2} \sin^2\phi_2 = 1 + \epsilon^2, \\ &\lambda^2 \sin^2\phi_2 + \lambda^{-2} \cos^2\phi_2 = 1, \\ &(\lambda^2 - \lambda^{-2}) \sin\theta_2 \cos\phi_2 = \epsilon. \end{aligned} \tag{1.32}$$

From the relations (1.31)–(1.32) we are able to deduce the following expressions for λ, ϕ_1 and ϕ_2

$$\lambda = \frac{\epsilon}{2} + \frac{\sqrt{4+\epsilon^2}}{2} = \cot\theta, \quad \phi_1 = \frac{\pi}{2} - \theta, \quad \phi_2 = \theta, \tag{1.33}$$

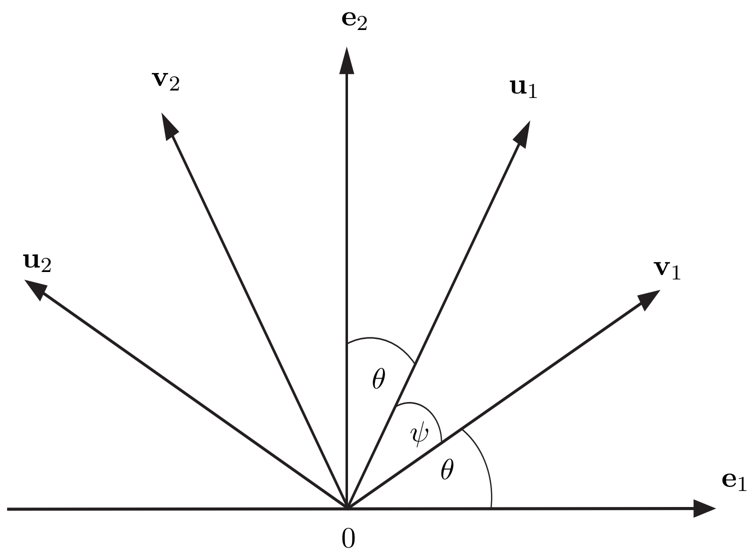


Figure 1.3: *Orientations of the Lagrangean axes* $(\mathbf{u}_1, \mathbf{u}_2)$ *and Eulerian axes* $(\mathbf{v}_1, \mathbf{v}_2)$ *in the basis vector system* $(\mathbf{e}_1, \mathbf{e}_2)$ *of the body for the simple shear deformation. The angle between the Lagrangean and Eulerian axes is given by* $\psi = \frac{\pi}{2} - 2\theta$.

where

$$\theta = \frac{1}{2}\tan^{-1}(\frac{2}{\epsilon}), \quad \tan(2\theta) = \frac{2}{\epsilon}, \quad (0 < \theta < \frac{\pi}{4}). \tag{1.34}$$

Shown in the Figure 1.3 are the Lagrangean and Eulerian axes relative to the base vectors $\mathbf{e}_1$, $\mathbf{e}_2$, taking into account the connections between ϕ_1 and ϕ_2 and θ shown in (1.33). We note that the principal stretches may be represented in terms of θ, thus

$$\lambda_1 = \lambda = \cot\theta, \quad \lambda_2 = \lambda^{-1} = \tan\theta, \quad \lambda_3 = 1. \tag{1.35}$$

Taking into account the incompressibility condition (1.26) and the connection between the angle θ and amount of shear, given in (1.34), the following relations are established

$$\lambda_1^2\lambda_2^2 = 1, \quad \lambda^2 + \lambda^{-2} = 2 + \epsilon^2, \quad \epsilon = \lambda - \lambda^{-1}. \tag{1.36}$$

In addition if we introduce the material parameters

$$\alpha = B_{1212}, \quad \gamma = B_{2121}, \quad 2\beta = B_{1111} + B_{2222} - 2B_{1122} - 2B_{1221}, \tag{1.37}$$

then the following connections are valid

$$\alpha = \gamma\lambda^4, \quad \lambda = \frac{\alpha^{1/4}}{\gamma^{1/4}}, \quad \epsilon = \frac{\lambda^2 - 1}{\lambda} = \frac{\alpha^{1/2} - \gamma^{1/2}}{\alpha^{1/4}\gamma^{1/4}}. \tag{1.38}$$

The unit vectors specifying the Lagrangean axes may now be written in the form

$$\mathbf{u}_1 = \sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_2, \quad \mathbf{u}_2 = -\cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2, \quad \mathbf{u}_3 = \mathbf{e}_3, \tag{1.39}$$

and the Eulerian axes given by

$$\mathbf{v}_1 = \cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2, \quad \mathbf{v}_2 = -\sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_2, \quad \mathbf{v}_3 = \mathbf{e}_3. \tag{1.40}$$

The rotation tensor $\mathbf{R}$ represents a rotation of amount $-\psi = -(\pi/2 - 2\theta)$ about $\mathbf{e}_3$ axis, see Figure 1.3. As the Lagrangean axes are rotated into Eulerian axes we are able to specify a rotation tensor in term of vectors $\mathbf{u}_i$ and $\mathbf{v}_i$

$$\mathbf{R} = \mathbf{v}_i \otimes \mathbf{u}_i = \sin 2\theta(\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2) + \mathbf{e}_3 \otimes \mathbf{e}_3 + \cos 2\theta(\mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1). \tag{1.41}$$

The components of the rotation tensor $\mathbf{R}$ may be represented in matrix form through

$$\boldsymbol{R} = \begin{bmatrix} \sin 2\theta & \cos 2\theta & 0 \\ -\cos 2\theta & \sin 2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{1.42}$$

To conclude this section we remark that instead of using the principal stretches $\lambda_1, \lambda_2, \lambda_3$ as independent measures of deformation, one can introduce the invariants I_1, I_2, I_3 defined as the following symmetric functions of stretches

$$\begin{aligned} I_1 &= tr(\mathbf{B}) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \\ I_2 &= \frac{1}{2}(I_1^2 - tr(\mathbf{B^2})) = \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 + \lambda_1^2\lambda_2^2, \\ I_3 &= det\mathbf{B} = \lambda_1^2\lambda_2^2\lambda_3^2. \end{aligned} \tag{1.43}$$

Taking into account definitions (1.43) and relations (1.35)–(1.36) the invariants I_1, I_2 and I_3 are given by

$$I_1 = I_2 = 3 + \epsilon^2, \quad I_3 = 1, \tag{1.44}$$

emphasizing that the simple shear is an isochoric deformation.

We note from the relations (1.33) and (1.35) that when $\epsilon = 0$, the principal stretches are $\lambda_1 = \lambda_2 = \lambda_3 = 1$.

1.2.2 The relation between Eulerian axes and the natural axes of the layer

In this subsection we introduce the natural coordinate system of the layer to describe the propagation of travelling harmonic wave along its longitudinal direction. A complicating feature of the simple shear deformation is that no principal axis is normal to the faces of the layer. We remark that a similar phenomenon is also possible with anisotropy and with the specific orientation of principal axes relative to a free surface, see for example Fu (2005), Triantafyllidis and Abeyarante (1983). Initially the simplified governing equations were established relative to principal axes of deformation, which corresponds to Eulerian coordinates. Therefore in order to describe wave propagation in a layer we transfer all the equations from Eulerian to natural coordinate system of the layer. The basis vectors for the Eulerian coordinates form an angle θ with the natural coordinate system of the layer, see Figure 1.4. Therefore, the basis vectors of the natural coordinate system of the layer (x_1, x_2) and Eulerian coordinate system (x_1', x_2') are connected via the following relations

$$\mathbf{x} = \mathbf{R}\mathbf{x}', \;\; \mathbf{x}' = \mathbf{R}^T\mathbf{x}, \;\; \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \;\; \mathbf{x}' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}, \;\; \mathbf{R} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{1.45}$$

We consider an incompressible elastic layer of finite thickness h and infinite lateral extent that has origin on its upper face and occupies the domain $-\infty \leq x_1 \leq \infty, -h \leq x_2 \leq 0$. In addition, the layer is subject to primary simple shear deformation, further details may be found in the papers by Connor and Ogden (1996), Destrade and Ogden (2005). It was shown by Destrade and

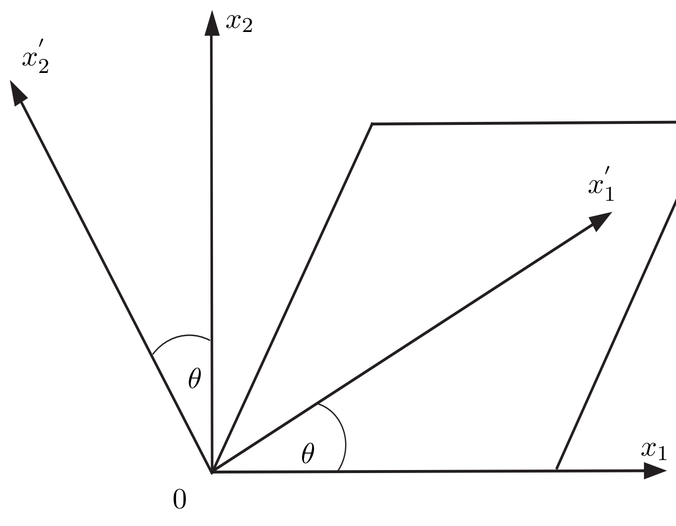


Figure 1.4: *The angle θ between Eulerian coordinates (x_1', x_2') and natural coordinate system of the layer (x_1, x_2).*

Ogden (2005) that within the Eulerian coordinate system (x_1', x_2'), coincident with the principal axes of the primary deformation, there are 6 non zero components of the elasticity tensor $\mathbf{B}$, namely

$$B_{1111}, \quad B_{2222}, \quad B_{1122}, \quad B_{1221}, \quad B_{1212}, \quad B_{2121}. \tag{1.46}$$

In contrast, within the natural coordinate system of the layer (x_1, x_2) there are 10 non zero components of the elasticity tensor $\mathbf{B}$, namely

$$\begin{aligned} &\hat{B}_{1111}, \quad \hat{B}_{2222}, \quad \hat{B}_{1122}, \quad \hat{B}_{1221}, \quad \hat{B}_{1212}, \quad \hat{B}_{2121}, \\ &\hat{B}_{1112}, \quad \hat{B}_{1121}, \quad \hat{B}_{1222}, \quad \hat{B}_{2122}. \end{aligned} \tag{1.47}$$

These two sets of components of the elasticity tensor $\mathbf{B}$ satisfy the symmetry properties

$$B_{ijkl} = B_{klij}, \quad \hat{B}_{ijkl} = \hat{B}_{klij}, \tag{1.48}$$

and are related by

$$\hat{B}_{ijkl} = \Omega_{ip}\Omega_{jq}\Omega_{kr}\Omega_{ls}B_{pqrs}, \tag{1.49}$$

where Ω_{ij} denotes the components of the rotation matrix connecting the natural coordinates of the layer x_i and the Eulerian coordinates x'_i, thus

$$x_i = \Omega_{ij}x'_j, \quad \Omega_{ij} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{1.50}$$

1.3 Linearized equations of motion in the natural coordinate system of the layer

Our interest is long-wave motion in a layer of finite thickness and infinite lateral extent. The layer is composed of an incompressible elastic material and subject to a simple shear primary deformation. Wave propagation in homogeneously pre-stressed rectangular plates has been discussed in detail by a number of authors, see for example Ogden (1984), Rogerson (1997), Rogerson and Fu (1995) in respect of incompressible elastic plates, Roxburg and Ogden (1994), Nolde et al. (2004) for compressible elastic plates and Rogerson (1998) for anisotropic plates. In respect of a half-space subject to a simple shear, wave propagation along a principal plane of deformation was studied by Hussain and Odgen (1999).

In this section, we consider two dimensional motions in the plane of shear defined by coordinates (x_1, x_2). To begin with, for the propagation of a harmonic plane wave within a layer of incompressible elastic material we introduce the strong ellipticity condition to ensure physically realistic response. An important requirement of the equations of motions is that they provide physically realistic response and uniqueness of solution. For our purposes we shall assume that the elasticity tensor satisfies the strong ellipticity condition

$$B_{ijkl}n_j n_l m_i m_k > 0, \tag{1.51}$$

for all non-zero vectors $\mathbf{m}$ and $\mathbf{n}$, such that $\mathbf{m}\cdot\mathbf{n} = 0$, for the further details see Ogden (1984), Dowaikh and Ogden (1990). In this thesis we are concerned with two dimensional motions, therefore in (1.51) we assume that

$$m_3 = n_3 = 0, \quad m_1 = n_2 = \cos\theta, \quad m_2 = -n_1 = \sin\theta, \tag{1.52}$$

equation (1.51) then becomes

$$B_{1212}\sin^4\theta + (B_{1111} + B_{2222} - 2B_{1122} - 2B_{1221})\sin^2\theta\cos^2\theta + B_{2121}\cos^4\theta > 0. \tag{1.53}$$

Necessary and sufficient conditions for (1.53) to be satisfied are provided by

$$B_{1212} > 0, \quad B_{2121} > 0, \quad B_{1111} + B_{2222} - 2B_{1122} - 2B_{1221} > -(B_{1212}B_{2121})^{1/2}. \tag{1.54}$$

If we employ the notation

$$\alpha = B_{1212}, \quad \gamma = B_{2121}, \quad 2\beta = B_{1111} + B_{2222} - 2B_{1122} - 2B_{1221}, \tag{1.55}$$

adopted by Ogden (1984), the strong ellipticity condition (1.54) may be presented in the form

$$\alpha > 0, \quad \gamma > 0, \quad \beta > -\sqrt{\alpha\gamma}. \tag{1.56}$$

The linearized equations of two dimensional motion superimposed on the primary simple shear deformation and an associated linearized incompressibility condition will be established relative to the principal axes of the simple shear deformation, which corresponds to the Eulerian coordinate system. Then the linearized equations of motion and incompressibility condition will be transferred to the natural coordinate system of the layer.

We remark that there are two possible ways to relate all the equations from Eulerian coordinates into the natural coordinates of the layer, either using representations of spatial derivatives of displacement and incremental pressure components given by (1.45) or employing the components of elasticity tensor $\mathbf{B}$ established in (1.49). We choose the first less algebraically complex way.

According to Connor and Ogden (1996), the equations of motion (1.13) take the following form within the Eulerian coordinate system

$$
\begin{aligned}
&(B_{1111} - B_{1122} + \bar{p})u'_{1,11} - p'_{t,1} + B_{2121}u'_{1,22} + (B_{2121} - \sigma_2)u'_{2,12} = \rho\ddot{u}'_1,\\
&(B_{1111} - B_{1122} + \bar{p})u'_{2,22} - p'_{t,2} + B_{1212}u'_{2,11} + (B_{2121} - \sigma_2)u'_{1,21} = \rho\ddot{u}'_2,
\end{aligned} \tag{1.57}
$$

where

$$\bar{p} = B_{2121} - B_{2112} - \sigma_2. \tag{1.58}$$

We remark here that the quantity σ_2 in the above formula (1.58) is a second principal component of Cauchy stress tensor $\mathbf{T} = \mathbf{FS}/J$. The representation (1.58) can be obtained from the general relation

$$\bar{p} = B_{ijij} - B_{ijji} - \sigma_i\,, \qquad i \neq j\,, \qquad i,j = 1,2,3, \tag{1.59}$$

which can be used to represent static pressure $\bar{p}$ in terms of the one of principal Cauchy stress components σ_i, $i = 1,2,3$. We eliminate static pressure $\bar{p}$ in favor of σ_2 to establish the equations of motions in the following form

$$
\begin{aligned}
&(B_{1111} - B_{1122} + B_{2121} - B_{2112} - \sigma_2)u'_{1,11} - p'_{t,1} + B_{2121}u'_{1,22} + (B_{2121} - \sigma_2)u'_{2,12} = \rho\ddot{u}'_1,\\
&(B_{1111} - B_{1122} + B_{2121} - B_{2112} - \sigma_2)u'_{2,22} - p'_{t,2} + B_{1212}u'_{2,11} + (B_{2121} - \sigma_2)u'_{1,21} = \rho\ddot{u}'_2.
\end{aligned} \tag{1.60}
$$

Using formulations (1.45) we relate the spatial derivatives of displacement components and incremental pressure component p_t from the Eulerian coordinate system to the natural coordinate system of the layer. Derivatives of dashed quantities will always be with respect to Eulerian coordinates, whilst non-dashed quantities are always differentials with respect to the natural coordinates of the layer. According to relations (1.45) the displacement components in two coordinate systems are connected through

$$
\begin{aligned}
u_1 &= u'_1\cos\theta - u'_2\sin\theta, & u'_1 &= u_1\cos\theta + u_2\sin\theta,\\
u_2 &= u'_1\sin\theta + u'_2\cos\theta, & u'_2 &= u_2\cos\theta - u_1\sin\theta.
\end{aligned} \tag{1.61}
$$

To relate the spatial derivatives between Eulerian and the layer coordinates we use the following standard formulations

$$
\begin{aligned}
\frac{\partial}{\partial x'_1} &= \cos\theta\frac{\partial}{\partial x_1} + \sin\theta\frac{\partial}{\partial x_2},\\
\frac{\partial}{\partial x'_2} &= -\sin\theta\frac{\partial}{\partial x_1} + \cos\theta\frac{\partial}{\partial x_2},
\end{aligned} \tag{1.62}
$$

$$
\begin{aligned}
\frac{\partial^2}{\partial {x'_1}^2} &= (\cos\theta\frac{\partial}{\partial x_1} + \sin\theta\frac{\partial}{\partial x_2})^2 = \cos^2\theta\frac{\partial^2}{\partial x_1^2} + \sin 2\theta\frac{\partial^2}{\partial x_1\partial x_2} + \sin^2\theta\frac{\partial^2}{\partial x_2^2},\\
\frac{\partial^2}{\partial {x'_2}^2} &= (-\sin\theta\frac{\partial}{\partial x_1} + \cos\theta\frac{\partial}{\partial x_2})^2 = \sin^2\theta\frac{\partial^2}{\partial x_1^2} - \sin 2\theta\frac{\partial^2}{\partial x_1\partial x_2} + \cos^2\theta\frac{\partial^2}{\partial x_2^2},\\
\frac{\partial^2}{\partial x'_1\partial x'_2} &= -\frac{\sin 2\theta}{2}\frac{\partial^2}{\partial x_1^2} + \cos 2\theta\frac{\partial^2}{\partial x_1\partial x_2} + \frac{\sin 2\theta}{2}\frac{\partial^2}{\partial x_2^2}.
\end{aligned}
\tag{1.63}
$$

Using the relations (1.62)–(1.63) the equations of motion (1.60) can be represented in the natural coordinate system of the layer in the following form

$$
\begin{aligned}
&- p_{t,1}\cos\theta - p_{t,2}\sin\theta - \rho\ddot{u}_1\cos\theta - \rho\ddot{u}_2\sin\theta\\
&- \cos\theta\left(\cos^2\theta(\gamma+\mathcal{K}_1) - 2\,\gamma + \sigma_2\right)u_{1,11}\\
&+ \cos\theta\left(\cos^2\theta(\mathcal{K}_1+\gamma) - \mathcal{K}_1\right)u_{1,22}\\
&- \sin\theta\left(2\cos^2\theta(\mathcal{K}_1+\gamma) + \sigma_2 - \gamma\right)u_{1,12}\\
&- \sin\theta\left(\cos^2\theta(\mathcal{K}_1+\gamma) - \gamma\right)u_{2,11}\\
&- \sin\theta\left(\mathcal{K}_1 - \cos^2\theta(\gamma+\mathcal{K}_1) - \gamma + \sigma_2\right)u_{2,22}\\
&+ \cos\theta\left(2\cos^2\theta(\mathcal{K}_1+\gamma) - \gamma - \sigma_2 - 2\,\mathcal{K}_1\right)u_{2,12} = 0,
\end{aligned}
\tag{1.64}
$$

$$
\begin{aligned}
&p_{t,1}\sin\theta - p_{t,2}\cos\theta + \rho\ddot{u}_1\sin\theta - \rho\ddot{u}_2\cos\theta\\
&- \sin\theta\left(\cos^2\theta(\alpha+\mathcal{K}_2) + \gamma - \sigma_2 - \mathcal{K}_2\right)u_{1,11}\\
&+ \sin\theta\left(-\alpha + \cos^2\theta(\mathcal{K}_2+\alpha)\right)u_{1,22}\\
&+ \cos\theta\left(-2\,\alpha + 2\cos^2\theta(\alpha+\mathcal{K}_2) - \sigma_2 - 2\,\mathcal{K}_2 + \gamma\right)u_{1,12}\\
&+ \cos\theta\left(-\mathcal{K}_2 + \cos^2\theta(\mathcal{K}_2+\alpha)\right)u_{2,11}\\
&- \cos\theta\left(\cos^2\theta(\alpha+\mathcal{K}_2) + \sigma_2 - \gamma - \alpha\right)u_{2,22}\\
&+ \sin\theta\left(2\cos^2\theta(\mathcal{K}_2+\alpha) - \gamma + \sigma_2\right)u_{2,12} = 0.
\end{aligned}
\tag{1.65}
$$

The quantities $\mathcal{K}_1$ and $\mathcal{K}_2$ in the formulas (1.64)–(1.65) are given by

$$
\mathcal{K}_1 = B_{2112} + B_{1122} - B_{1111}, \quad \mathcal{K}_2 = B_{2112} + B_{1122} - B_{2222}, \quad \mathcal{K}_1 + \mathcal{K}_2 = -2\beta. \tag{1.66}
$$

The properties of the simple shear deformation dictate that

$$
\cot\theta = \lambda, \quad \sin\theta = \frac{1}{\sqrt{\lambda^2+1}}, \quad \cos\theta = \frac{\lambda}{\sqrt{\lambda^2+1}}. \tag{1.67}
$$

Employing the relations (1.67) the equations of motion (1.64)–(1.65) can be expressed as

$$
\begin{aligned}
&-\lambda(1+\lambda^2)p_{t,1} - (1+\lambda^2)p_{t,2} - \rho\lambda(1+\lambda^2)\ddot{u}_1 - \rho(1+\lambda^2)\ddot{u}_2 \\
&\quad + \lambda\{(\gamma - \mathcal{K}_1 - \sigma_2)\lambda^2 + 2\gamma - \sigma_2\}u_{1,11} - \lambda\{\mathcal{K}_1 - \gamma\lambda^2\}u_{1,22} \\
&\qquad - \{(2\,\mathcal{K}_1 + \sigma_2 + \gamma)\lambda^2 + \sigma_2 - \gamma\}u_{1,12} - \{\mathcal{K}_1\lambda^2 - \gamma\}u_{2,11} \\
&\quad - \{(\sigma_2 - 2\,\gamma)\lambda^2 - \gamma + \mathcal{K}_1 + \sigma_2)\}u_{2,22} \\
&- \lambda\{(\sigma_2 - \gamma)\lambda^2 + 2\mathcal{K}_1 + \sigma_2 + \gamma\})u_{2,12} = 0,
\end{aligned} \tag{1.68}
$$

$$
\begin{aligned}
&(1+\lambda^2)p_{t,1} - \lambda(1+\lambda^2)p_{t,2} + \rho(1+\lambda^2)\ddot{u}_1 - \rho\lambda(1+\lambda^2)\ddot{u}_2 \\
&\quad - \{(\gamma - \sigma_2)(\lambda^2+1) - \mathcal{K}_2 + \alpha\lambda^2\}u_{1,11} - \{\alpha - \mathcal{K}_2\lambda^2\}u_{1,22} \\
&\qquad - \lambda\{(\sigma_2 - \gamma)(\lambda^2+1) + 2(\mathcal{K}_2 + \alpha)\}u_{1,12} + \lambda\{\alpha\lambda^2 - \mathcal{K}_2\}u_{2,11} \\
&- \lambda\{(\sigma_2 - \gamma)(\lambda^2+1) + \mathcal{K}_2\lambda^2 - \alpha\}u_{2,22} \\
&+ \{2(\mathcal{K}_2 + \alpha)\lambda^2 + (\sigma_2 - \gamma)(\lambda^2 - 1)\}u_{2,12} = 0.
\end{aligned} \tag{1.69}
$$

Finally we remark that the linearized incompressibility condition can be represented in the Eulerian coordinates in the following form

$$u'_{1,1} + u'_{2,2} = 0, \tag{1.70}$$

and in the natural coordinates of the layer

$$u_{1,1} + u_{2,2} = 0. \tag{1.71}$$

1.4 Characteristic equation, displacements and incremental pressure in the natural coordinate system of the layer

In this section we will derive a characteristic equation and establish the form of solution for the displacement components u_1, u_2 and incremental pressure component p_t in the natural coordinate system of the layer. To derive a characteristic equation we specify solutions of the linearized equations of motion (1.68)–(1.69), and incompressibility condition (1.71), in the form of travelling harmonic wave

$$(u_1, u_2, p_t) = (U_1, U_2, kP)e^{ikqx_2}e^{ik(vt-x_1)}, \tag{1.72}$$

where k is a wave number, v is a wave speed and the parameter q is to be determined. From the representation (1.72), together with incompressibility condition (1.71), we obtain the following relation between amplitudes of the displacement components

$$U_1 = qU_2. \tag{1.73}$$

We now substitute the relation (1.73) into equations of motion (1.64)–(1.65) to establish the following homogeneous system of two equations in the unknown amplitudes P and U_2

$$c_{11}U_2 + c_{12}P = 0, \quad c_{21}U_2 + c_{22}P = 0, \tag{1.74}$$

$$c_{11} = k^2(C_1^{(3)}q^3 + C_1^{(2)}q^2 + C_1^{(1)}q + C_1^{(0)}), \quad c_{12} = k^3 v\,(q\sin\theta - \cos\theta)\,,$$
$$c_{21} = k^2(C_2^{(3)}q^3 + C_2^{(2)}q^2 + C_2^{(1)}q + C_2^{(0)}), \quad c_{22} = k^3 v\,(q\cos\theta + \sin\theta)\,,$$

$$\begin{aligned}
C_1^{(0)} &= \sin\theta\left(\mathcal{K}_1\cos^2\theta - \gamma\sin^2\theta + \bar{v}^2\right),\\
C_1^{(1)} &= \cos\theta\left(\mathcal{K}_1\cos^2\theta - (3\,\gamma + 2\mathcal{K}_1)\sin^2\theta + \bar{v}^2\right),\\
C_1^{(2)} &= \sin\theta\left(-\mathcal{K}_1\sin^2\theta - (3\gamma + 2\mathcal{K}_1)\cos^2\theta - 2\mathcal{K}_1\cos^2\theta\right),\\
C_1^{(3)} &= \cos\theta\left(\mathcal{K}_1\sin^2\theta - \gamma\cos^2\theta\right),\\
C_2^{(0)} &= \cos\theta\left(\mathcal{K}_2\sin^2\theta + \bar{v}^2 - \alpha\cos^2\theta\right),\\
C_2^{(1)} &= \sin\theta\left((3\,\alpha + 2\,\mathcal{K}_2)\cos^2\theta - \mathcal{K}_2\sin^2\theta - \bar{v}^2\right),\\
C_2^{(2)} &= \cos\theta\left(\mathcal{K}_2\cos^2\theta - (3\,\alpha + 2\mathcal{K}_2)\sin^2\theta\right),\\
C_2^{(3)} &= \sin\theta\left(\alpha\sin^2\theta - K_2\cos^2\theta\right),
\end{aligned} \tag{1.75}$$

where we introduce the notation $\bar{v}^2 = \rho v^2$.

A non-trivial solution of the system (1.74) will exist provided the following fourth order characteristic equation in q

$$\begin{aligned}
&(\alpha\sin^4\theta + 2\beta\sin^2\theta\cos^2\theta + \gamma\cos^4\theta)q^4\\
&+ 4\sin\theta\cos\theta(\gamma\cos^2\theta - \beta\left(\cos^2\theta - \sin^2\theta\right) - \alpha\sin^2\theta)q^3\\
&+ (2\beta + 6\,(\alpha + \gamma - 2\,\beta)\sin^2\theta\cos^2\theta - \bar{v}^2)q^2\\
&+ 4\sin\theta\cos\theta\left(\gamma\sin^2\theta + \beta\left(\cos^2\theta - \sin^2\theta\right) - \alpha\cos^2\theta\right)q\\
&+\gamma\sin^4\theta + 2\beta\sin^2\theta\cos^2\theta + \alpha\cos^4\theta - \bar{v}^2 = 0,
\end{aligned} \tag{1.76}$$

is satisfied. Equation (1.76) was previously derived by Connor and Ogden (1996) employing a stream function. The motivation for our differing approach is to retain the incremental pressure component p_t explicitly within the governing equations. This will later allow comparison of its relative order with the two displacement components within the long wave high and low frequency regimes.

Connor and Ogden (1996) showed that with the help of relations (1.67), and the properties of the simple shear deformation (1.36), the characteristic equation (1.76) can be simplified to the following

$$\begin{aligned} & q^4 - 2\,\epsilon\, q^3 + \left(4\,\delta + 2 + \epsilon^2 - (1+\delta)\,\hat{v}\right) q^2 - 2\,(1 + 2\,\delta)\,\epsilon\, q \\ & + 1 + (1+\delta)\,\epsilon^2 - (1+\delta)\,\hat{v} = 0, \end{aligned} \tag{1.77}$$

where $\hat{v}$ is a non-dimensional squared wave speed and δ is a non-dimensional material parameter, which may be represented as

$$\hat{v} = \frac{\rho v^2}{\sqrt{\alpha\gamma}}, \quad \delta = \frac{\alpha + \gamma - 2\beta}{2(\beta + \sqrt{\alpha\gamma})}. \tag{1.78}$$

We note in passing that strong ellipticity dictates that $\alpha > 0, \gamma > 0, \beta > -\sqrt{\alpha\gamma}$, thus $\delta + 1 > 0$.

There are relations to connect the roots of fourth order polynomial equation with the real coefficients, see for example G. Korn and T. Korn (1968) for details. Accordingly the characteristic equation (1.77) may have four complex roots, two real and two complex roots, or four real roots. Which of these particular cases occur is dependent on the numerical values of the coefficients of (1.77), which in turn are dependent on material parameters, amount of shear and non-dimensional squared wave speed.

Connor and Ogden (1996) showed that the characteristic equation factorizes for a certain class of materials for which $\delta = 0$, implying that $2\beta = \alpha + \gamma$. A neo-Hookean material model can be considered as an example of this class. Employing the condition $\delta = 0$ in the characteristic equation (1.77), we obtain

$$q^4 - 2\,\epsilon\, q^3 + \left(2 + \epsilon^2 - \hat{v}\right) q^2 - 2\,\epsilon\, q + 1 + \epsilon^2 - \hat{v} = 0, \tag{1.79}$$

with the four explicit roots given by

$$q_1 = i, \quad q_2 = -i, \quad q_3 = \epsilon + i\kappa, \quad q_4 = \epsilon - i\kappa, \quad \kappa^2 = 1 - \hat{v}. \tag{1.80}$$

We remark that when κ is real ($\hat{v} < 1$), four complex roots exist and when κ is imaginary ($\hat{v} > 1$) two complex and two real exist, however the case of four real roots is not possible.

As the determinant (1.76) of coefficients of homogeneous system (1.74) vanishes it is possible to express the amplitude P in terms of U_2. Below we specify a representation of P to recover, in the next section, results for the components of incremental traction obtained by Connor and Ogden (1996). To do this we multiply equation $(1.74)_1$ by $\cos\theta$ and subtract it from equation $(1.74)_2$ multiplied by $\sin\theta$, yielding

$$
\begin{aligned}
& P = U_2 \frac{C_3^{(p)} q^3 + C_2^{(p)} q^2 + C_1^{(p)} q + C_0^{(p)}}{kv} \\
& C_3^{(p)} = \left(\sin^2\theta\,(\mathcal{K}_2 + \alpha) + \mathcal{K}_1\right)\cos^2\theta - \alpha\sin^2\theta - (\mathcal{K}_1 + \gamma)\cos^4\theta \\
& C_2^{(p)} = \sin\theta\cos\theta((2\,\mathcal{K}_2 + 3\alpha + \mathcal{K}_1) - 3\,(\gamma + \mathcal{K}_2 + \mathcal{K}_1 + \alpha)\cos^2\theta) \\
& C_1^{(p)} = 3\,(\mathcal{K}_1 + \gamma)\cos^4\theta - \left(3\,(\alpha + \mathcal{K}_2)\sin^2\theta + 2\,\mathcal{K}_1 + 3\,\gamma - \bar{v}^2\right)\cos^2\theta \\
& + \left(\mathcal{K}_2 + \bar{v}^2\right)\sin^2\theta \\
& C_0^{(p)} = \sin\theta\cos\theta((\mathcal{K}_2 + \gamma + \mathcal{K}_1 + \alpha)\cos^2\theta - (\mathcal{K}_2 + \gamma)).
\end{aligned} \tag{1.81}
$$

If we make use of relations (1.67) in the representation (1.81), then the amplitude P can be simplified

$$
\begin{aligned}
& P = \mathcal{P}(q) U_2, \quad \mathcal{P}(q) = \frac{\mathcal{P}^{(3)} q^3 + \mathcal{P}^{(2)} q^2 + \mathcal{P}^{(1)} q + \mathcal{P}^{(0)}}{(\lambda^2 + 1)^2 kv}, \\
& \mathcal{P}^{(3)} = -\lambda^4\gamma - 2\,\beta\,\lambda^2 - \alpha, \\
& \mathcal{P}^{(2)} = (\mathcal{K}_1 + 3\,\alpha + 2\mathcal{K}_2 - (2\mathcal{K}_1 + \mathcal{K}_2 + 3\gamma)\lambda^2)\lambda, \\
& \mathcal{P}^{(1)} = (\bar{v}^2 + \mathcal{K}_1)\lambda^4 - (3\gamma - 4\beta - 2\bar{v}^2 + 3\alpha)\lambda^2 + \mathcal{K}_2 + \bar{v}^2, \\
& \mathcal{P}^{(0)} = ((\alpha + \mathcal{K}_1)\lambda^2 - \mathcal{K}_2 + \gamma))\lambda.
\end{aligned} \tag{1.82}
$$

With the help of the relations (1.72)–(1.73) and (1.82) the solutions for two displacement components u_1, u_2 and incremental pressure component p_t may now be expressed as the following linear combination of four solutions corresponding to four roots of equation (1.77)

$$
\begin{aligned}
& u_1 = (\sum_{j=1}^{4} q_j A_j e^{ikq_j x_2}) e^{ik(vt - x_1)}, \quad u_2 = (\sum_{j=1}^{4} A_j e^{ikq_j x_2}) e^{ik(vt - x_1)}, \\
& p_t = k(\sum_{j=1}^{4} \mathcal{P}(q_j) A_j e^{ikq_j x_2}) e^{ik(vt - x_1)},
\end{aligned} \tag{1.83}
$$

where $A_j, j = 1, 2, 3, 4$ are the constants.

1.5 Components of linearized incremental traction in the natural coordinate system of the layer

In this section we obtain components of incremental traction associated with the upper and lower faces of the layer. We begin with a derivation of a linearized measure of incremental traction, for further details see Ogden (1984), Dowaikh and Ogden (1990) and Rogerson (1997).

Let us consider an elementary surface $(dS, \mathbf{N})$ with area dS and outward unit normal $\mathbf{N}$ in the unstressed configuration B_0. The deformations $\bar{\mathbf{F}}$ and $\tilde{\mathbf{F}}$ are then imposed on B_0, transforming $(dS, \mathbf{N})$ into $(d\bar{s}, \bar{\mathbf{n}})$ at B_e and finally $(ds, \mathbf{n})$ at B_t, see Figure 1.5. The contact force on the

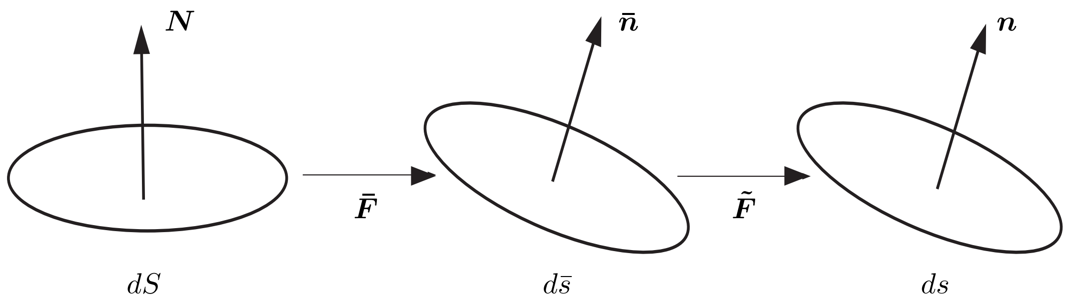


Figure 1.5: *Deformation of an elementary surface.*

surface $(d\bar{s}, \bar{\mathbf{n}})$ is given by $\bar{\sigma}^T \bar{\mathbf{n}} d\bar{s}$ and the contact force on $(ds, \mathbf{n})$ is given by $\sigma^T \mathbf{n} ds$, hence we consider the increment of contact force associated with the secondary deformation $B_e \to B_t$ written as

$$\Delta \mathbf{f}_c = \sigma^T \mathbf{n} ds - \bar{\sigma}^T \bar{\mathbf{n}} d\bar{s} \,. \tag{1.84}$$

By considering the volumes formed by the elementary surfaces and their associated normals before and after deformation, we use Nanson's formula, see Ogden (1984, p. 88), which in our case takes the form

$$\mathbf{N} dS = \bar{\mathbf{F}}^T \bar{\mathbf{n}} d\bar{s} = \bar{\mathbf{F}}^T \tilde{\mathbf{F}}^T \mathbf{n} ds = \mathbf{F}^T \mathbf{n} ds \,. \tag{1.85}$$

We can now use (1.85) to represent increment of contact force (1.84) with respect to the surface in the statically deformed configuration B_e, yielding

$$\Delta \mathbf{f}_c = \left(\left(\tilde{\mathbf{F}}^{-1} \sigma \right)^T - \bar{\sigma}^T \right) \bar{\mathbf{n}} \; d\bar{s} = \boldsymbol{\tau}_{\bar{\boldsymbol{n}}} d\bar{s} \,, \tag{1.86}$$

where $\boldsymbol{\tau}_{\bar{\boldsymbol{n}}}$ is the increment of surface traction associated with the deformation $B_e \to B_t$. We may now use a connection between the Cauchy and nominal stress, see Chadwick (1999, p. 99), to conclude

$$\bar{\boldsymbol{\sigma}}^T \bar{\mathbf{n}} d\bar{s} = \bar{\mathbf{S}}^T \mathbf{N} dS \,, \qquad \boldsymbol{\sigma}^T \mathbf{n} ds = \mathbf{S}^T \vec{N} dS. \tag{1.87}$$

Equation (1.87) with use of (1.85) and (1.86), gives, when referred to the equilibrium configuration

B_e a measure of the incremental surface traction in the form

$$\boldsymbol{\tau}_{\bar{\boldsymbol{n}}} = \left(\mathbf{S}^T - \bar{\mathbf{S}}^T\right)\bar{\mathbf{F}}^T\bar{\mathbf{n}}\,, \tag{1.88}$$

with $\bar{\mathbf{n}}$ assumed to be the outward normal to the surface in the equilibrium configuration B_e. Once again expanding the nominal stress tensor as a Taylor series about the equilibrium configuration B_e, we obtain a representation of the surface traction increment in coordinate form

$$\tau_{\bar{n}_i} = B_{milk}u_{k,l}\bar{n}_m + \bar{p}\,u_{m,i}\bar{n}_m - p_t\bar{n}_i, \tag{1.89}$$

Taking into account the Cauchy stress $\mathbf{T} = \mathbf{FS}/J$, and in view of the relations (1.5) and (1.11), we obtain the following representation of incremental pressure component $\bar{p}$

$$\bar{p} = B_{ijij} - B_{ijji} - \sigma_i\,, \qquad i \neq j\,, \qquad i,j = 1,2,3, \tag{1.90}$$

which we will use to eliminate $\bar{p}$ in favor of one of the principal Cauchy stress components σ_i, $i = 1,2,3$. In our two dimensional case we apply the general formula (1.89) with the following relations

$$i = 1,2, \quad \bar{p} = B_{2121} - B_{2112} - \sigma_2, \quad \bar{n}_1 = \sin\theta, \quad \bar{n}_2 = \cos\theta. \tag{1.91}$$

Relative to the principal axes of the simple shear deformation, the Eulerian coordinate system, in the two dimensional case the components of incremental traction associated with the upper and lower faces of the layer are therefore given by

$$\begin{aligned}
\tau_1' &= (B_{1111}u_{1,1}' + B_{1122}u_{2,2}')\sin\theta + (B_{2112}u_{2,1}' + B_{2121}u_{1,2}')\cos\theta + \bar{p}(u_{1,1}'\sin\theta + u_{2,1}'\cos\theta) \\
&\quad - p_t\sin\theta, \\
\tau_2' &= (B_{1212}u_{2,1}' + B_{1221}u_{1,2}')\sin\theta + (B_{2211}u_{1,1}' + B_{2222}u_{2,2}')\cos\theta + \bar{p}(u_{1,2}'\sin\theta + u_{2,2}'\cos\theta) \\
&\quad - p_t\cos\theta.
\end{aligned} \tag{1.92}$$

We will relate the components of incremental traction (1.92) to the natural coordinate system of the layer. Using equation (1.62) the components of incremental traction within the natural coordinate system of the layer are given by

$$
\begin{aligned}
\tau_1 = & -p_t \sin\theta + \sin\theta\cos^2\theta\,(B_{1111} - B_{1221})\,u_{1,1} \\
& + \sin\theta\,\big(B_{1122} - \cos^2\theta(B_{1122} + \gamma)\big)\,u_{1,1} \\
& + \sin\theta\,\big(\gamma + B_{1111} + B_{1122}\cos^2\theta - B_{1221} - \sigma_2\big)\,u_{2,2} \\
& + \sin\theta\cos^2\theta\,(\gamma + B_{1221} - B_{1111})\,u_{2,2} \\
& + \cos\theta\,\big(B_{1111} - B_{1221} + \cos^2\theta(B_{1221} + \gamma)\big)\,u_{1,2} \\
& + \cos\theta\,\big(\cos^2\theta(B_{1122} - B_{1111}) - B_{1122}\big)\,u_{1,2} \\
& + \cos\theta\,\big(B_{1221}\cos^2\theta - B_{1221} - \sigma_2 - B_{1122}\big)\,u_{2,1} \\
& + \cos\theta\,\big(\cos^2\theta(B_{1122} - B_{1111} + \gamma) + B_{1111}\big)\,u_{2,1},
\end{aligned}
\tag{1.93}
$$

$$
\begin{aligned}
\tau_2 = & -p_t \cos\theta + \cos\theta\,\big(B_{2222} - B_{1221} + \cos^2\theta(B_{1122} - B_{2222})\big)\,u_{1,1} \\
& + \cos\theta\,\big(\cos^2\theta(\alpha + B_{1221}) - \alpha\big)\,u_{1,1} \\
& + \cos\theta\,\big(B_{2222}\cos^2\theta + B_{1122} - \sigma_2 + \alpha\big)\,u_{2,2} \\
& + \cos\theta\,\big(\gamma - (B_{1221} + \alpha + B_{1122})\cos^2\theta\big)\,u_{2,2} \\
& + \sin\theta\cos^2\theta\,(B_{1221} - B_{2222} + \alpha)\,u_{1,2} \\
& + \sin\theta\,\big(B_{1122}\cos^2\theta - \alpha\big)\,u_{1,2} \\
& + \sin\theta\cos^2\theta\,(B_{1221} - B_{2222} + B_{1122})\,u_{2,1} \\
& + \sin\theta\,\big(\alpha\cos^2\theta - \gamma + \sigma_2\big)\,u_{2,1}.
\end{aligned}
\tag{1.94}
$$

With the help of relations (1.67), the components of incremental traction (1.93)–(1.94) can be established in the following form

$$
\begin{aligned}
\tau_1(\lambda^2+1)^{5/2} = & -\big(\lambda^2+1\big)^2 p_t \\
& + \big((B_{1111} - B_{1221} - \gamma)\,\lambda^4 + (2B_{1122} - \mathcal{K}_1 - \gamma)\,\lambda^2 + B_{1122}\big)\,u_{1,1} \\
& + \big((2\,\gamma - \sigma_2 + B_{1122})\,\lambda^2 - \mathcal{K}_1 - 2\,\sigma_2 + 3\,\gamma + 2\,B_{1122}\big)\,\lambda^2 u_{2,2} \\
& + (\gamma + B_{1122} - \sigma_2 - \mathcal{K}_1)\,u_{2,2} + \big(\lambda^4\gamma + (\gamma - K_1)\,\lambda^2 - \mathcal{K}_1\big)\,\lambda u_{1,2} \\
& + \big((\gamma - \sigma_2)\,\lambda^4 + (\gamma - 2\,\sigma_2 - \mathcal{K}_1)\,\lambda^2 - \mathcal{K}_1 - \sigma_2\big)\,\lambda u_{2,1},
\end{aligned}
\tag{1.95}
$$

$$
\begin{aligned}
&\tau_2(\lambda^2+1)^{5/2} = -\lambda\left(\lambda^2+1\right)^2 p_t \\
&+\left(B_{1122}\lambda^4+\left(2\,B_{1122}-\mathcal{K}_2-\alpha\right)\lambda^2-\alpha+B_{1122}-\mathcal{K}_2\right)\lambda u_{1,1} \\
&+\left(\left(\gamma+B_{1122}-\sigma_2-\mathcal{K}_2\right)\lambda^4-\sigma_2+\alpha+\gamma+B_{1122}\right)\lambda u_{2,2} \\
&+\left(\alpha+2\,\gamma-\mathcal{K}_2-2\,\sigma_2+2\,B_{1122}\right)\lambda^3 u_{2,2}+\left(\mathcal{K}_2\lambda^4-\alpha\left(\mathcal{K}_2-\alpha\right)\lambda^2-\alpha\right)u_{1,2} \\
&+\left(\left(\alpha+\mathcal{K}_2+\sigma_2-\gamma\right)\lambda^4+\left(\mathcal{K}_2+\alpha+2(\sigma_2-\gamma)\right)\lambda^2+\sigma_2-\gamma\right)u_{2,1}.
\end{aligned} \tag{1.96}
$$

The components of incremental traction (1.95)–(1.96) are represented in terms of incremental pressure component p_t and spatial derivatives of displacement components u_1, u_2. Solutions for u_1, u_2 and p_t are established in the form of the travelling wave (1.72). Hence we can represent components of incremental traction in a representation similar to (1.72),

$$
(\tau_1, \tau_2) = (T_1^{(a)}, T_2^{(a)})e^{ikqx_2}e^{ik(vt-x_1)}. \tag{1.97}
$$

If we use relations (1.72) and (1.81) together with formulations (1.93)–(1.94) the amplitudes $T_1^{(a)}, T_2^{(a)}$ are expressible as

$$
\begin{aligned}
T_1^{(a)} &= U_2(C_3^{(1)}q^3+C_2^{(1)}q^2+C_1^{(1)}q+C_0^{(1)}), \\
T_2^{(a)} &= U_2(C_3^{(2)}q^3+C_2^{(2)}q^2+C_1^{(2)}q+C_0^{(2)}),
\end{aligned} \tag{1.98}
$$

$$
\begin{aligned}
C_3^{(1)} &= \sin\theta((\cos^2\theta-1)^2\alpha+\cos^4\theta\gamma+\cos^2\theta(1-\cos^2\theta)2\beta), \\
C_2^{(1)} &= \cos\theta(\cos^2\theta(4-3\cos^2\theta)\gamma+2\beta(3\cos^4\theta-5\cos^2\theta+2)-3\alpha(\cos^2\theta-1)^2), \\
C_1^{(1)} &= \sin\theta((3\cos^2\theta(1-\cos^2\theta)\alpha+(5\cos^2\theta+1-3\cos^4\theta)\gamma) \\
&+\sin\theta((3\cos^4\theta-4\cos^2\theta+1)2\beta-\bar{v}^2-\sigma_2), \\
C_0^{(1)} &= \cos\theta(\cos^2\theta(\cos^2\theta-1)\alpha+(\cos^4\theta-3\cos^2\theta+1)\gamma+\sigma_2-(\cos^2\theta-1)^2 2\beta),
\end{aligned} \tag{1.99}
$$

$$
\begin{aligned}
C_3^{(2)} &= \cos\theta((\cos^2\theta-1)^2\alpha+\cos^4\theta\gamma+2\beta\cos^2\theta(1-\cos^2\theta)), \\
C_2^{(2)} &= \sin\theta((3\cos^4\theta-2\cos^2\theta-1)\alpha+3\cos^4\theta\gamma+2\beta\cos^2\theta(1-3\cos^2\theta)), \\
C_1^{(2)} &= \cos\theta((\cos^2\theta+2-3\cos^4\theta)\alpha+(1+3\cos^2\theta-3\cos^4\theta)\gamma) \\
&+\cos\theta(\cos^2\theta(3\cos^2\theta-2)2\beta-\bar{v}^2-\sigma_2), \\
C_0^{(2)} &= \sin\theta((1+\cos^2\theta-\cos^4\theta)\gamma-\cos^2\theta(1+\cos^2\theta)\alpha+2\beta\cos^4\theta-\sigma_2).
\end{aligned} \tag{1.100}
$$

Employing relations (1.67) in formulations (1.98), with the coefficients (1.99)–(1.100), the simplified amplitudes of the incremental traction components can be written as the following

$$T_1^{(a)} = U_2\mathcal{C}\mathcal{T}_1(q), \qquad \mathcal{T}_1(q) = \mathcal{T}_1^{(3)}q^3 + \mathcal{T}_1^{(2)}q^2 + \mathcal{T}_1^{(1)}q + \mathcal{T}_1^{(0)},$$
$$T_2^{(a)} = U_2\mathcal{C}\mathcal{T}_2(q), \qquad \mathcal{T}_2(q) = \mathcal{T}_2^{(3)}q^3 + \mathcal{T}_2^{(2)}q^2 + \mathcal{T}_2^{(1)}q + \mathcal{T}_2^{(0)}, \tag{1.101}$$

$$\mathcal{T}_1^{(3)} = \gamma\lambda^4 + 2\,\beta\,\lambda^2 + \alpha, \quad \mathcal{T}_1^{(2)} = (\lambda^4\gamma - (2\beta - 4\gamma)\lambda^2 + 4\beta - 3\alpha)\lambda,$$
$$\mathcal{T}_1^{(1)} = (3\gamma - \sigma_2 - \bar{v}^2)\lambda^4 + (3\alpha - 2\bar{v}^2 - 2\sigma_2 + 7\gamma - 4\beta)\lambda^2 + 2\beta - \bar{v}^2 + \gamma - \sigma_2,$$
$$\mathcal{T}_1^{(0)} = ((\sigma_2 - \gamma)\lambda^4 + (2\sigma_2 - \alpha - \gamma)\lambda^2 - 2\beta + \gamma + \sigma_2)\lambda,$$
$$\mathcal{T}_2^{(3)} = (\gamma\lambda^4 + 2\,\beta\,\lambda^2 + \alpha)\lambda, \quad \mathcal{T}_2^{(2)} = (3\gamma - 4\beta)\lambda^4 + (2\beta - 4\alpha)\lambda^2 - \alpha,$$
$$\mathcal{T}_2^{(1)} = ((2\beta - \sigma_2 + \gamma - \bar{v}^2)\lambda^4 - (4\beta + 2\sigma_2 - 5\alpha + 2\bar{v}^2 - 5\gamma)\lambda^2 + 2\alpha - \bar{v}^2 + \gamma - \sigma_2)\lambda,$$
$$\mathcal{T}_2^{(0)} = (2\beta - 2\alpha + \gamma - \sigma_2)\lambda^4 + (3\gamma - \alpha - 2\sigma_2)\lambda^2 - \sigma_2 + \gamma, \tag{1.102}$$

where notations $\mathcal{C}$ represent the following constant

$$\mathcal{C} = (\lambda^2 + 1)^{-5/2}. \tag{1.103}$$

With the help of relations (1.97), (1.101)–(1.102) we now ready to establish solutions for the incremental traction components τ_1 and τ_2 as a linear combination of four solutions corresponding to four roots of equation (1.77), thus

$$\tau_1 = \mathcal{C}(\sum_{j=1}^{4} \mathcal{T}_1(q_j)A_j e^{ikq_jx_2})e^{ik(vt-x_1)}, \quad \tau_2 = \mathcal{C}(\sum_{j=1}^{4} \mathcal{T}_2(q_j)A_j e^{ikq_jx_2})e^{ik(vt-x_1)}, \tag{1.104}$$

where the functions $\mathcal{T}_1(q_j)$ and $\mathcal{T}_2(q_j)$ are given by (1.101) with the coefficients (1.102), constants $A_j, j = 1, 2, 3, 4$ are the same as in representation (1.83) and the constant $\mathcal{C}$ is given by equation (1.103).

At this stage it is possible to demonstrate that the amplitudes $T_1^{(a)}, T_2^{(a)}$ of the incremental traction components (1.98), with coefficients given by (1.99)–(1.100), and the amplitudes of the incremental traction components calculated by Connor and Ogden (1996) are equivalent. According to Connor and Ogden (1996) the amplitudes of the incremental traction components are given by

$$t_{11} = (F(q_1)A_1e^{ikq_1x_2} + F(q_2)A_2e^{ikq_2x_2} + F(q_3)B_1e^{ikq_3x_2} + F(q_4)B_2e^{ikq_4x_2})e^{ik(vt-x_1)},$$
$$t_{21} = (G(q_1)A_1e^{ikq_1x_2} + G(q_2)A_2e^{ikq_2x_2} + G(q_3)B_1e^{ikq_3x_2} + G(q_4)B_2e^{ikq_4x_2})e^{ik(vt-x_1)}. \tag{1.105}$$

Here A_i, B_i, $i=1,2$ are the constants and the functions $F(q)$ and $G(q)$ are the following

$$
\begin{aligned}
&F(q) = \left((\gamma-\sigma_2)\, m^3 - \gamma\, mn^2\right)\cos^2\theta \\
&+ \left(2\,\gamma\, n^3 - 2\,(\gamma-\sigma_2)\, m^2 n - \bar{v}^2 n\right)\sin\theta\cos\theta \\
&+ \left(\alpha\, m^3 + (2\,\beta+\gamma-\sigma_2)\, mn^2 - \bar{v}^2 m\right)\sin^2\theta, \\
&G(q) = \left(\gamma\, n^3 + (2\,\beta+\gamma-\sigma_2)\, m^2 n - \bar{v}^2 n\right)\cos^2\theta \\
&+ \left(2\,\alpha\, m^3 - 2\,(\gamma-\sigma_2)\, mn^2 - \bar{v}^2 m\right)\sin\theta\cos\theta \\
&+ \left((\gamma-\sigma_2)\, n^3 - \alpha\, m^2 n\right)\sin^2\theta, \\
&m = q\sin\theta - \cos\theta, \quad n = q\cos\theta + \sin\theta. \qquad (1.106)
\end{aligned}
$$

If we collect terms with the same power of q in (1.106) and multiply both $F(q), G(q)$ by the constant U_2, we obtain the amplitudes of incremental traction components established in (1.98) with the coefficients given by (1.99)–(1.100) and the following connection is valid

$$
T_1^{(a)} = U_2 F(q), \quad T_2^{(a)} = U_2 G(q). \qquad (1.107)
$$

We remark that with the help of relations (1.36), (1.67) and (1.78) Connor and Ogden (1996) represented (1.106) in the following simplified form

$$
F(q) = \sqrt{\alpha\gamma}\sin(\theta) f(q), \quad G(q) = \sqrt{\alpha\gamma}\cos(\theta) g(q), \qquad (1.108)
$$

$$
\begin{aligned}
f(q) &= \frac{(q-\lambda)\left(q+\lambda^{-1}\right)^2}{1+\delta} + 3\,q - \lambda + 2\,\lambda^{-1} + p\,(q-\lambda) - \hat{v}\,q, \\
g(q) &= \frac{(q-\lambda)^2\left(q+\lambda^{-1}\right)}{1+\delta} + 3\,q - 2\,\lambda + \lambda^{-1} + p\left(q+\lambda^{-1}\right) - \hat{v}\,q, \qquad (1.109)
\end{aligned}
$$

where σ_2 is eliminated in favor of p with the help of the following notation

$$
p = \frac{\gamma-\sigma_2}{\sqrt{\alpha\gamma}}. \qquad (1.110)
$$

Therefore with the help of relations (1.97), (1.107)–(1.109) we can represent the solutions for incremental traction components (1.104) in the following form

$$
\tau_1 = \mathcal{C}_1 \sum_{j=1}^{4} (f(q_j) A_j e^{ikq_j x_2}) e^{ik(vt-x_1)}, \quad \tau_2 = \mathcal{C}_2 \sum_{j=1}^{4} (g(q_j) A_j e^{ikq_j x_2}) e^{ik(vt-x_1)}, \qquad (1.111)
$$

where the functions $f(q_j)$ and $g(q_j)$ are given by equations (1.109), the constants $A_j, j=1,2,3,4$ are the same as in representation (1.83) and the constants $\mathcal{C}_1, \mathcal{C}_2$ are the following

$$
\mathcal{C}_1 = \frac{\gamma\lambda^4}{\sqrt{\lambda^2+1}}, \quad \mathcal{C}_2 = \frac{\gamma\lambda^5}{\sqrt{\lambda^2+1}}. \qquad (1.112)
$$

1.6 Some specific models for rubber-like materials subject to simple shear

In this section neo-Hookean and Varga material models are considered as examples of rubber-like materials. In the later chapters of this thesis we employ these strain energy functions to produce illustrative numerical calculations.

We remark that for an incompressible elastic layer subject to the primary simple shear deformation (1.36) in the two dimensional case the general form of any strain-energy function can be simplified

$$W(\lambda_1,\lambda_2,\lambda_3)=W^{(s)}(\lambda,\lambda^{-1},1)=\bar{W}(\epsilon). \tag{1.113}$$

In the case of a neo-Hookean material model the strain-energy function is given by

$$W_{(n)}=\frac{\mu}{2}\left({\lambda_1}^2+{\lambda_2}^2+{\lambda_3}^2-3\right)-p(\lambda_1\lambda_2\lambda_3-1). \tag{1.114}$$

Taking into account principal stretches of a two dimensional simple shear deformation (1.35) the following simplification of strain-energy function (1.114) becomes possible

$$W^{(s)}_{(n)}=\frac{\mu}{2}(\lambda^2+\lambda^{-2}-2). \tag{1.115}$$

Using relations (1.36) we can express the strain-energy function (1.115) in term of amount of shear ϵ

$$W^{(e)}_{(n)}=\frac{\mu\epsilon^2}{2}. \tag{1.116}$$

For the considered material model the non-zero components of the elasticity tensor $\mathbf{B}$ are given by

$$B_{1111}=B_{1212}=\mu\lambda_1^2=\mu\lambda^2=\alpha,\quad B_{2222}=B_{2121}=\mu\lambda_2^2=\frac{\mu}{\lambda^2}=\gamma, \tag{1.117}$$

and the material constants α,β,γ take the form

$$\alpha=\mu\lambda_1^2=\mu\lambda^2,\quad \gamma=\mu\lambda_2^2=\mu\lambda^{-2},\quad 2\beta=\mu(\lambda_1^2+\lambda_2^2)=\frac{\mu(\lambda^4+1)}{\lambda^2}. \tag{1.118}$$

From the above we conclude that for a neo-Hookean material

$$2\beta=\alpha+\gamma,\quad \delta=0. \tag{1.119}$$

With the help of the relations (1.117)–(1.119) and previously obtained connections (1.61),(1.62)–(1.63) we represent the equations of motion and components of incremental traction in the natural coordinate system of layer. Hence, for the case of the neo-Hookean strain energy function the equations of motion within the natural coordinate system of the layer take the form

$$\begin{aligned}
&\lambda^2 p_{t,1} + \lambda p_{t,2} + \rho\lambda^2 \ddot{u}_1 + \rho\lambda \ddot{u}_2 - \mu\left(\lambda^4 + \lambda^2(p-1) + 1\right)(u_{1,11} + u_{2,11}) \\
&- \mu\lambda\left(\lambda^2(p-2) + 2\right)(u_{1,12} + u_{2,12}) - \mu\lambda^2 u_{1,22} - \mu\lambda\,(1+p)\,u_{2,22} = 0, \\
&\lambda^2 p_{t,1} - \lambda^3 p_{t,2} + \rho\lambda^2 \ddot{u}_1 - \rho\lambda^3 \ddot{u}_2 - \mu\left(\lambda^4 + \lambda^2(p-1) + 1\right)(u_{1,11} + u_{2,11}) \\
&+ \mu\lambda\left(\lambda^2(p-2) + 2\right)(u_{1,12} + u_{2,12}) - \mu\lambda^2 u_{1,22} + \mu\lambda^3\,(1+p)\,u_{2,22} = 0,
\end{aligned} \tag{1.120}$$

with the components of incremental traction

$$\begin{aligned}
\tau_1 &= \frac{(\lambda^2+1)}{\lambda}(\mu(\lambda^2(\lambda^2-1)u_{1,1} + \lambda^2 u_{1,2} + (\lambda^2(p+1)-1)u_{2,1} + \lambda(p+1)u_{2,2}) - \lambda p_t), \\
\tau_2 &= \frac{(\lambda^2+1)}{\lambda}(\mu((\lambda^2-1)u_{1,1} - \lambda u_{1,2} + (\lambda^2-(p+1))u_{2,1} + \lambda^2(p+1)u_{2,2}) - \lambda p_t).
\end{aligned} \tag{1.121}$$

In case of a Varga material the strain-energy function is given by

$$W_{(v)} = \mu(\lambda_1 + \lambda_2 + \lambda_3 - 3) - p(\lambda_1\lambda_2\lambda_3 - 1). \tag{1.122}$$

Taking into account principal stretches of a simple shear deformation (1.35) the following simplification of strain-energy function (1.122) becomes possible

$$W_{(v)}^{(s)} = \mu(\lambda + \lambda^{-1} - 2). \tag{1.123}$$

With the help of relations (1.36) the Varga strain-energy function (1.123) can be expressed in term of amount of shear ϵ

$$W_{(v)}^{(e)} = \mu(\sqrt{\epsilon^2+4} - 2). \tag{1.124}$$

We note that in this case the material constants α, β, γ (1.37) are given by

$$\alpha = \frac{\mu\lambda_1^2}{\lambda_1+\lambda_2} = \frac{\mu\lambda^3}{\lambda^2+1}, \quad \gamma = \frac{\mu\lambda_2^2}{\lambda_1+\lambda_2} = \frac{\mu}{\lambda(\lambda^2+1)}, \quad \beta = \frac{\mu\lambda_1\lambda_2}{\lambda_1+\lambda_2} = \frac{\mu\lambda}{\lambda^2+1}. \tag{1.125}$$

Taking into account relations (1.36) and (1.125) we represent the dimensionless parameter δ (1.78) in the case of Varga material (1.123) in the following form

$$\beta = \sqrt{\alpha\gamma}, \quad \delta = \frac{[(\alpha/\gamma)^{1/4} - (\gamma/\alpha)^{-1/4}]^2}{4} = \frac{(\lambda - 1/\lambda)^2}{4} = \frac{\epsilon^2}{4}. \tag{1.126}$$

Chapter 2

Long wave motion in a layer with free faces

This chapter is devoted to the analysis of two dimensional long wave motion in an incompressible elastic layer subject to a primary simple shear deformation. Considering free faces boundary conditions we derive the dispersion relation for the most general incompressible strain energy function and then simplify it for a specific class. The influence of a primary simple shear deformation does not allow decomposition of dispersion relation into symmetric and anti-symmetric parts. In order to perform a long wave asymptotic analysis of the dispersion relation, some knowledge of the dispersive curves behavior is required. For the purpose of numerical calculations the neo-Hookean and Varga incompressible strain energy functions are employed to produce the dispersion curves. The numerical analysis of the dispersion relation reveals that, depending on the amount of shear and pre-stress, there may be non, one or two real long wave limits of the fundamental modes. Motivated by this numerical investigation, we proceed to the asymptotic analysis of the dispersion relation, considering both long wave low and long wave high frequency motion. As a result we derive approximations giving phase speed and frequency as an explicit function of wave and mode number. In addition, there is good agreement between numerical and asymptotic solutions over a relatively large wave number regime. Finally, the approximations are employed to establish the relative asymptotic orders of displacement components and incremental pressure, providing the theoretical framework for long wave asymptotic models to be derived later in this thesis. In respect of the long wave high frequency motion, the incremental pressure is asymptotically leading and the in-plane displacement component is asymptotically larger than its normal counterpart. For the long wave low frequency motion the results are different, with incremental pressure, in-plane and normal displacement components all having the same asymptotic orders. This fact demonstrates that at the presence of pre-stress in a form of simple shear there is neither bending nor extension, or analogues of their previously established pre-stressed counterparts.

2.1 The dispersion relation

We consider two dimensional harmonic wave propagation in an incompressible elastic layer of finite thickness h, infinite lateral extent and which is subject to a primary simple shear deformation. We assume that the incremental surface traction components vanish on the lower and upper surfaces of the layer. The appropriate free face boundary conditions, zero incremental traction on the surfaces, can be formulated through

$$\tau_1 = \tau_2 = 0 \quad \text{at} \quad x_2 = 0, -h. \tag{2.1}$$

We remark that specific values of x_2 coordinate in (2.1) were taken to demonstrate consistency in derivation of dispersion relations with previous work by Connor and Ogden (1996). Inserting the solutions for incremental traction components (1.111) into the boundary conditions (2.1) we obtain a homogeneous system of four linear equations in the four unknown constants $A_i, i = 1, 2, 3, 4$, given by

$$\begin{bmatrix} f_1 & f_2 & f_3 & f_4 \\ g_1 & g_2 & g_3 & g_4 \\ f_1 e^{-i\eta\, q_1} & f_2 e^{-i\eta\, q_2} & f_3 e^{-i\eta\, q_3} & f_4 e^{-i\eta\, q_4} \\ g_1 e^{-i\eta\, q_1} & g_2 e^{-i\eta\, q_2} & g_3 e^{-i\eta\, q_3} & g_4 e^{-i\eta\, q_4} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = 0, \tag{2.2}$$

where the functions $f(q)$ and $g(q)$ are expressed in (1.109). Here the non-dimensional parameter η is defined as the ratio of a layer thickness h to a wave length l, so $\eta \equiv h/l \equiv kh$ can be interpreted as a scaled wave number.

For the considered boundary value problem the dispersion relation is established by ensuring the homogeneous system of four equations (2.2) possesses a non-trivial solution. This condition is equivalent to vanishing of the following determinant

$$det \begin{bmatrix} f_1 & f_2 & f_3 & f_4 \\ g_1 & g_2 & g_3 & g_4 \\ f_1 e^{-i\eta\, q_1} & f_2 e^{-i\eta\, q_2} & f_3 e^{-i\eta\, q_3} & f_4 e^{-i\eta\, q_4} \\ g_1 e^{-i\eta\, q_1} & g_2 e^{-i\eta\, q_2} & g_3 e^{-i\eta\, q_3} & g_4 e^{-i\eta\, q_4} \end{bmatrix} = 0. \tag{2.3}$$

The dispersion relation is a transcendental equation connecting implicitly the phase speed and wave number. We remark that Connor and Ogden (1996) apparently first derived the dispersion relation (2.3) and showed that additional complexity arises because no principal axes is normal to the layer. To facilitate derivation of later analytic approximations the following simplification of the dispersion relation is possible by making some row and columns operations, resulting in

$$det \begin{bmatrix} f_1C_1 & f_2C_2 & f_3C_3 & f_4C_4 \\ g_1C_1 & g_2C_2 & g_3C_3 & g_4C_4 \\ f_1S_1 & f_2S_2 & f_3S_3 & f_4S_4 \\ g_1S_1 & g_2S_2 & g_3S_3 & g_4S_4 \end{bmatrix} = 0, C_j = \cos\left(q_j\eta/2\right), S_j = \sin\left(q_j\eta/2\right), j = 1, 2, 3, 4. \quad (2.4)$$

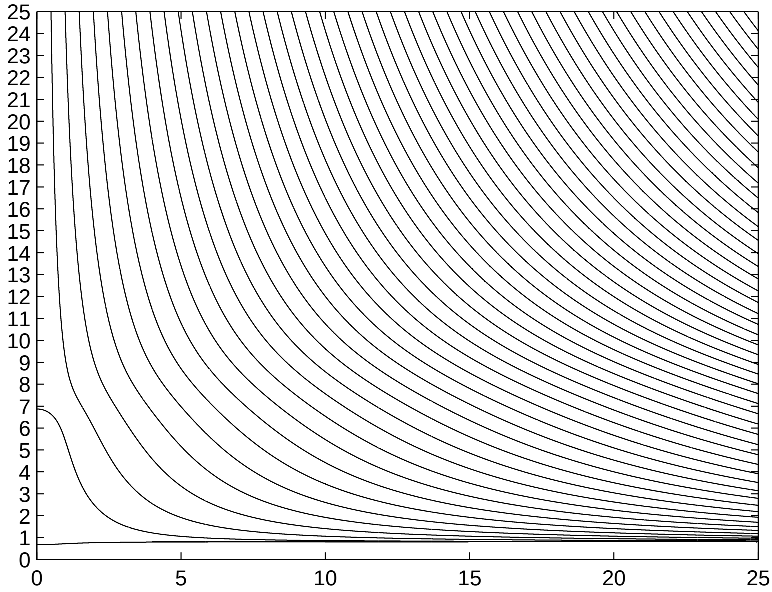
We note that the dispersion relation (2.4) cannot be decoupled into symmetric and anti-symmetric parts. This is related to lack of symmetry and something similar occurs in two-layer structures investigated in Rogerson and Sandiford (2000). We deduce from equation (2.4), taking into account the representations of the functions $f(q)$ and $g(q)$ given by (1.109), that the dispersion relation may be shown to always provide a real equation for all possible types of roots of the characteristic equation (1.77).

According to Connor and Ogden (1996), for a specific class of strain energy functions for which $2\beta = \alpha + \gamma$, implying that $\delta = 0$, for example neo-Hookean and Mooney-Rivlin materials, the characteristic equation (1.79) factorizes yielding four explicit roots (1.80). Taking this into account the dispersion relation (2.4) then can be expressed in the following simplified form obtained by Connor and Ogden (1996)

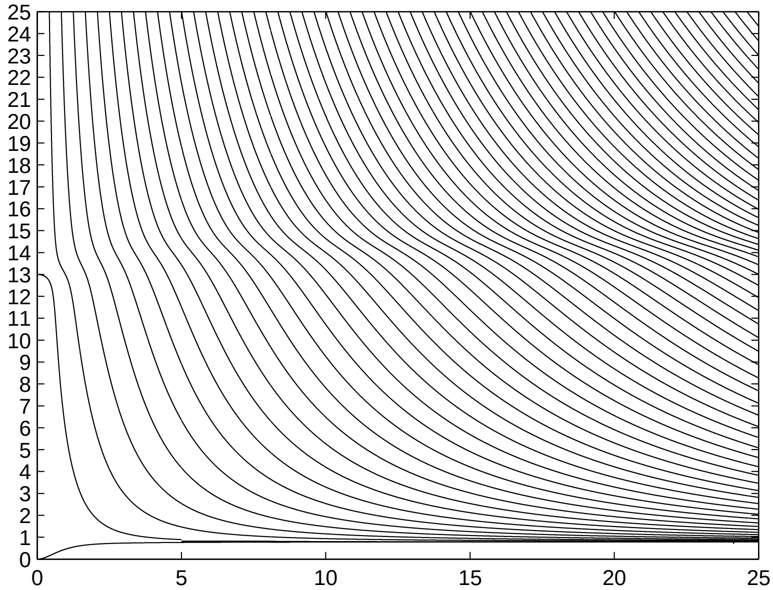
$$\begin{aligned} &\left(q_0\left(p^2-\kappa^2\right)^2 + q_0\kappa^2\left(q_0+2\,p\right)^2 + 4\,\kappa^2\left(p^2-\kappa^2\right)\left(q_0+2\,p\right)\right)\sinh\left(\eta\right)\sinh\left(\eta\,\kappa\right) \\ &+ 2\,\kappa\,\left(pq_0+p^2+\kappa^2\right)^2\left(\cos\left(\eta\,\epsilon\right) - \cosh\left(\eta\right)\cosh\left(\eta\,\kappa\right)\right) = 0, \\ &q_0 = 1+\epsilon^2+\kappa^2, \quad \kappa^2 = 1-\hat{v}. \end{aligned} \quad (2.5)$$

2.2 Numerical analysis of the dispersion relation

In this section we include some numerical solutions of the dispersion relation (2.4). Different cases, dependent on the amount of shear ϵ and pressure p (1.110) are represented in Figures 2.1 and 2.2. To produce these dispersive curves the Varga strain-energy function (1.122) is employed. In Figures 2.1 and 2.2 the plots of the dispersion relation representing the scaled squared phase speed $\hat{v}$ as a function of scaled wave number η.

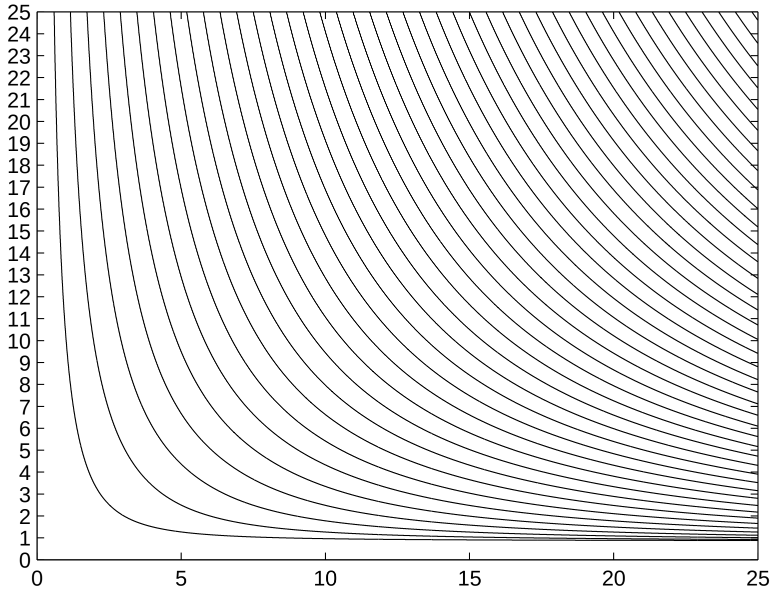


(a)

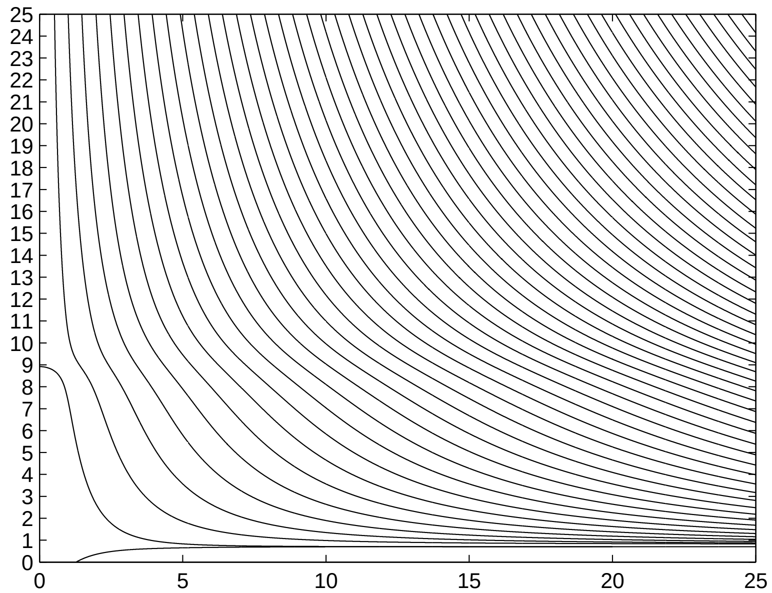


(b)

Figure 2.1: *Numerical solution of the dispersion relation (2.3). Shown scaled squared phase speed* $\hat{v}$ *(vertical scale) against scaled wave number* η *(horizontal scale) for the Varga material (1.124) with (a)* $\epsilon=2$*,* $p=0.5$*; (b)* $\epsilon=3$*,* $p=1$*.*



(a)



(b)

Figure 2.2: *Numerical solution of the dispersion relation (2.3). Shown scaled squared phase speed* $\hat{v}$ *(vertical scale) against scaled wave number* η *(horizontal scale) for the Varga material (1.124) with (a)* ϵ*=1, p=-2; (b)* ϵ*=2, p=1.5.*

As $\eta \to 0$ the scaled wave number is small and the wave length l is therefore large in comparison to layer thickness h. This type of motion is termed *long wave motion.* In respect of the plots of $\hat{v}$ against η, we remark that commonly two branches revealing the slowest speed and generally having finite limit as $\eta \to 0$ are labelled *fundamental modes*, with all the other branches termed as *harmonics.*

Figures 2.1 and 2.2 demonstrate the fact that the phase speeds of the harmonics tend to infinity as $\eta \to 0$. In addition, Figure 2.1 illustrates the fact that two fundamental modes have distinct finite phase speed limits as $\eta \to 0$. In particular, Figure 2.1 (b) demonstrates the situation when one long wave limit of fundamental mode is equal to zero. However, depending on the numerical values of ϵ and p there may be non, one or two real long wave limits of fundamental modes, shown in the Figure 2.2(a),(b) and Figure 2.1 respectively. According to terminology seemingly first suggested by Kaplunov et al. (1998), motion in the vicinity of long wave limits of fundamental modes will be termed *long wave low frequency motion.*

Plots of the dispersion relation, giving scaled frequency $\Omega = \sqrt{\hat{v}}\eta$ as a function of a scaled wave number η, are shown in Figures 2.3 and 2.4 for a layer composed of Varga material. The plots in Figure 2.3 demonstrate the fact that as $\eta \to 0$ the limit of each fundamental mode is zero. As $\eta \to 0$ the limit of each harmonic is non-zero, this limit is associated with the *cut-off (resonance) frequency* and the motion in the vicinity of cut-off frequencies will be termed as *long wave high frequency motion.*

Various wave fronts may occur for propagation of dispersive waves in a layer of elastic material, see for example Mindlin (1960). Figures 2.1(b), 2.3(a)-(b) and 2.4(b) illustrate the existence of wave fronts in the dispersive curves calculated for Varga materials. However we remark that our aim is to analyze long wave motion therefore these wave fronts will not be investigated in the thesis.

2.3 Analysis of the dispersion relation for long wave low frequency motion

Our first consideration is so-called *long wave low frequency motion*, for which $\hat{v}$ generally remains finite as $\eta \to 0$. For long wave low frequency motion we assume that all the roots of the characteristic equation are all of order $O(1)$. With the help of the above assumptions we derive long wave low frequency approximations of the dispersion relation (2.4). The approximate form

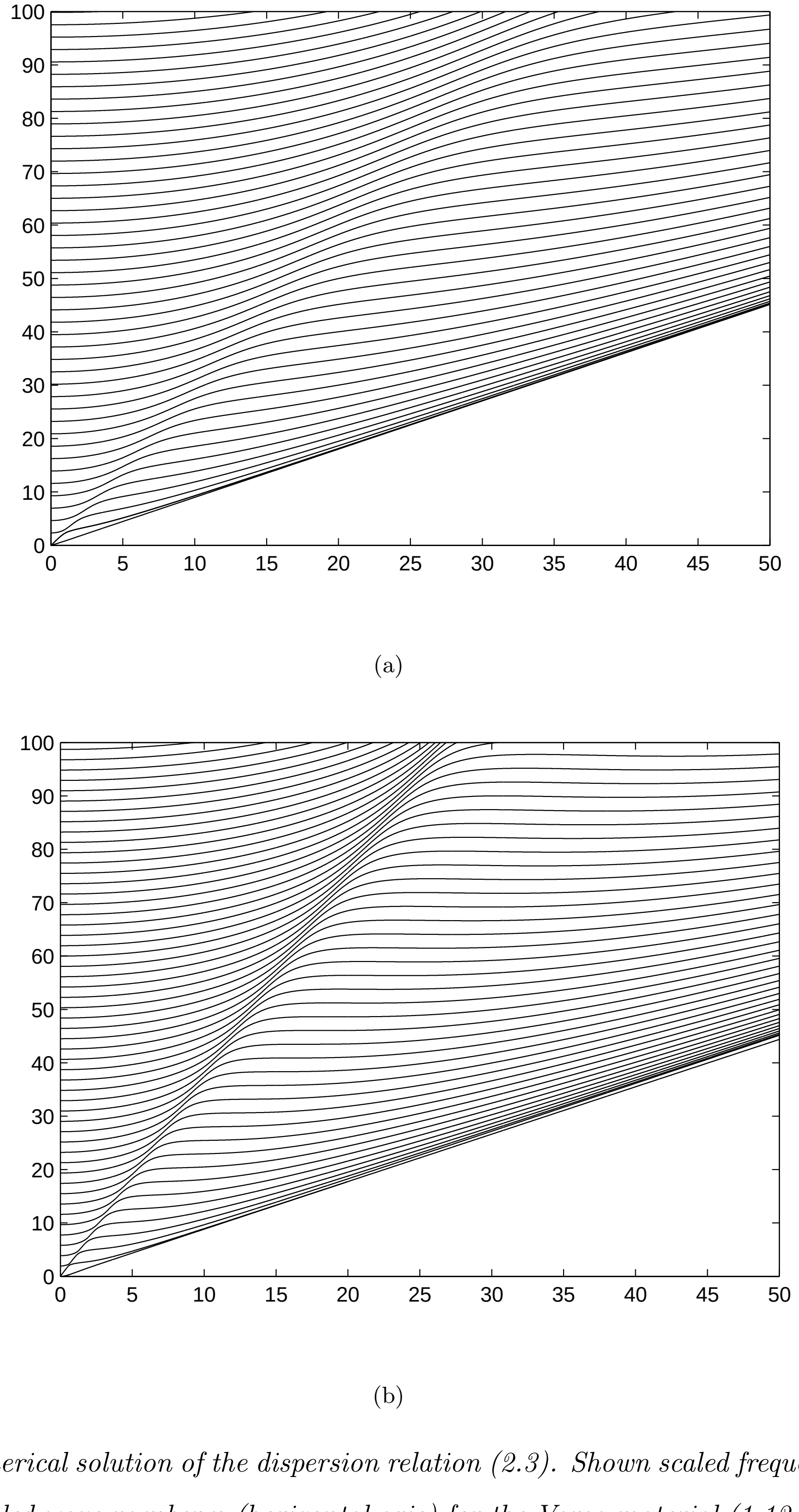
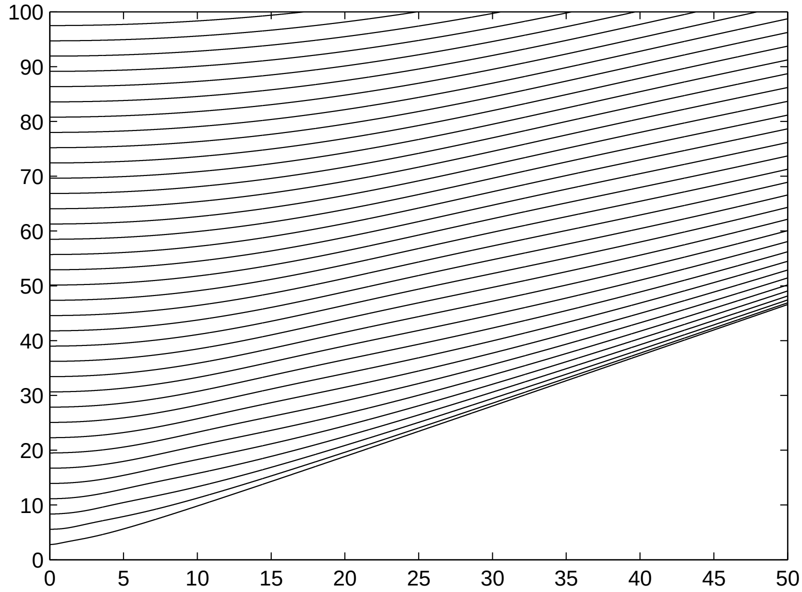
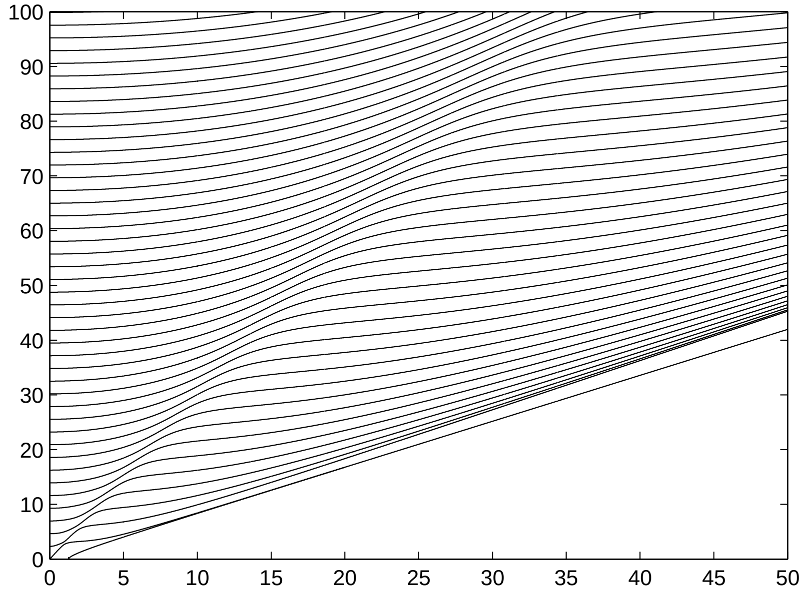


Figure 2.3: *Numerical solution of the dispersion relation (2.3). Shown scaled frequency* Ω *(vertical axis) against scaled wave number* η *(horizontal axis) for the Varga material (1.124) with (a)* ϵ*=2, p=0.5; (b)* ϵ*=3, p=1.*



(a)



(b)

Figure 2.4: *Numerical solution of the dispersion relation (2.3). Shown scaled frequency* Ω *(vertical axis) against scaled wave number* η *(horizontal axis) for the Varga material (1.124) with (a)* ϵ*=1, p=-2; (b)* ϵ*=2, p=1.5.*

of dispersion relation (2.4) allows us to analyze the two associated fundamental modes. For each of these two fundamental modes we derive approximations giving scaled squared phase speed $\hat{v}$ as an explicit function of scaled wave number η.

2.3.1 Leading order approximation

To obtain the leading order approximation of the dispersion relation (2.4) in the long wave low frequency regime we employ the Taylor series for the trigonometric functions S_j and C_j $j = 1, 2, 3, 4$ with respect to η up to $O(\eta^2)$. The corresponding leading order term in the approximation of the dispersion relation (2.4) is then given by

$$det \begin{bmatrix} f(q_1) & f(q_2) & f(q_3) & f(q_4) \\ g(q_1) & g(q_2) & g(q_3) & g(q_4) \\ q_1 f(q_1) & q_2 f(q_2) & q_3 f(q_3) & q_4 f(q_4) \\ q_1 g(q_1) & q_2 g(q_2) & q_3 g(q_3) & q_4 g(q_4) \end{bmatrix} = 0. \tag{2.6}$$

At this point we employ the following relations, which are valid for the roots of fourth order polynomial equation, see for example G. Korn and T. Korn (1968), and note that

$$\begin{aligned} & q^4 + Aq^3 + Bq^2 + Cq + D = 0, \\ & q_1 + q_2 + q_3 + q_4 = -A, \\ & q_1q_2 + q_1q_3 + q_1q_4 + q_2q_3 + q_2q_4 + q_3q_4 = B, \\ & q_1q_2q_3 + q_1q_2q_4 + q_1q_3q_4 + q_2q_3q_4 = -C, \\ & q_1q_2q_3q_4 = D. \end{aligned} \tag{2.7}$$

We substitute the relations (2.7) into the characteristic equation (1.77) to facilitate calculation of the 4×4 determinant (2.6), which results in the following quadratic equation in $\hat{v}$

$$\hat{v}^2 + (p^2(\delta+1) - 2p(\delta+1) + \delta - 3 - \epsilon^2)\hat{v} - (p^2-1)(2p(1+\delta) - 2(\delta-1) + 2 + \epsilon^2) = 0, \tag{2.8}$$

with the two roots taking the explicit forms

$$\begin{aligned} & \hat{v}_{1p}^{(0)} = \frac{-(p-1)^2(\delta+1) + \epsilon^2 + 4 - \sqrt{R}}{2}, \quad \hat{v}_{2p}^{(0)} = \frac{-(p-1)^2(\delta+1) + \epsilon^2 + 4 + \sqrt{R}}{2}, \\ & R = \left((p-1)^2\,\delta + (p+1)^2 - \epsilon^2\right)^2 + 4\,\epsilon^2\,(p+1)^2 . \end{aligned} \tag{2.9}$$

Hence we conclude that there are two fundamental modes associated with dispersion relation and the corresponding long wave phase speed limits are given by (2.9). In passing we note that the discriminant of the quadratic equation (2.8) is a perfect square in each of the cases in which δ or ϵ vanish or $p = -1$.

2.3.2 Second order approximation

To obtain a second order approximation of the dispersion relation (2.4) we employ the Taylor series for the trigonometric functions $C_j, S_j, j = 1, 2, 3, 4$ on η up to $O(\eta^4)$. Then a higher order approximation of the dispersion relation (2.4) can be expressed as

$$\mathcal{D}_l = \mathcal{D}_l^{(0)} + \eta^2 \mathcal{D}_l^{(2)} + O(\eta^4). \tag{2.10}$$

We note that both terms $\mathcal{D}_l^{(0)}$ given by (2.6) and $\mathcal{D}_l^{(2)}$ are quadratic in $\hat{v}$, and the explicit representations of $\mathcal{D}_l^{(2)}$ was obtained using Maple (1996). Notwithstanding this, an improved approximation for the scaled squared phase speeds of both fundamental modes may be readily established and written in the form

$$\hat{v}_i = \hat{v}_i^{(0)} - \eta^2 \frac{\mathcal{D}^{(2)}(\hat{v}_i^{(0)})}{\mathcal{D}'^{(0)}(\hat{v}_i^{(0)})} + O(\eta^4), \tag{2.11}$$

where the indexes $i = 1, 2$ represent each of two fundamental modes and the scaled squared phase speeds $\hat{v}_i^{(0)}$ are given by (2.9). The approximations afforded by (2.11) can still be readily generated and compared with the numerical solution.

In Figure 2.5 both numerical and approximate solutions for the fundamental modes are shown in respect of a layer composed of Varga material. Figure 2.5(a) illustrates the fact that two long wave limits of the fundamental modes exist with appropriate numerical parameters whereas the parameters ϵ and p on the Figure 2.5(b) correspond to the only one long wave limit of the fundamental mode. The plots in the Figure 2.5 clearly demonstrate very good agreement between the numerical solution and the second order approximations (2.11) over a relatively large wave number regime.

2.3.3 Higher order approximations in the case $\delta = 0$

We now consider a specific class of strain energy functions for which $2\beta = \alpha + \gamma$, implying that $\delta = 0$, with the dispersion relation expressible in the particulary simple form (2.5). Taking into account such simplification the long wave low frequency approximations of the dispersion relation (2.5) are derived to establish scaled squared phase speeds as an explicit function of scaled wave number for each of the two fundamental modes.

Motivated by the general case (2.11), the approximation of the scaled squared phase speed of fundamental mode can be represented as

$$\hat{v} = \hat{v}^{(0)} + \eta^2 \hat{v}^{(2)} + \eta^4 \hat{v}^{(4)} + O(\eta^6), \tag{2.12}$$

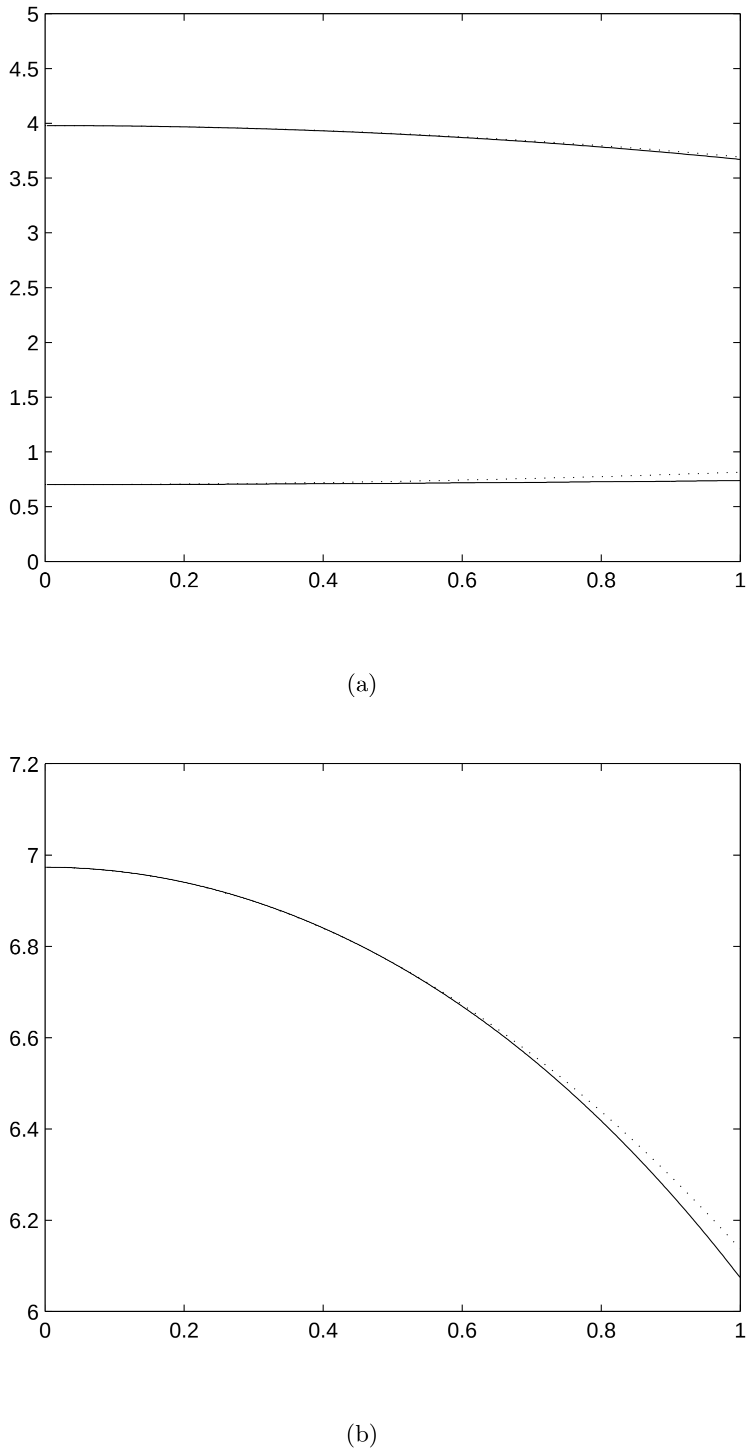


Figure 2.5: *Comparison of numerical solutions and approximations (2.11) for the fundamental modes in the case of the Varga material (1.124) with (a) ϵ=1, p=0.5; (b) ϵ=1, p=2. Plots demonstrate scaled squared phase speed $\hat{v}$ (vertical axis) as a function of scaled wave number η (horizontal axis). Dotted curves represent the second order approximations (2.11) and solid curves the numerical solution.*

where notation $\hat{v}^{(0)}$ represents long wave limit of fundamental mode, second and third order corrections to which are denoted by $\hat{v}^{(2)}$ and by $\hat{v}^{(4)}$ respectively. Then we introduce the following expansions

$$\begin{aligned}
\kappa &= \sqrt{1-\hat{v}^{(0)}} - \frac{1}{2}\frac{\hat{v}^{(2)}\eta^2}{\sqrt{1-\hat{v}^{(0)}}} - \frac{4\hat{v}^{(4)}(1-\hat{v}^{(0)}) + (\hat{v}^{(2)})^2}{8(1-\hat{v}^{(0)})^{3/2}}\eta^4 + O(\eta^5),\\
\sinh(\eta\kappa) &= \sqrt{1-\hat{v}^{(0)}}\eta + \frac{(\hat{v}^{(0)}-1)^2 - 3\hat{v}^{(2)}}{6\sqrt{1-\hat{v}^{(0)}}}\eta^3 + \frac{(\hat{v}^{(0)}-1)^2}{8}\eta^4 + O(\eta^5),\\
\cosh(\eta\kappa) &= 1 + \frac{1}{2}\left(1-\hat{v}^{(0)}\right)\eta^2 - \frac{1}{12}\left(6\hat{v}^{(2)} + (\hat{v}^{(0)}-1)^2\right)\eta^4 + O(\eta^5). \qquad (2.13)
\end{aligned}$$

If we substitute expansions (2.13) together with Taylor series for the functions $\sinh(\eta)$, $\cosh(\eta)$, $\cos(\eta\epsilon)$ for η into the dispersion relation (2.5) and collect terms of same order of η we obtain the following approximation of the dispersion relation (2.5) up to $O(\eta^6)$

$$\mathcal{D}_{nl} = \mathcal{D}_{nl}^{(0)} + \eta^2\mathcal{D}_{nl}^{(2)} + \eta^4\mathcal{D}_{nl}^{(4)} + O(\eta^6). \qquad (2.14)$$

The condition that the leading order term $\mathcal{D}_{nl}^{(0)}$ is equal to zero can be expressed as

$$\mathcal{D}_{nl}^{(0)} = \sqrt{1-\hat{v}^{(0)}}(1-p^2-\hat{v}^{(0)})((\hat{v}^{(0)})^2 - 2\epsilon^2\hat{v}^{(0)} + \epsilon^4 + 4\epsilon^2)(2+\epsilon^2-\hat{v}^{(0)}+2p) = 0. \qquad (2.15)$$

The spurious solution $\hat{v}^{(0)} = 1$ of equation (2.15) is associated with a double root of the characteristic equation and the third factor in (2.15) cannot give any real solutions. Therefore we conclude that

$$\hat{v}_1^{(0)} = 1 - p^2, \quad \hat{v}_2^{(0)} = \epsilon^2 + 2p + 2. \qquad (2.16)$$

Results (2.16) are equivalent to the general formulas (2.9) provided $\delta = 0$. In addition, the numerical long wave limits of the fundamental modes observed in Figure 2.6 coincide with the leading order approximations (2.16) at the equivalent set of numerical parameters.

It may be shown that the second order term $\mathcal{D}_{nl}^{(2)}$ provides a linear function in $\hat{v}$, which is too long to be of value to be written here, however its explicit representation was obtained using Maple (1996). Employing the condition $\mathcal{D}_{nl}^{(2)} = 0$ we establish the following second order corrections

$$\hat{v}_1^{(2)} = \frac{1}{12}(\epsilon^2 + (p+1)^2)p^2, \quad \hat{v}_2^{(2)} = -\frac{1}{12}(\epsilon^2 + (p+1)^2). \qquad (2.17)$$

Therefore an approximation for the scaled squared phase speeds associated with the fundamental modes may be expressed as

$$\hat{v}_m = \hat{v}_m^{(0)} + \eta^2\hat{v}_m^{(2)} + O(\eta^4), \qquad (2.18)$$

where $\hat{v}_m^{(n)}$, $m = 1, 2, n = 0, 2$ are given by equations (2.16) and (2.17), respectively.

The third term in the dispersion relation $\mathcal{D}_{nl}^{(4)}$ is a linear function of $\hat{v}^{(2)}$, the explicit representation of $\mathcal{D}_{nl}^{(4)}$ was obtained with the help of Maple (1996). We make use of the condition $\mathcal{D}_{nl}^{(4)} = 0$, together with leading order (2.17) and second order (2.18) approximations to obtain the following third order corrections to scaled squared phase speeds (2.16)

$$\hat{v}_1^{(4)} = \frac{p^2 \mathcal{Z}_1}{576(\epsilon^2 + 1 + 3\,p^2 - 4\,p)}, \quad \hat{v}_2^{(4)} = \frac{\mathcal{Z}_2}{576(2p + \epsilon^2 + 1)(7p + 3\epsilon^2 + 3)}, \tag{2.19}$$

$$\mathcal{Z}_1 = \epsilon^6 + (3p^2 - 12p - 13)\epsilon^4 - (13p^4 - 24p^3 - 10p^2 + 24p + 29)\epsilon^2$$
$$- 15p^6 + 4p^5 + 55p^4 + 88p^3 + 23p^2 - 12p - 15,$$
$$\mathcal{Z}_2 = -24(p^2 + p + 1)\epsilon^6 - (136p^3 + 168p^2 - 73p - 72)\epsilon^4$$
$$- (256p^4 + 366p^3 - 28p^2 - 218p - 72)\epsilon^2$$
$$- 159p^5 - 252p^4 - 34p^3 + 172p^2 + 121p + 24.$$

Hence for the class of materials considered, third order approximations for the scaled squared phase speeds associated with two fundamental modes may be determined by

$$\hat{v}_m = \hat{v}_m^{(0)} + \eta^2 \hat{v}_m^{(2)} + \eta^4 \hat{v}_m^{(4)} + O(\eta^6), \tag{2.20}$$

where $\hat{v}_m^{(n)}$, $m = 1, 2, n = 0, 2, 4$ are provided by equations (2.16), (2.17) and (2.19), respectively.

We remark that for the specific sets of parameters, these approximations are not valid because third order corrections become very large within the vicinity of certain combinations of ϵ and p. These combinations of parameters may be identified as

$$\epsilon^2 + 1 + 3p^2 - 4p = 0, \quad 2p + \epsilon^2 + 1 = 0, \quad 7p + 3\epsilon^2 + 3 = 0. \tag{2.21}$$

In addition to (2.21) the leading order approximations should stay positive. Taking that into account and using the conditions (2.21) we obtain the following regions in which the approximations (2.20) are not valid

$$(a) \quad 0 \le \epsilon \le \frac{1}{\sqrt{3}} \quad \text{with} \quad p = \frac{2 \pm \sqrt{1 - 3\epsilon^2}}{3},$$
$$(b) \quad 0 \le \epsilon \le 1 \quad \text{with} \quad p = \frac{(\epsilon^2 + 1)}{2},$$
$$(c) \quad 0 \le \epsilon \le \frac{2}{\sqrt{3}} \quad \text{with} \quad p = \frac{3(\epsilon^2 + 1)}{7}. \tag{2.22}$$

We conclude this section with some illustrative numerical results, for which a neo-Hookean strain-energy function is employed. Shown in Figure 2.6 are second order approximations (2.18) and third order approximations (2.20) for the fundamental modes, which are compared with numerical results for the neo-Hookean material. Figure 2.6(a) represents two real long wave limits of the fundamental modes, whereas Figure 2.6(b) demonstrates the only one real long wave limit of the fundamental mode. All the plots in Figure 2.6 clearly demonstrate a good agreement between the numerical solution and approximations (2.18) and (2.20) over a relatively large wave number regime.

2.4 Analysis of the dispersion relation for long wave high frequency motion

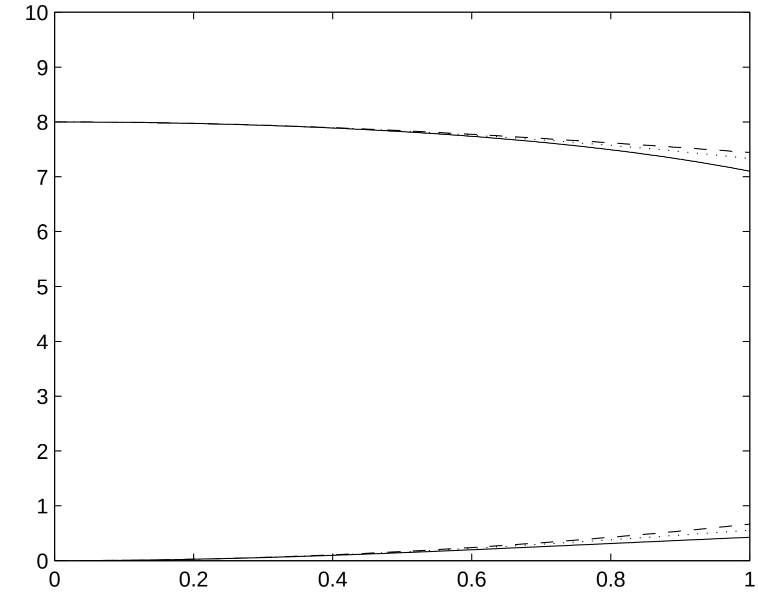
We now consider the propagation of waves associated with harmonics of the dispersion relation (2.4). The first attempt to analyze high frequency motion was made by Lamb (1917). In addition in certain boundary value problems, such as motion in an elastic layer with zero displacement boundary conditions and mixed boundary conditions to be considered in later chapters of the thesis, there are no fundamental modes. Hence in the such cases all the energy propagates with the harmonics only, see Kaplunov and Nolde (2002).

Long wave high frequency motion can be characterized by the fact that the scaled frequency $\Omega = \sqrt{\hat{v}}\eta$ is of order $O(1)$ as $\eta \to 0$. Hence the scaled squared phase speed $\hat{v}$ is of order $O(1/\eta^2)$ and $\hat{v} \to \infty$ as $\eta \to 0$.

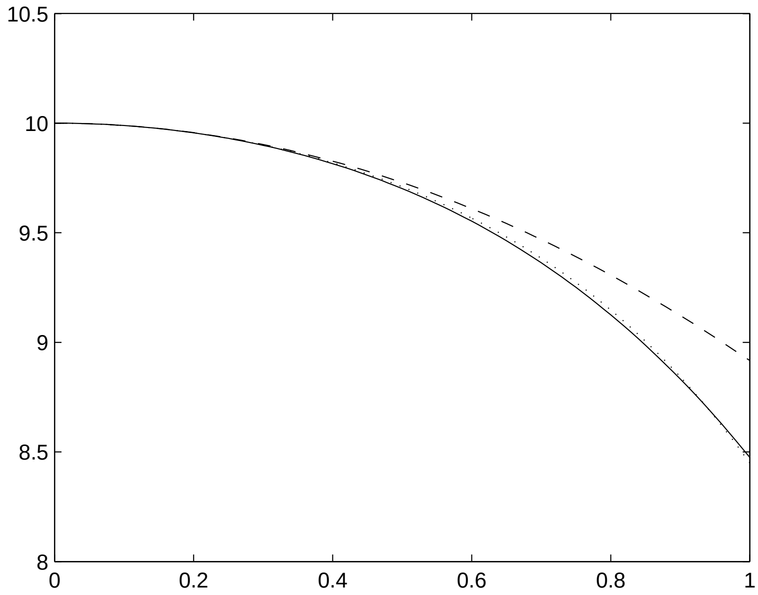
We begin by examining the four roots of the characteristic equation (1.77) when $\hat{v} >> 1$ and consider expansions of its roots in the following form

$$q = s_{01}\tilde{v} + s_0 + \frac{s_1}{\tilde{v}} + \frac{s_2}{\tilde{v}^2} + \frac{s_3}{\tilde{v}^3} + \frac{s_4}{\tilde{v}^4}, \quad \tilde{v} = \frac{\Omega}{\eta}. \tag{2.23}$$

We substitute the expansion (2.23) into the characteristic equation (1.77), collect terms of the same order in η, and then determine the coefficients s_i, $i = 1, 2, 3, 4$ to establish the following



(a)



(b)

Figure 2.6: *Comparison of numerical solution and approximations (2.18) and (2.20) for the fundamental modes in the case of the neo-Hookean material (1.116) with (a) ϵ=2, p=1; (b) ϵ=2, p=2. Plots represent scaled squared phase speed $\hat{v}$ (vertical axis) as a function of scaled wave number η (horizontal axis). Dotted curves correspond to third order approximations (2.20), dashed curves to second order approximations (2.18) and solid curves to numerical results.*

approximations for each of four roots up to $O(\eta^5)$

$$
\begin{aligned}
q_1 &= i - \frac{i\delta\,(\epsilon - 2\,i)^2\,\eta^2}{2\,(1+\delta)\,\Omega^2} \\
&- \frac{i\delta\,\left(\epsilon^4(\delta+4) - \epsilon^3 24i + \epsilon^2 24(\delta-2) - 32i\epsilon(2\delta-1) - 48\delta\right)\eta^4}{8\,(1+\delta)^2\,\Omega^4} + O(\eta^5), \\
q_2 &= -i + \frac{i\delta\,(\epsilon + 2\,i)^2\,\eta^2}{2\,(1+\delta)\,\Omega^2} \\
&+ \frac{i\delta\,\left(\epsilon^4(\delta+4) + \epsilon^3 24i + \epsilon^2 24(\delta-2) + 32i\epsilon(2\delta-1) - 48\delta\right)\eta^4}{8\,(1+\delta)^2\,\Omega^4} + O(\eta^5), \\
q_3 &= \frac{\sqrt{1+\delta}\Omega}{\eta} + \epsilon - \frac{1}{2}\frac{(4\,\delta+1)\,\eta}{\sqrt{1+\delta}\Omega} + 2\,\frac{\epsilon\,\delta\,\eta^2}{(1+\delta)\,\Omega^2} - \frac{1}{8}\frac{\left(16\delta^2 + 4\delta(5\epsilon^2-2)+1\right)\eta^3}{(1+\delta)^{3/2}\,\Omega^3} \\
&+ \frac{\epsilon\,\delta\,\left(8\,\delta - 4 + 3\,\epsilon^2\right)\eta^4}{(1+\delta)^2\,\Omega^4} + O(\eta^5), \\
q_4 &= -\frac{\sqrt{1+\delta}\Omega}{\eta} + \epsilon + \frac{1}{2}\frac{(4\,\delta+1)\,\eta}{\sqrt{1+\delta}\Omega} + 2\,\frac{\epsilon\,\delta\,\eta^2}{(1+\delta)\,\Omega^2} + \frac{1}{8}\frac{\left(16\delta^2 + 4\delta(5\epsilon^2-2)+1\right)\eta^3}{(1+\delta)^{3/2}\,\Omega^3} \\
&+ \frac{\epsilon\,\delta\,\left(8\,\delta - 4 + 3\,\epsilon^2\right)\eta^4}{(1+\delta)^2\,\Omega^4} + O(\eta^5).
\end{aligned} \tag{2.24}
$$

With the help of expansions (2.24) we will examine the long wave high frequency regime. Each harmonic can be characterized by its mode number n and corresponding cut-off (resonance) frequency. We will start our analysis with the derivation of appropriate long wave high frequency asymptotic approximations of dispersion relation (2.4). Then we will establish the cut-off frequencies associated with harmonics of the dispersion relation. In the case of $\delta = 0$ we obtain second and third order corrections to cut-off frequencies.

2.4.1 Leading order approximation

To obtain the leading order approximation of dispersion relation (2.4) within the long wave high frequency regime we use reduced expansions (2.24) up to and including $O(\eta^2)$. Then the following approximations for the functions $f(q)$ and $g(q)$ in (1.109) are

$$
\begin{aligned}
f(q_1) &\approx -\frac{((1+\delta)\,p+\delta)\,\lambda^2 + 1 - 2\,\delta}{(1+\delta)\,\lambda} - i\left(\frac{\Omega^2}{\eta^2} - \frac{1 + ((1+\delta)\,p + 3\,\delta)\,\lambda^2}{(1+\delta)\,\lambda^2}\right) + O(\eta^3), \\
f(q_3) &\approx \frac{\epsilon - \left(\epsilon^2 + (1+\delta)\,(p+1)\right)\lambda^3 + \left(\epsilon^3 + ((1+\delta)\,p + 3\,\delta + 1)\,\epsilon\right)\lambda^2 + \left(2\,\epsilon^2 + 2\,\delta + 1\right)\lambda}{(1+\delta)\,\lambda^2} \\
&+ \left(\frac{1 - 2\,\epsilon\,\lambda^3 + \left(3\,\epsilon^2 + 1 + (1+\delta)\,p + 3\,\delta\right)\lambda^2 + 4\,\epsilon\,\lambda}{\sqrt{1+\delta}\lambda^2}\right)\frac{\Omega}{\eta} + \left(\frac{2\,\epsilon\,\lambda - \lambda^2 + 2}{\lambda}\right)\frac{\Omega^2}{\eta^2} + O(\eta^3),
\end{aligned} \tag{2.25}
$$

$$
\begin{aligned}
g(q_1) &\approx \frac{(1-2\,\delta)\,\lambda^2+(1+\delta)\,p+\delta}{(1+\delta)\,\lambda}-i\left(\frac{\Omega^2}{\eta^2}-\frac{3\,\delta+\lambda^2+(1+\delta)\,p}{1+\delta}\right)+O(\eta^3),\\
g(q_3) &\approx \frac{\epsilon\,\lambda^3-\left(2\,\epsilon^2+2\,\delta+1\right)\lambda^2+\left(\epsilon^3+((1+\delta)\,p+3\,\delta+1)\,\epsilon\right)\lambda+\epsilon^2+\delta+(1+\delta)\,p+1}{(1+\delta)\,\lambda}\\
&+\left(\frac{\lambda^3-4\,\epsilon\,\lambda^2+\left(3\,\epsilon^2+1+(1+\delta)\,p+3\,\delta\right)\lambda+2\,\epsilon}{\sqrt{1+\delta}\lambda}\right)\frac{\Omega}{\eta}+\left(\frac{2\,\epsilon\,\lambda-2\,\lambda^2+1}{\lambda}\right)\frac{\Omega^2}{\eta^2}+O(\eta^3).
\end{aligned}
\tag{2.26}
$$

Taking into account expansions (2.24) we substitute the following approximations for trigonometrical functions $S_j, C_j, j = 1, 2, 3, 4$ to insert into the dispersion relation (2.4)

$$
\begin{aligned}
S_1 &= \frac{i\eta}{2}+O(\eta^2), \quad S_3=\sin\left(\frac{1}{2}\sqrt{1+\delta}\Omega\right)+\frac{1}{2}\cos\left(\frac{1}{2}\sqrt{1+\delta}\Omega\right)\epsilon\,\eta+O(\eta^2),\\
C_1 &= 1+O(\eta^2), \quad C_3=\cos\left(\frac{1}{2}\sqrt{1+\delta}\Omega\right)-\frac{1}{2}\sin\left(\frac{1}{2}\sqrt{1+\delta}\Omega\right)\epsilon\,\eta+O(\eta^2).
\end{aligned}
\tag{2.27}
$$

We use the above expansions (2.25)–(2.27) to obtain the following leading order approximation of the dispersion relation (2.4)

$$
\mathcal{D}_h = i(1+\delta)^4\Omega^8\sin\left(\sqrt{1+\delta}\Omega\right)+O(\eta^2). \tag{2.28}
$$

The condition that leading order term is equal to zero provides the following cut-off frequencies associated with each harmonic of dispersion relation (2.4)

$$
\Omega_g^{(0)} = \frac{n\pi}{\sqrt{1+\delta}}. \tag{2.29}
$$

We remark here that the relation (2.29) gives an approximate form of cut-off frequencies and due to the algebraic complexity of dispersion relation (2.4) it is not possible to derive their exact representations in the general case.

2.4.2 Higher order approximations in the case $\delta = 0$

For the class of strain energy functions for which $\delta = 0$, our intension is to derive long wave high frequency approximations of the dispersion relation (2.5) up to and including $O(\eta^4)$. To begin, we consider the third order expansion for the frequency Ω in the vicinity of cut-off frequencies $\Omega^{(0)}$ in the following form

$$
\Omega = \Omega^{(0)}+\Omega^{(2)}\eta^2+\Omega^{(4)}\eta^4+O(\eta^6). \tag{2.30}
$$

Then taking into account expansion (2.30) we insert into equation (2.5) the Taylor series expansions for the functions $\sinh(\eta), \cosh(\eta), \cos(\eta\epsilon)$ up to $O(\eta^5)$, together with the following expansions

$$
\begin{aligned}
&\kappa=\sqrt{1-\frac{\Omega^2}{\eta^2}}=i\Omega^{(0)}\eta^{-1}-\frac{i\left(1-2\Omega^{(0)}\Omega^{(2)}\right)}{2\Omega^{(0)}}\eta\\
&+\frac{i\left(4\Omega^{(0)}(\Omega^{(2)}+2\Omega^{(4)}(\Omega^{(0)})^2)-1\right)}{8(\Omega^{(0)})^3}\eta^3+O(\eta^5),\\
&\sinh(\eta\kappa)=i\sin\Omega^{(0)}-\frac{i\cos\Omega^{(0)}\left(1-2\Omega^{(0)}\Omega^{(2)}\right)}{2\Omega^{(0)}}\eta^2\\
&+\left(\frac{i\left(4\Omega^{(0)}(\Omega^{(2)}+2\Omega^{(4)}(\Omega^{(0)})^2)-1\right)\cos\Omega^{(0)}}{8(\Omega^{(0)})^3}\right)\eta^4\\
&+\left(\frac{i\Omega^{(0)}\left(4\Omega^{(0)}(\Omega^{(2)}-\Omega^{(0)}(\Omega^{(1)})^2)-1\right)\sin\Omega^{(0)}}{8(\Omega^{(0)})^3}\right)\eta^4+O(\eta^6),\\
&\cosh(\eta\kappa)=\cos\Omega^{(0)}+\frac{\sin\Omega^{(0)}\left(1-2\Omega^{(0)}\Omega^{(2)}\right)}{2\Omega^{(0)}}\eta^2\\
&-\left(\frac{\Omega^{(0)}(2\Omega^{(0)}\Omega^{(2)}-1)^2\cos\Omega^{(0)}}{8(\Omega^{(0)})^3}+\frac{\left(4\Omega^{(0)}(\Omega^{(1)}+2(\Omega^{(0)})^2\Omega^{(4)})-1\right)\sin\Omega^{(0)}}{8(\Omega^{(0)})^3}\right)\eta^4+O(\eta^6).
\end{aligned}
\tag{2.31}
$$

The dispersion relation (2.5) can now be expressed in the following form

$$\mathcal{D}_{nh}=\mathcal{D}_{nh}^{(0)}+\mathcal{D}_{nh}^{(2)}\eta^2+\mathcal{D}_{nh}^{(4)}\eta^4+O(\eta^6). \tag{2.32}$$

The condition that the following leading order term is equal to zero

$$\mathcal{D}_{nh}^{(0)}=i(\Omega^{(0)})^8\sin\Omega^{(0)}=0, \tag{2.33}$$

provides the cut-off frequencies expressed as

$$\Omega^{(0)}=n\pi. \tag{2.34}$$

Taking into account expansions (2.30), the cut-off frequencies (2.34) and the facts that $\cos(n\pi)=(-1)^n$ in the expansion (2.32), the second order term $\mathcal{D}_{nh}^{(2)}$ is the following linear function of second order correction $\Omega^{(2)}$

$$\mathcal{D}_{nh}^{(2)}=\frac{in^5\pi^5}{2}\Big(4\left(1-(-1)^n\right)(p^2+1)+8\left(1-(-1)^n\right)p+(-1)^n\pi^2n^2(2\pi n\Omega^{(2)}-1)\Big). \tag{2.35}$$

The condition that $\mathcal{D}_{nh}^{(2)}=0$ enables us to express the second order corrections in the following form

$$\Omega^{(2)}=\frac{4\,(p+1)^2\left((-1)^n-1\right)+(-1)^n\pi^2n^2}{2\pi^3n^3\,(-1)^n}. \tag{2.36}$$

Therefore for the scaled frequency Ω in the vicinity of the cut-off frequencies $n\pi$ the following second order approximation is valid

$$\Omega = n\pi + \Omega^{(2)}\eta^2 + O(\eta^4), \tag{2.37}$$

where the corrections $\Omega^{(2)}$ are given by (2.36).

Substituting expansions (2.30), the cut-off frequencies $n\pi$ and the second order corrections (2.36) into relation (2.32) we obtain the third order term $\mathcal{D}_{nh}^{(4)}$ as the following linear function of $\Omega^{(4)}$

$$\begin{aligned}
\mathcal{D}_{nh}^{(4)} &= \frac{i\pi^3 n^3}{24}(192\,(-1)^n\,\pi^4 n^4 (\Omega^{(2)})^2 + c_1^{(4)}\Omega^{(2)} + c_0^{(4)} + 24\,\pi^5 n^5\,(-1)^n\,\Omega^{(4)}),\\
c_1^{(4)} &= 240\,\pi\,n\,(1-(-1)^n)\,p^2 + \big(5(1-(-1)^n) - (-1)^n\,\pi^2 n^2\big)\,96\pi n p\\
&+ 4\,\pi^5 n^5\,(-1)^n - 36\pi^3 n^3 (-1)^n (2\epsilon^2+5) + 240\,\pi\,n(1-(-1)^n),\\
c_0^{(4)} &= 96\,((-1)^n-1)\,p^3 + \big(\big(4((-1)^n-1) - \pi^2 n^2\big)\,\epsilon^2 - (-1)^n\,\pi^2 n^2 + 13((-1)^n-1)\big)\,24p^2\\
&+ \big(7((-1)^n-1) + \big(2((-1)^n-1) - \pi^2 n^2\big)\,\epsilon^2\big)\,48p\\
&- 2\,(-1)^n\,\pi^4 n^4 + 12\epsilon^2\pi^2 n^2 (3(-1)^n - 2) + 21\pi^2 n^2\,(-1)^n + 120((-1)^n-1).
\end{aligned} \tag{2.38}$$

Similarly, from the condition $\mathcal{D}_{nh}^{(4)} = 0$ we establish the following third order corrections

$$\begin{aligned}
\Omega^{(4)} &= \frac{c_e^{(2)}\epsilon^2 + c_e^{(0)}}{24 n^7 \pi^7},\\
c_e^{(2)} &= \big((-1)^n\pi^2 n^2 - 2((-1)^n-1)\big)\,24\pi^2 n^2 p^2\\
&+ \big((-1)^n\pi^2 n^2 + 4(1-(-1)^n)\big)\,48\pi^2 n^2 p + (6(1-(-1)^n) + (-1)^n\pi^2 n^2)24\pi^2 n^2,\\
c_e^{(0)} &= ((-1)^n-1)\,576p^4 + (24-\pi^2 n^2)((-1)^n-1)96p^3\\
&+ \big(\pi^4 n^4(((-1)^n+2) + 3(144-7\pi^2 n^2)((-1)^n-1)\big)\,8p^2\\
&+ \big(\pi^4 n^4(2+(-1)^n) + 3(48-\pi^2 n^2)((-1)^n-1)\big)\,16p\\
&+ \pi^4 n^4 (8(-1)^n + 13) + 24(24+\pi^2 n^2)((-1)^n-1).
\end{aligned} \tag{2.39}$$

Therefore, for the each harmonic of dispersion relation (2.5) the associated scaled frequency Ω in the vicinity of cut-off frequencies $n\pi$ is determined by the following third order approximation

$$\Omega = n\pi + \Omega^{(2)}\eta^2 + \Omega^{(4)}\eta^4 + O(\eta^6), \tag{2.40}$$

where the corrections $\Omega^{(2)}$ and $\Omega^{(4)}$ are given by representations (2.36) and (2.39) respectively.

At the end of this section we include some illustrative numerical results produced for a neo-Hookean material. In Figure 2.7 comparisons of the numerical solutions of (2.5) and approximations (2.37), (2.40) are shown for harmonics in the vicinity of the first two cut-off frequencies.

The accuracy of the approximations (2.37) and (2.40) can be demonstrated by representing each harmonic on a separated plot. All the plots in Figure 2.7 illustrate the fact that the third order approximations (2.40) are the closest to numerical solution.

2.5 Relative asymptotic orders of displacements and incremental pressure

In this section we establish the relative asymptotic orders of displacement components u_1, u_2 and incremental pressure p_t within both the long wave low and high frequency regimes. With the help of the notation $\zeta = x_2/h$ the solutions for displacement and incremental pressure (1.83) may be represented in the following form

$$
\begin{aligned}
&u_1 = (\sum_{j=1}^{4} q_j A_j e^{i\eta q_j \zeta}) e^{ik(vt-x_1)}, \quad u_2 = (\sum_{j=1}^{4} A_j e^{i\eta q_j \zeta}) e^{ik(vt-x_1)},\\
&p_t = k(\sum_{j=1}^{4} \mathcal{P}(q_j) A_j e^{i\eta q_j \zeta}) e^{ik(vt-x_1)}.
\end{aligned} \tag{2.41}
$$

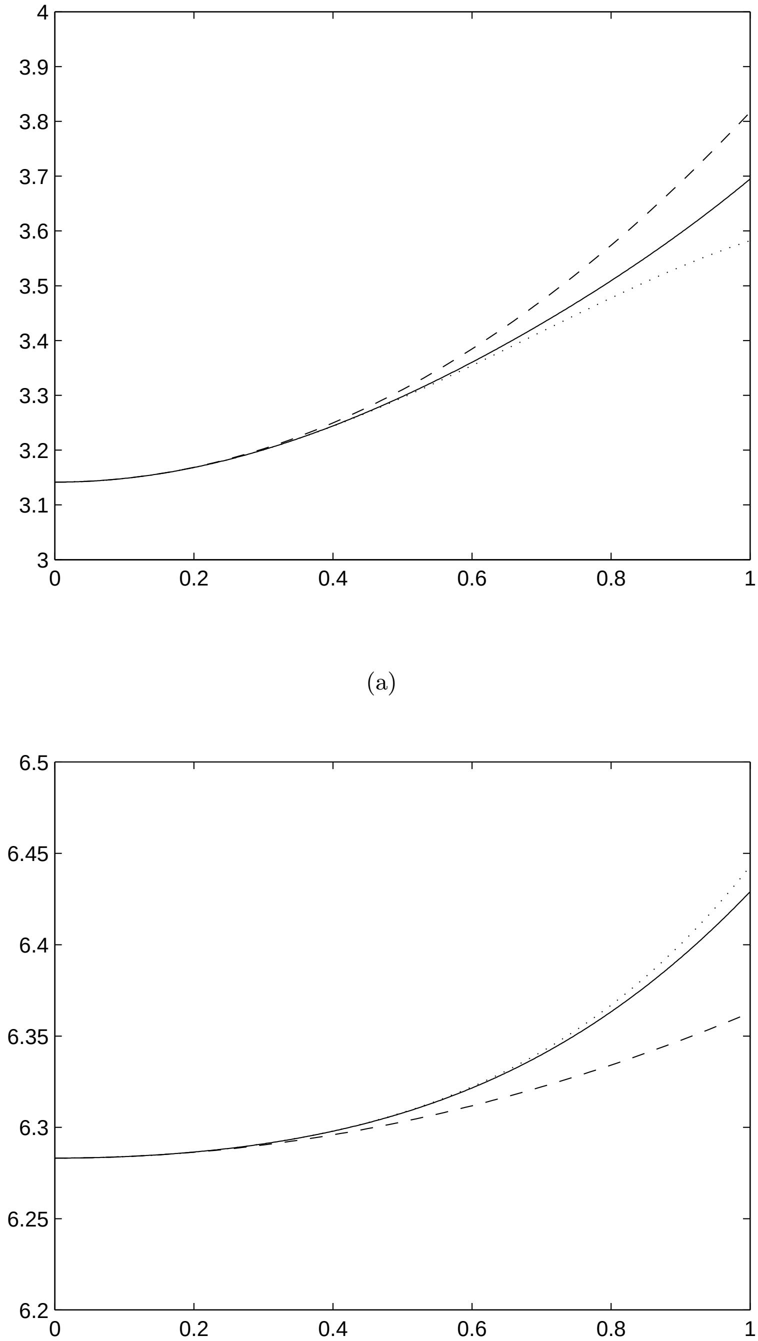
Using the boundary conditions (2.2) the coefficients $A_j, j = 1, 2, 3, 4$ may be related through

$$
\begin{aligned}
&f(q_1)A_1 + f(q_2)A_2 + f(q_3)A_3 + f(q_4)A_4 = 0,\\
&g(q_1)A_1 + g(q_2)A_2 + g(q_3)A_3 + g(q_4)A_4 = 0,\\
&f(q_1)e^{-iq_1\eta}A_1 + f(q_2)e^{-iq_2\eta}A_2 + f(q_3)e^{-iq_3\eta}A_3 + f(q_4)e^{-iq_4\eta}A_4 = 0,\\
&g(q_1)e^{-iq_1\eta}A_1 + g(q_2)e^{-iq_2\eta}A_2 + g(q_3)e^{-iq_3\eta}A_3 + g(q_4)e^{-iq_4\eta}A_4 = 0.
\end{aligned} \tag{2.42}
$$

The system (2.42) enable us to express the coefficients $A_j, j = 1, 2, 3, 4$ in terms of one arbitrary constant $\tilde{U}$. Then if we substitute A_j into the relations (2.41), we are able to compare the relative asymptotic orders of the displacement components u_1, u_2 and incremental pressure p_t. However, to do this we will first have to obtain the orders of all functions occurring in relations (2.41) and (2.42).

2.5.1 Long wave low frequency regime

We recall that within the long wave low frequency regime there are no large or small parameters in the characteristic equation (1.77) and therefore all its roots are of order $O(1)$. Taking into account the above assumptions we employ the expansions for the exponential functions $\exp(-i\eta q_j)$, $j =$



(a)

(b)

Figure 2.7: *Scaled frequency Ω (vertical axis) against scaled wave number η (horizontal axis), numerical solution and approximations (2.37) and (2.40), obtained for the neo-Hookean material (1.116) with ϵ=2, p=1; (a) first harmonic, (b) second harmonic. Dotted curves correspond to third order approximations (2.40), dashed curves to second order approximations (2.37) and solid curves to numerical results.*

$1, 2, 3, 4$ up to and including $O(\eta)$ to establish the following leading order approximations for the displacement components u_1, u_2 and incremental pressure p_t

$$u_1 \approx \mathcal{C}\sum_{j=1}^{4} q_j A_j, \quad u_2 \approx \mathcal{C}\sum_{j=1}^{4} A_j, \quad p_t \approx \mathcal{C}k\sum_{j=1}^{4} \mathcal{P}(q_j) A_j, \tag{2.43}$$

where $\mathcal{C}$ is the constant.

Then we insert into the relations (2.42) the expansions for the exponential functions $\exp(-i\eta q_j)$ up to and including $O(\eta)$. Hence we obtain the dependance of the coefficients A_j, and thereby the displacement components u_1, u_2 and incremental pressure p_t, on a scaled wave number η. Finally the two displacement components at leading order can be expressed in the following form

$$\begin{aligned} &u_1 = F_1 D_1 \tilde{U}\eta + O(\eta^2), \quad u_2 = F_1 D_2 \lambda^{-1}\tilde{U}\eta + O(\eta^2), \\ &F_1 = \frac{\left(\lambda^2+1\right)(q_1-q_2)(q_1-q_3)(q_1-q_4)(q_2-q_3)(q_2-q_4)(-q_4+q_3)}{(1+\delta)^3\lambda^3}, \end{aligned} \tag{2.44}$$

where the exponential function $e^{ik(x_1-vt)}$ has been incorporated into the definition of the constant $\tilde{U}$.

In the representations (2.44) we employ long wave limits of two fundamental modes given by equations (2.9). We remark that for each of the two fundamental modes the coefficients in representation (2.44) are different but the asymptotic orders are equivalent. For the first case, with associated scaled phase speed $v_{1p}^{(0)}$ given by the equation $(2.9)_1$, the corresponding coefficients in representation (2.44) are the following

$$\begin{aligned}
&\textbf{Case 1: } D_1 = D_{11} = 0.5\,(1+\delta)\,\lambda^2\epsilon^2 \\
&+ \left(\left((-4\,\delta-4)\,\lambda^3+\lambda^2\,(1+\delta)+(4\,\delta+4)\,\lambda\right)p-\lambda^2\delta-2\,\lambda^3+2\,\lambda\right)\epsilon \\
&+ \left(-0.5\delta^2-\delta-0.5\right)\lambda^2p^2+\left((3\,\delta+3)\,\lambda^4+\left(\delta^2-5\,\delta-6\right)\lambda^2+2\,\delta+2\right)p \\
&+ \left(-3.5-2\,\delta-0.5\delta^2\right)\lambda^2+0.5\,(1+\delta)\,\lambda^2\sqrt{R}+(2+\delta)\,\lambda^4+1, \\
&D_2 = D_{12} = (1+\delta)\,\lambda^3\epsilon^3+\left((-\delta-1)\,\lambda^4+0.5\,(1+\delta)\,\lambda^2\right)\epsilon^2 \\
&+ \left(\left(2\,\delta^2+2\,\delta\right)\lambda^3p+2\,\lambda^5+\left(-1-\delta^2-2\,\delta\right)\lambda^3p^2\right)\epsilon \\
&+ \left(-\lambda^4+2\,\lambda+(-\delta-1)\,\lambda^3\sqrt{R}+\lambda^2+\left(-3-\delta^2\right)\lambda^3\right)\epsilon \\
&+ \left(\left(2\,\delta+\delta^2+1\right)\lambda^4+\left(-0.5\delta^2-\delta-0.5\right)\lambda^2\right)p^2 \\
&+ \left(\delta+3+\delta^2\right)\lambda^4+\left(\left(-\delta+1-2\,\delta^2\right)\lambda^4+\left(\delta^2+\delta\right)\lambda^2\right)p \\
&- \lambda^6+\left((1+\delta)\,\lambda^4-0.5\,(\delta+1)\,\lambda^2\right)\sqrt{R}+\left(-0.5\delta^2-2.5\right)\lambda^2+1, \\
&R = \left((p-1)^2\,\delta+(p+1)^2-\epsilon^2\right)^2+4\,\epsilon\,(p+1)^2\,.
\end{aligned} \tag{2.45}$$

In respect to the second case, with associated scaled phase speed $v_{2p}^{(0)}$ given by the equation $(2.9)_2$, the corresponding coefficients in representation (2.44) are expressed as

$$
\begin{aligned}
&\textbf{Case 2: } D_1 = D_{21} = 0.5\,(1+\delta)\,\lambda^2\epsilon^2\\
&+\big(\big((-4\,\delta-4)\,\lambda^3+\lambda^2\,(1+\delta)+(4\,\delta+4)\,\lambda\big)\,p-\lambda^2\delta-2\,\lambda^3+2\,\lambda\big)\,\epsilon\\
&+\big(-0.5\delta^2-\delta-0.5\big)\,\lambda^2p^2+\big((3\,\delta+3)\,\lambda^4+\big(\delta^2-5\,\delta-6\big)\,\lambda^2+2\,\delta+2\big)\,p\\
&+\big(-3.5-2\,\delta-0.5\delta^2\big)\,\lambda^2-0.5\,(\delta+1)\,\lambda^2\sqrt{R}+(2+\delta)\,\lambda^4+1,\\
&D_2 = D_{22} = (1+\delta)\,\lambda^3\epsilon^3+\big((-\delta-1)\,\lambda^4+0.5\,(1+\delta)\,\lambda^2\big)\,\epsilon^2\\
&+\big(\big(2\,\delta^2+2\,\delta\big)\,\lambda^3p+2\,\lambda^5+\big(-1-\delta^2-2\,\delta\big)\,\lambda^3p^2\big)\,\epsilon\\
&+\Big(-\lambda^4+2\,\lambda+(1+\delta)\,\lambda^3\sqrt{R}+\lambda^2+\big(-3-\delta^2\big)\,\lambda^3\Big)\,\epsilon\\
&+\big(\big(2\,\delta+\delta^2+1\big)\,\lambda^4+\big(-0.5\delta^2-\delta-0.5\big)\,\lambda^2\big)\,p^2\\
&+\big(\delta+3+\delta^2\big)\,\lambda^4+\big(\big(-\delta+1-2\,\delta^2\big)\,\lambda^4+\big(\delta^2+\delta\big)\,\lambda^2\big)\,p\\
&-\lambda^6+\big((-\delta-1)\,\lambda^4+0.5\,(1+\delta)\,\lambda^2\big)\,\sqrt{R}+\big(-0.5\delta^2-2.5\big)\,\lambda^2+1.
\end{aligned}
\tag{2.46}
$$

For the class of strain energy functions for which $\delta = 0$, the representations of the displacement components are given by

$$
\begin{aligned}
&\textbf{Case 1: } \hat{v}^{(0)} = \hat{v}_1^{(0)} = 1-p^2,\\
&u_1 = \left(\frac{16\mathcal{M}_1^1\left((p-1)^2+\epsilon^2\right)\left((p+1)^2+\epsilon^2\right)p\epsilon}{\left(\epsilon+\sqrt{\epsilon^2+4}\right)^3}\right)\tilde{U}\eta+O(\eta^2),\\
&u_2 = \left(\frac{32\mathcal{M}_2^1\left((p-1)^2+\epsilon^2\right)\left((p+1)^2+\epsilon^2\right)(p+1)\,p}{\left(\epsilon+\sqrt{\epsilon^2+4}\right)^4}\right)\tilde{U}\eta+O(\eta^2),\\
&\mathcal{M}_1^1 = \sqrt{\epsilon^2+4}\left(2(p+1)+\epsilon^2(p+4)+\epsilon^4\right)+\epsilon^5+\epsilon^3(p+6)+4\epsilon(p+2),\\
&\mathcal{M}_2^1 = \left(\epsilon^5+\epsilon^3(p+5)+\epsilon(3p+5)\right)\sqrt{\epsilon^2+4}+\epsilon^6+\epsilon^4(p+7)+\epsilon^2(5p+13)+4(p+1),
\end{aligned}
\tag{2.47}
$$

$$
\begin{aligned}
&\textbf{Case 2: } \hat{v}^{(0)} = \hat{v}_2^{(0)} = \epsilon^2 + 2p + 2, \\
&u_1 = \left(\frac{64\mathcal{M}_1^2 \left((p+1)^2 + \epsilon^2 \right) (p+1) \sqrt{1+\epsilon^2+2p}}{\left(\epsilon + \sqrt{\epsilon^2+4} \right)^3} \right) \tilde{U}\eta + O(\eta^2), \\
&u_2 = -\left(\frac{128\mathcal{M}_2^2 \left((p+1)^2 + \epsilon^2 \right) \epsilon \sqrt{1+\epsilon^2+2p}}{\left(\epsilon + \sqrt{\epsilon^2+4} \right)^4} \right) \tilde{U}\eta + O(\eta^2), \\
&\mathcal{M}_1^2 = \left(\epsilon^2 p + 3p + 1 \right) \epsilon \sqrt{\epsilon^2+4} + \epsilon^4 p + \epsilon^2(5p+1) + 4, \\
&\mathcal{M}_2^2 = \left(\epsilon^4 p + \epsilon^2(4p+1) + 2(p+1) \right) \sqrt{\epsilon^2+4} + \epsilon^5 p + \epsilon^3(6p+1) + 4\epsilon(2p+1).
\end{aligned} \tag{2.48}
$$

We remark that in the representation (2.47), the in-plane displacement component u_1 has a factor ϵ at leading order, whereas in the formulation (2.48), the normal displacement component u_2 has an equivalent factor. We may therefore conclude that if $\epsilon \sim O(1)$ the orders of displacement components u_1 and u_2 are equivalent. This contrasts with the classical cases, see for example Kaplunov et al. (1998), in which the following relation are valid in the first case (bending) $u_2 >> u_1$ and in the second case (extension) $u_1 >> u_2$. It also contrasts with their pre-stress counterparts for which one principal axis is normal to the plate, see Kaplunov et al. (2000), within which the classical structure is preserved.

However if we consider small amount of shear, i.e. $\epsilon \sim O(\eta^n)$, where $n = 1, 2, ...$ the representations (2.47) and (2.48) the asymptotic orders of displacements will gradually approach the relations typical for in the classical case. If we consider $\epsilon \sim O(\eta)$ in the first case (2.47) one can see that $u_2 > u_1$ due to factor η of in-plane displacement component and in the second case (2.48) $u_1 > u_2$ as normal displacement component has an equivalent factor . The difference in displacement orders is gradually increasing with $n = 2, 3...$ Therefore qualitatively the transition from the case $\epsilon \sim O(1)$ to the classical case when $\epsilon = 0$ goes smoothly.

We remark that in the case of $\epsilon = 0$ the general results (2.44)–(2.46) as well as simplified formulations (2.47)–(2.48) reduce to

$$
\begin{aligned}
&\textbf{Case 1: Bending} \quad \hat{v}_1^{(0)} = 1 - p^2 : \quad \mathcal{M}_1^2 = \mathcal{M}_2^2 = 4(p+1), \\
&u_1 = O(\eta^2), \quad u_2 = 8\,(p+1)^4\,(p-1)^2\, p\tilde{U}\eta + O(\eta^2); \\
&\textbf{Case 2: Extension} \quad \hat{v}_2^{(0)} = 2p + 2 : \quad \mathcal{M}_1^1 = \mathcal{M}_2^1 = 4(p+1), \\
&u_1 = 32\,(p+1)^4\, \tilde{U}\sqrt{1+2p}\eta + O(\eta^2), \quad u_2 = O(\eta^2).
\end{aligned} \tag{2.49}
$$

The relative orders of displacement components given by (2.49) agree with previously published results, see Kaplunov et al. (2000).

The asymptotic order of the incremental pressure p_t in the general case may be established by use of (2.43) and shown to be of the following form for both fundamental modes

$$p_t = k\mathcal{Q}^{(i)}\tilde{U}\eta + O(\eta^2), \tag{2.50}$$

with the coefficient $\mathcal{Q}^{(i)}$ generally of order $O(1)$ and the indexes $i = 1, 2$ represent each of the two fundamental modes. For each associated fundamental modes, the coefficient $\mathcal{Q}^{(i)}$ is algebraic complicated both in the general case and for $\delta = 0$, however the explicit representation for $\mathcal{Q}^{(i)}$ for both cases was obtained using Maple (1996).

Hence, according to relations (2.44) and (2.50) the relative asymptotic orders of displacement components u_1, u_2 and incremental pressure p_t for both fundamental modes are equivalent

$$p_t \sim u_1 \sim u_2. \tag{2.51}$$

However, in the case of $\epsilon = 0$ the results are different. The coefficients in representation (2.50) become

$$\begin{aligned} &\textbf{Case 1: Bending } \hat{v}_1^{(0)} = 1 - p^2: \quad \mathcal{Q}_0^{(1)} = 0, \\ &\textbf{Case 2: Extension } \hat{v}_2^{(0)} = 2p + 2: \quad \mathcal{Q}_0^{(2)} = 32\mu\sqrt{1+2p}(1+p)^5\tilde{U}. \end{aligned} \tag{2.52}$$

Therefore, from relations (2.49), (2.52) we can conclude that in the case of $\epsilon = 0$ the relative asymptotic orders of displacement components and incremental pressure for both bending and extension motions are connected through

$$p_t \sim u_1 \sim \eta u_2. \tag{2.53}$$

The above relations (2.53) coincide with previously published results, see Kaplunov et al. (2000).

2.5.2 Long wave high frequency regime

In the long wave high frequency regime the representation for the roots q_i is given by (2.24). We employ (2.24) to obtain expansions for the exponential functions $e^{i\eta q_i \zeta}$ up to $O(\eta)$ and insert them into the relations (2.41). Then the displacement components u_1, u_2 and incremental pressure p_t can be represented in the following forms

$$\begin{aligned} &u_1 \approx \mathcal{C}(\sum_{j=1}^{2} q_j A_j + \sum_{j=3}^{4} q_j A_j e^{-i\zeta n\pi}), \quad u_2 \approx \mathcal{C}(\sum_{j=1}^{2} A_j + \sum_{j=3}^{4} A_j e^{-i\zeta n\pi}), \\ &p_t \approx \mathcal{C}k(\sum_{j=1}^{2} \mathcal{P}(q_j) A_j + \sum_{j=3}^{4} \mathcal{P}(q_j) A_j e^{i\zeta n\pi}), \end{aligned} \tag{2.54}$$

where $\mathcal{C}$ is the constant.

Similarly, using relations (2.24), we employ the following approximations for the exponential functions up to $O(\eta^2)$

$$
\begin{aligned}
&e^{-i\eta q_1} = 1 + \eta + O(\eta^2), \quad e^{-i\eta q_2} = 1 - \eta + O(\eta^2), \\
&e^{-i\eta q_3} = (-1)^n(1 - i\eta\epsilon) + O(\eta^2), \quad e^{-i\eta q_4} = (-1)^n(1 - i\eta\epsilon) + O(\eta^2).
\end{aligned} \tag{2.55}
$$

We remark that if we consider the reduced approximations (2.55) up to $O(\eta)$ the representation (2.42) will give identically zero coefficients $A_j, j = 1, 2, 3, 4$. Then we substitute expansions (2.55) into relations (2.42) to obtain the dependance of the coefficients A_j on scaled wave number η. After that we make use of (2.54), take into account the dependance of coefficients A_j on η and normalize order by η^7 to establish the asymptotic orders of displacement components u_1, u_2 and incremental pressure p_t giving by

$$
\begin{aligned}
&u_1 = C_6^{(1)}\eta + C_5^{(1)}\eta^2 + O(\eta^3), \quad u_2 = C_5^{(2)}\eta^2 + O(\eta^3), \qquad (2.56)\\
&C_6^{(1)} = \frac{4n^7\pi^7\,(-1)^n\left(\lambda^2+1\right)\cosh(\zeta n\pi)\tilde{U}}{\lambda\,(1+\delta)^3}, \quad C_5^{(1)} \neq 0,\\
&C_5^{(2)} = \frac{2\left(\lambda^2+1\right)\pi^5 n^5 \mathcal{C}_u \tilde{U}}{\lambda^3\,(1+\delta)^3}, \quad \mathcal{C}_u = -2\pi\,\lambda^2 n\,(-1)^n\sinh(\zeta n\pi) + \\
&+ 2\,(1-(-1)^n)\left(\lambda^4 - 2\,\lambda^3\epsilon - \lambda^2\left(p\,(1+\delta) + \epsilon^2 + 6 + 2\,\delta\right) + 2\,\epsilon + 1\right),
\end{aligned}
$$

where the exponential function $e^{ik(x_1-vt)}$ has been incorporated into the definition of constant $\tilde{U}$ and the explicit expression of the coefficient $C_5^{(1)}$ was shown to be non-zero using Maple (1996).

The incremental pressure p_t takes the following form

$$
\begin{aligned}
&p_t = kc^{(7)} + kc^{(6)}\eta + O(\eta^2),\\
&c_7 = c_1^{(7)}\epsilon + c_0^{(7)}, \quad c_6 = c_2^{(6)}\epsilon^2 + c_1^{(6)}\epsilon + c_0^{(6)}.
\end{aligned} \tag{2.57}
$$

In the representation (2.57) all the coefficients $c_1^{(7)}, c_0^{(7)}, c_2^{(6)}$ and $c_1^{(6)}$ are quite lengthy both within general case and in the case of $\delta = 0$, however their explicit expressions have been obtained using Maple (1996) and were shown to be of order $O(1)$.

From the above representations (2.56) and (2.57) we are now ready to deduce that relative asymptotic orders of the displacement components u_1, u_2 and incremental pressure p_t are connected through

$$
p_t \sim \eta u_1 \sim \eta^2 u_2. \tag{2.58}
$$

Thus, the incremental pressure p_t is asymptotically leading, with the in-plane displacement component u_1 very much larger than its normal counterpart u_2.

In the case of $\epsilon = 0$ the displacement components u_1 and u_2 remain of orders $O(\eta^{-6})$ and $O(\eta^{-5})$ respectively with the following coefficients in representation (2.56)

$$C_6^{(1)}(\epsilon = 0, \delta = 0, \lambda = 1) = \tilde{U} 8 n^7 \pi^7 \, (-1)^n \cosh(\zeta n \pi), \quad C_5^{(1)}(\epsilon = 0, \delta = 0, \lambda = 1) \neq 0,$$
$$C_5^{(2)}(\epsilon = 0, \delta = 0, \lambda = 1) = \tilde{U} 8 \pi^5 n^5 (((-1)^n - 1)(p + 4) - n\pi(-1)^n \sinh(\zeta n \pi)). \tag{2.59}$$

Taking into account $\epsilon = 0$ the incremental pressure p_t is of order $O(\eta^{-6})$ and the corresponding coefficients in the representation (2.57) are given by

$$c_0^{(7)}(\epsilon = 0, \Omega = \Omega^{(0)}) = 0, \quad c_0^{(6)}(\epsilon = 0, \Omega = \Omega^{(0)}) = 8\mu\pi^7 n^7 (-1)^n (p + 1)\tilde{U}. \tag{2.60}$$

Therefore we conclude that in the case of $\epsilon = 0$ representations (2.56)–(2.57) and (2.60)–(2.59) yield the following relation between asymptotic orders of displacements u_1, u_2 and incremental pressure p_t

$$u_1 \sim p_t \sim \eta u_2, \tag{2.61}$$

where the incremental pressure p_t and in-plane displacement u_1 are of the same asymptotic order. The relations (2.61) coincide with previously published results, see Kaplunov et al. (2002).

Chapter 3

Long wave motion in a layer subject to some non-classical boundary conditions

In this chapter we analyze long wave motion in a sheared pre-stressed incompressible elastic layer with some non-classical boundary conditions, namely, fixed faces and one fixed and one free face boundary conditions. To begin with, we derive the associated dispersion relations first in respect of the most general incompressible strain-energy function and then simplified forms for specific class of materials. Numerical dispersion curves are produced for the neo-Hookean and Varga material models. The numerical analysis of the dispersion relations reveals that there are no fundamental modes for both problems. Guided by the numerical results we proceed with a long wave asymptotic analysis of the dispersion relations. In respect of long wave high frequency motion in a layer with fixed faces, there are two families of cut-off frequencies. For each of these families second and third order corrections to the cut-off frequencies are obtained. In respect of a layer with one fixed and one free face, the cut-off frequencies are established. For a specific class of materials their second and third order corrections are derived. Finally, the relative asymptotic orders of the displacement components and incremental pressure are established. For both problems the incremental pressure is asymptotically leading, with the in-plane displacement component very much larger than its normal counterpart.

3.1 The dispersion relations

3.1.1 Layer with fixed faces

We consider fixed face boundary conditions, i.e. zero displacement on the lower and upper surfaces of the layer, which can be expressed as

$$u_1 = u_2 = 0 \quad \text{at} \quad x_2 = 0, -h. \tag{3.1}$$

Inserting the solutions for displacement components (1.83) into the boundary conditions (3.1) we obtain the following homogeneous system of four linear equations in the four unknown constants $A_i, i = 1, 2, 3, 4$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ q_1 & q_2 & q_3 & q_4 \\ e^{-i\eta\, q_1} & e^{-i\eta\, q_2} & e^{-i\eta\, q_3} & e^{-i\eta\, q_4} \\ q_1 e^{-i\eta\, q_1} & q_2 e^{-i\eta\, q_2} & q_3 e^{-i\eta\, q_3} & q_4 e^{-i\eta\, q_4} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = 0. \tag{3.2}$$

A non-trivial solution of the systems (3.2) will exist provided

$$D_f = det \begin{bmatrix} 1 & 1 & 1 & 1 \\ q_1 & q_2 & q_3 & q_4 \\ e^{-iq_1\eta} & e^{-iq_2\eta} & e^{-iq_3\eta} & e^{-iq_4\eta} \\ q_1 e^{-iq_1\eta} & q_2 e^{-iq_2\eta} & q_3 e^{-iq_3\eta} & q_4 e^{-iq_4\eta} \end{bmatrix} = 0. \tag{3.3}$$

Performing operations with the rows and columns, the dispersion relation (3.3) may be expressed as

$$D_f = det \begin{bmatrix} C_1 & C_2 & C_3 & C_4 \\ q_1 C_1 & q_2 C_2 & q_3 C_3 & q_4 C_4 \\ S_1 & S_2 & S_3 & S_4 \\ q_1 S_1 & q_2 S_2 & q_3 S_3 & q_4 S_4 \end{bmatrix} = 0, C_j = \cos\left(q_j \eta/2\right), S_j = \sin\left(q_j \eta/2\right), j = 1, 2, 3, 4. \tag{3.4}$$

We deduce from the representation (3.4) that the dispersion relation may be shown to always provide a real equation for all possible types of the roots of characteristic equation (1.77). For a specific class of materials for which $\delta = 0$ the dispersion relation (3.4) takes the following simplified form

$$\begin{aligned} &D_{fs} = 2\,\kappa\, \left(\cosh\left(\eta\right)\cosh\left(\eta\,\kappa\right) - \cos\left(\eta\,\epsilon\right)\right) - \left(1 + \epsilon^2 + \kappa^2\right)\sinh\left(\eta\right)\sinh\left(\eta\,\kappa\right) = 0, \\ &\kappa = \sqrt{1 - \hat{v}}. \end{aligned} \tag{3.5}$$

3.1.2 Layer with one free and one fixed faces

Our consideration now is the analogous problem with one free and one fixed face. For practical applications of this system we specify thin coatings and cite the study by Fu (2007). This type of boundary conditions were studied largely numerically in the paper by Connor and Ogden (1996) and can be expressed as

$$\begin{aligned} &\tau_1 = \tau_2 = 0 \quad \text{at} \quad x_2 = -h, \\ &u_1 = u_2 = 0 \quad \text{at} \quad x_2 = 0. \end{aligned} \tag{3.6}$$

Taking into account the solutions for displacement components (1.83) the boundary conditions (3.6) yield the following homogeneous system in the four unknown constants $A_i, i = 1, 2, 3, 4$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ q_1 & q_2 & q_3 & q_4 \\ f_1 e^{-i\eta\, q_1} & f_2 e^{-i\eta\, q_2} & f_3 e^{-i\eta\, q_3} & f_4 e^{-i\eta\, q_4} \\ g_1 e^{-i\eta\, q_1} & g_2 e^{-i\eta\, q_2} & g_3 e^{-i\eta\, q_3} & g_4 e^{-i\eta\, q_4} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = 0, \tag{3.7}$$

resulting in the following dispersion relation

$$D_m = det \begin{bmatrix} 1 & 1 & 1 & 1 \\ q_1 & q_2 & q_3 & q_4 \\ f_1 e^{-i\eta q_1} & f_2 e^{-i\eta q_2} & f_3 e^{-i\eta q_3} & f_4 e^{-i\eta q_4} \\ g_1 e^{-i\eta q_1} & g_2 e^{-i\eta q_2} & g_3 e^{-i\eta q_3} & g_4 e^{-i\eta q_4} \end{bmatrix} = 0. \tag{3.8}$$

Employing various theorems from linear algebra, for details see G. Korn and T. Korn (1968), Beklemishev (1971), and using Maple (1996) it may be shown that the dispersion relation (3.8) always provides a real equation for all possible types of the roots of characteristic equation (1.77). For a specific class of materials for which $\delta = 0$, the dispersion relation (3.8) takes the simplified form previously derived by Connor and Ogden (1996), namely

$$\begin{aligned} \mathcal{D}_{ms} = &\left((2\,p+2)\,\kappa^3 + \left(2\,p^2 + 2\,\left(1+\epsilon^2\right)p\right)\kappa\right)\cos\left(\epsilon\,\eta\right) + \\ &\left(\kappa^4 + \left(p^2 + 4\,p + \epsilon^2 + 1\right)\kappa^2 + \left(1+\epsilon^2\right)p^2\right)\sinh\left(\eta\,\kappa\right)\sinh\left(\eta\right) + \\ &\left(-\kappa^5 + \left(-2\,p - 2\,\epsilon^2\right)\kappa^3 + \left(-2\,p^2 - \left(2\,\epsilon^2+2\right)p - \epsilon^4 - 1 - 2\,\epsilon^2\right)\kappa\right)\cosh\left(\eta\,\kappa\right)\cosh\left(\eta\right) = 0, \\ &\kappa = \sqrt{1-\hat{v}}. \end{aligned} \tag{3.9}$$

3.2 Numerical analysis of the dispersion relations

3.2.1 Layer with fixed faces

In this subsection, we investigate the numerical solution of the dispersion relation for a layer with fixed faces. The plots are calculated for the neo-Hookean material (1.116) and show the scaled squared phase speed $\hat{v}$ and scaled frequency Ω against scaled wave number η, see Figure 3.1(a) and (b), respectively. The plots in the Figures 3.1 indicate that there is no low frequency motion in a layer with fixed faces, which means that the only type of motion is the high frequency motion. In addition, the plots in Figure 3.1 reveal that in the vicinity of the cut-off frequencies, the harmonics with even and odd mode number have positive and negative gradients respectively. The negative gradients are associated with the existence of a negative group velocity.

3.2.2 Layer with one free and one fixed faces

In this subsection we discuss the numerical solution of the dispersion (3.8) for a layer with one fixed and one free face. For these numerical calculations the Varga material model (1.122) has been employed. The plots of the dispersion relation show scaled squared phase speed $\hat{v}$ and scaled frequency Ω against scaled wave number η in Figure 3.2(a) and (b), respectively. Similar to a layer with fixed faces, the plots indicate that there is no low frequency motion in a layer with one fix and one free faces.

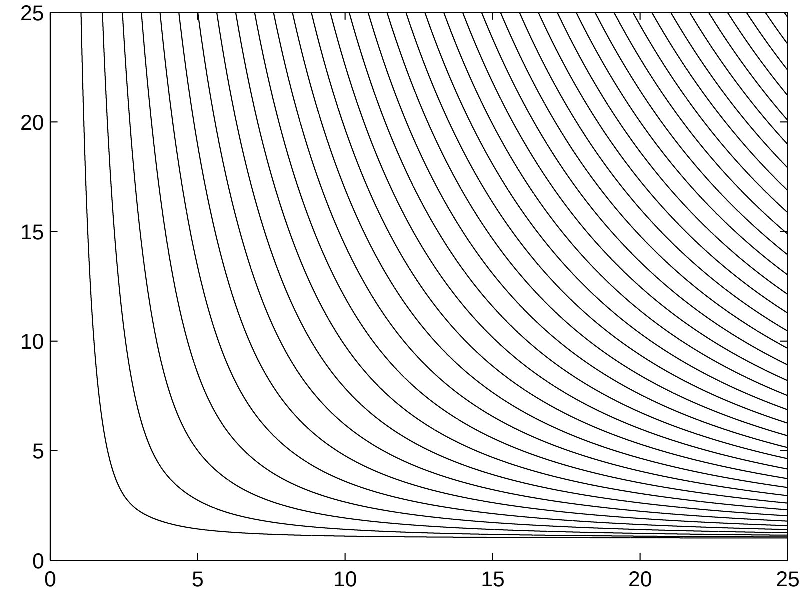
3.3 Analysis of the dispersion relations for long wave high frequency motion

In this section we will examine the long wave regime of the harmonics. We begin with a derivation of appropriate long wave high frequency asymptotic expansions of the dispersion relation (3.4). With the help of these approximations we establish the cut-off frequencies and represent scaled frequency as an explicit function of scaled wave number η.

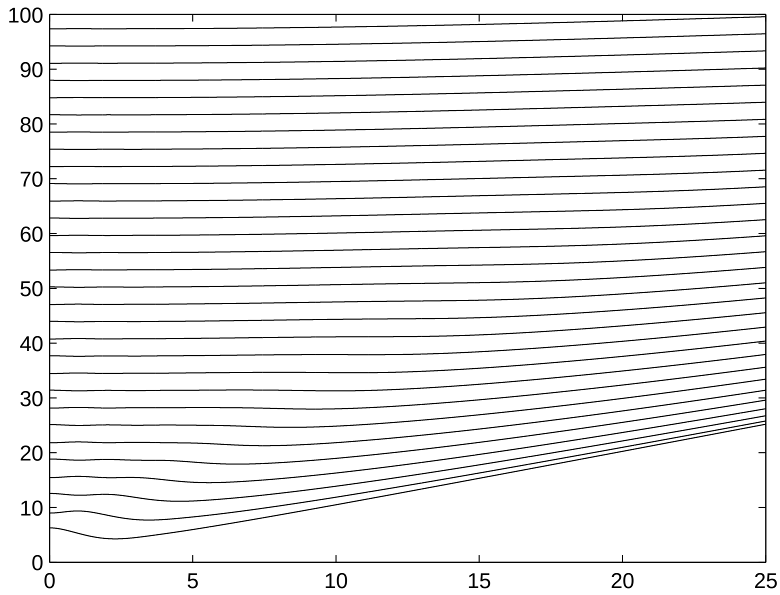
3.3.1 Layer with fixed faces

Long wave high frequency approximations

Our consideration now is long wave high frequency motion in a layer with fixed faces in respect of the most general incompressible strain-energy function. To derive approximations of the disper-

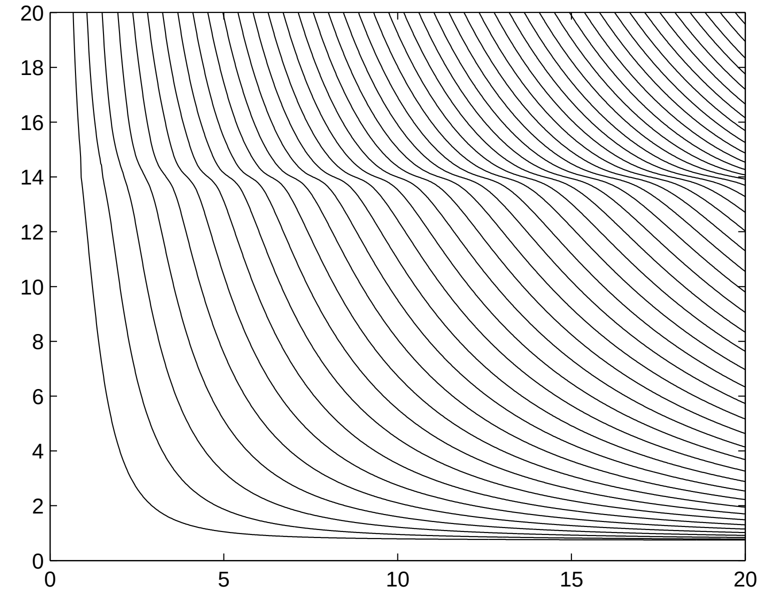


(a)

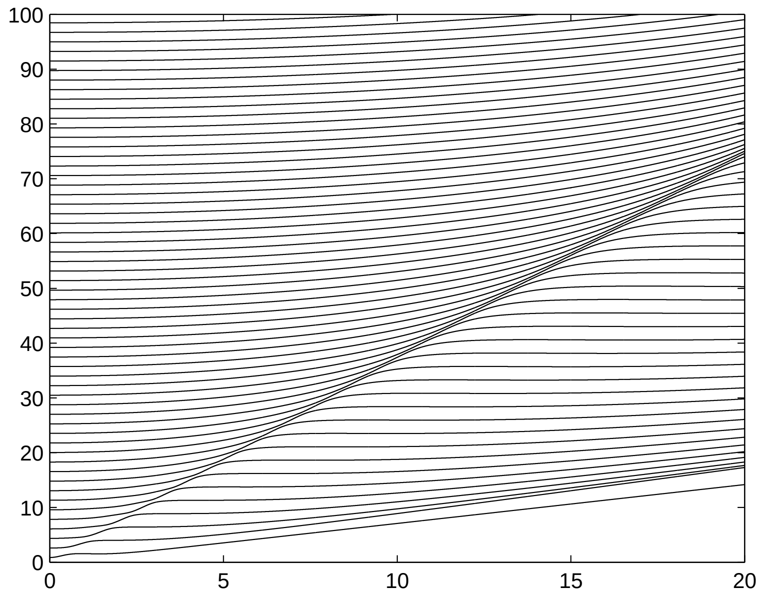


(b)

Figure 3.1: *Layer with fixed faces: numerical solution of the dispersion relation for the neo-Hookean material (1.116) with parameters* $\epsilon = 3, p = 2$. *The scaled squared phase speed* $\hat{v}$ *(vertical axis) and scaled frequency* Ω *(vertical axis) against scaled wave number* η *are shown in (a) and (b), respectively.*



(a)



(b)

Figure 3.2: *Layer with one free and one fixed faces: numerical solution of the dispersion relation for a Varga material (1.124) with parameters* $\epsilon = 3, p = 2$*. The scaled squared phase speed* $\hat{v}$ *(vertical axis) and scaled frequency* Ω *(vertical axis) against scaled wave number* η *are shown in (a) and (b) respectively.*

sion relation we employ a third order expansion for the scaled frequency Ω (2.30) in the vicinity of cut-off frequencies $\Omega^{(0)}$ and consider the dispersion relation in the form (3.4). Using relations (2.24) and (2.30) we obtain expansions for the trigonometric functions $S_j, C_j, j = 1,2,3,4$ up to $O(\eta^5)$. We remark that these expansions are too algebraically complex to be represented here, nevertheless their explicit representations were obtained using Maple (1996) and employed to establish the approximation of dispersion relation (3.4) in the following form

$$D_f = \mathcal{D}_f^{(0)} + \mathcal{D}_f^{(2)}\eta^2 + \mathcal{D}_f^{(4)}\eta^4 + O(\eta^6). \tag{3.10}$$

The condition that the following leading order term $\mathcal{D}_{fh}^{(0)}$ is equal to zero, reveals that

$$\mathcal{D}_f^{(0)} = 2\,i\left(\sqrt{1+\delta}\Omega^{(0)} - 2\,\tan\left(\frac{1}{2}\sqrt{1+\delta}\Omega^{(0)}\right)\right)\Omega^{(0)}\sqrt{1+\delta}\tan\left(\frac{1}{2}\sqrt{1+\delta}\Omega^{(0)}\right) = 0, \tag{3.11}$$

providing two families of cut-off frequencies, the first family given by

$$\Omega_{1f}^{(0)} = \frac{2n\pi}{\sqrt{1+\delta}}, \tag{3.12}$$

and the second family expressed as the solution of the following transcendent equation

$$\Omega_{2f}^{(0)} = \Omega_0, \quad \tan\left(\frac{\sqrt{1+\delta}\Omega_0}{2}\right) = \frac{\sqrt{1+\delta}\Omega_0}{2}. \tag{3.13}$$

The second order term $\mathcal{D}_f^{(2)}$ is too algebraic complex to be represented here, however its explicit representation was obtained with the help of Maple (1996). Using the equation $\mathcal{D}_f^{(2)} = 0$ we establish the following second order corrections to cut-off frequencies (3.12)

$$\Omega_{1f}^{(2)} = -\frac{2\epsilon^2 - 4\delta + 1}{4\pi n\sqrt{1+\delta}}, \tag{3.14}$$

with the correction to (3.13) given by

$$\Omega_{2f}^{(2)} = \frac{12\,\delta + 1 + 6\,\epsilon^2}{6\Omega_0\,(1+\delta)}. \tag{3.15}$$

Therefore, for the scaled frequency Ω in the vicinity of cut-off frequencies $\Omega_{if}^{(0)}$ the following second order approximation is valid

$$\Omega = \Omega_{if}^{(0)} + \Omega_{if}^{(2)}\eta^2 + O(\eta^4), \tag{3.16}$$

where $i = 1, 2$, cut-off frequencies $\Omega_{if}^{(0)}$ are given by (3.12)–(3.13) and the second order corrections $\Omega_{if}^{(2)}$ are expressed in relations (3.14)–(3.15), respectively.

Finally, taking into account the explicit representation of the third order term $\mathcal{D}_f^{(4)}$ obtained in Maple (1996), the second order approximation (3.16), and the condition $\mathcal{D}_f^{(4)} = 0$, we obtain the following third order corrections for cut-off frequencies (3.12)

$$\Omega_{1f}^{(4)} = \frac{1}{192}\frac{8\epsilon^4(\pi^2n^2-3)+4\epsilon^2(4\pi^2n^2-12-15\delta)+8\pi^2n^2-48\delta^2-120\delta-15}{\pi^3n^3\sqrt{1+\delta}}, \tag{3.17}$$

and for (3.13) given by

$$\begin{aligned}
&\Omega_{2f}^{(4)} = \frac{\mathcal{F}_0}{72(1+\delta)^3\Omega_0^5},\\
&\mathcal{F}_0 = -144{\Omega_0}^2\delta^3 + (6\epsilon^4(6-\Omega_0^4)+12\epsilon^2(12+50\Omega_0^2-\Omega_0^4)+2(288-132\Omega_0^2-\Omega_0^4))\delta^2\\
&-(12\epsilon^4(8\Omega_0^2+12+\Omega_0^4)+12\epsilon^2(52-39\Omega_0^2+12\Omega_0^4)-192+117\Omega_0^2+4\Omega_0^4)\delta\\
&-\Omega_0^2(6(\Omega_0^2+16)\epsilon^4+12(\Omega_0^2+11)\epsilon^2+2\Omega_0^2-3).
\end{aligned} \tag{3.18}$$

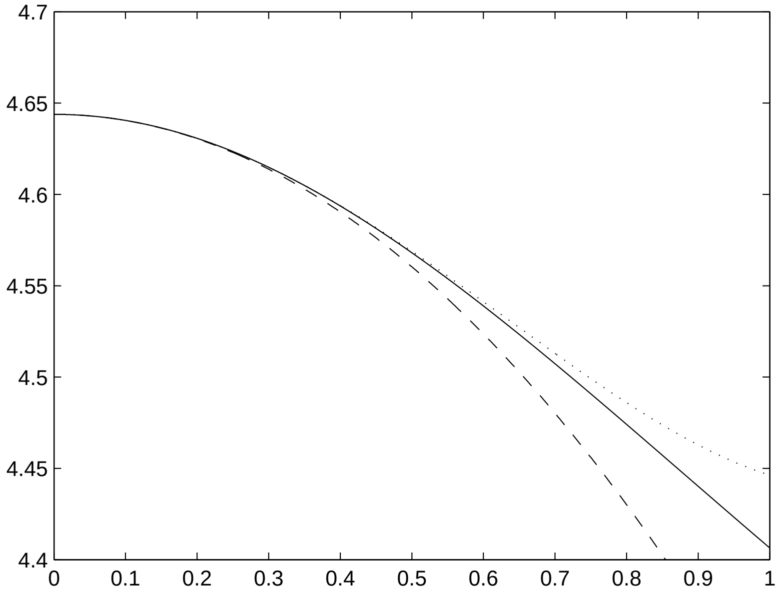
Therefore for each harmonic of dispersion relation (3.4) the associated scaled frequency Ω in the vicinity of cut-off frequencies $\Omega_{if}^{(0)}$ is determined by the following third order approximation

$$\Omega = \Omega_{if}^{(0)} + \Omega_{if}^{(2)}\eta^2 + \Omega_{if}^{(4)}\eta^4 + O(\eta^6), \tag{3.19}$$

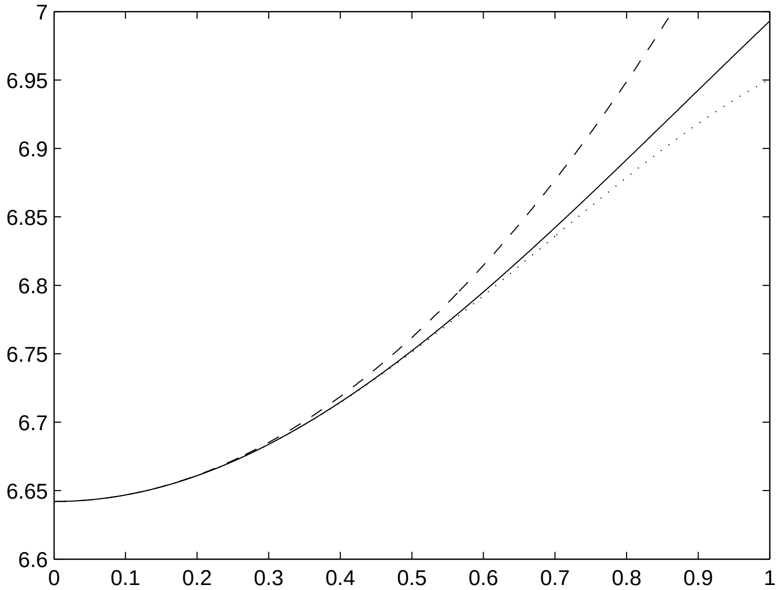
where $i = 1, 2$, cut-off frequencies $\Omega_{if}^{(0)}$ are given by equations (3.12)–(3.13), second order corrections $\Omega_{if}^{(2)}$ are expressed in relations (3.14)–(3.15) and third order corrections $\Omega_{if}^{(4)}$ are represented by (3.17)–(3.18).

We now represent some illustrative numerical results, for which the Varga strain-energy function (1.124) is employed. In Figure 3.3 comparisons of a numerical solution and approximations (3.16), (3.19) are presented for the first two harmonics in the vicinity of associated cut-off frequencies. Each harmonic is represented on a separate plot to illustrate the good quality of the approximations (3.16) and (3.19), with the third order approximations (3.19) being the closest to the numerical solution.

The numerical analysis of the dispersion relation (3.4) reveals the positive and negative gradients exist in the vicinity of cut-off frequencies. Now we are able to investigate this phenomenon by use of the explicit representations of second order corrections $\Omega_{if}^{(2)}$ (3.14)–(3.15), which indicate that the sign of $\Omega_{if}^{(2)}$, $i = 1, 2$ depends on the amount of shear ϵ. To illustrate this phenomenon we have plotted the second order corrections $\Omega_{if}^{(2)}$, $i = 1, 2$ as a functions of ϵ for a Varga material in the Figure 3.4. We conclude that for a fixed face boundary value problem negative group velocity exists for certain values of ϵ and p. The group velocity is the parameter which governs the motion of a pulse for dispersive waves and can be defined as the gradient of the dispersion



(a)



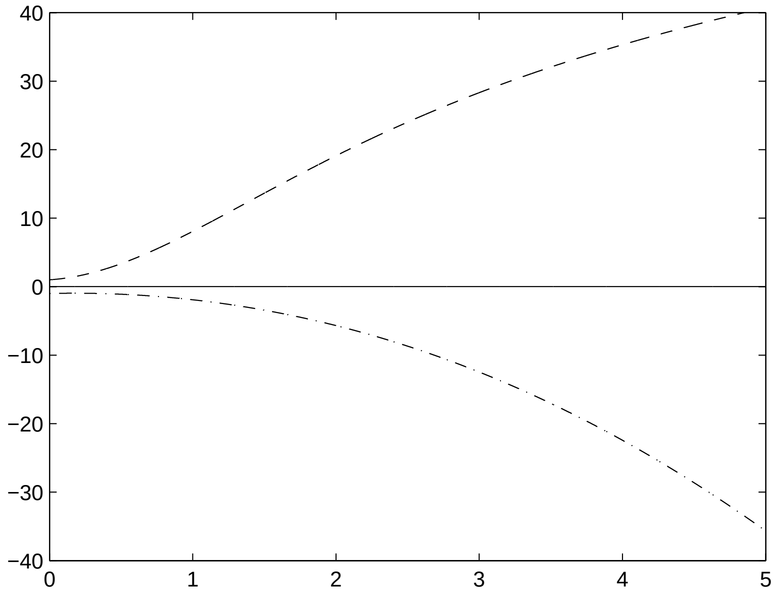
(b)

Figure 3.3: *Layer with fixed faces: scaled frequency Ω (vertical axis) against scaled wave number η (horizontal axis) for long wave high frequency motion, numerical solution together with second order approximations (3.16) and third order approximations (3.19) calculated for the Varga material (1.124) with ϵ=2, p=1; (a) first harmonic, (b) second harmonic. Dotted curves correspond to third order approximations (3.19), dashed curves to second order approximations (3.16) and solid curves to numerical results.*

curves

$$v_g = \frac{\partial \omega}{\partial kh} \approx \frac{A}{\sqrt{\alpha\gamma}\Omega^{(0)}} kh. \tag{3.20}$$

We remark that the phenomena of negative group velocity can be found in a number of papers for example by Fang et al. (2006), Picu (2002) and Nolde (2007).



(a)

Figure 3.4: *Layer with fixed faces: Second order frequency corrections* $\Omega_{if}^{(2)}$, $i = 1, 2$ *(vertical axis) as a functions of* ϵ *(horizontal axis) for Varga material (1.124). Upper dashed curve corresponds to second order corrections* $\Omega_{2f}^{(2)}$ *and lower dashed curve corresponds to second order correction* $\Omega_{1f}^{(2)}$.

Long wave high frequency approximations in the case $\delta = 0$

For a specific class of material for which $\delta = 0$, we will derive third order expansions for the scaled frequency in the vicinity of cut-off frequencies. We substitute the expansions (2.31), together with Taylor series expansions for the functions $\sinh(\eta), \cosh(\eta), \cos(\eta\epsilon)$ up to $O(\eta^5)$, into the dispersion relation (3.5), its approximation then being given by

$$D_{fh} = \mathcal{D}_{fh}^{(0)} + \mathcal{D}_{fh}^{(2)}\eta^2 + \mathcal{D}_{fh}^{(4)}\eta^4 + O(\eta^6). \tag{3.21}$$

The leading order term $\mathcal{D}^{(0)}_{fh}$ is of the following form

$$\mathcal{D}^{(0)}_{fh} = i\Omega^{(0)} \left(\Omega^{(0)} \sin\Omega^{(0)} + 2(\cos\Omega^{(0)} - 1)\right). \tag{3.22}$$

If we express $sin\Omega^{(0)}$ and $cos\Omega^{(0)}$ in terms of $tan(\Omega_0/2)$, the representation (3.22) takes the following form

$$\mathcal{D}^{(0)}_{fhs} = \frac{2\tan(\Omega^{(0)}/2)(\Omega^{(0)} - 2\tan(\Omega^{(0)}/2))}{1+\tan^2(\Omega^{(0)}/2)}. \tag{3.23}$$

The condition $\mathcal{D}^{(0)}_{fhs} = 0$ gives the two following families of cut-off frequencies

$$\tan\frac{\Omega^{(0)}}{2} = 0, \quad \Omega^{(0)}_{1fh} = 2n\pi, \tag{3.24}$$

$$\tan\frac{\Omega^{(0)}}{2} = \frac{\Omega^{(0)}}{2}, \quad \Omega^{(0)}_{2fh} = \Omega_0, \tag{3.25}$$

The representations (3.24)–(3.25) coincide with general formulations for cut-off frequencies (3.12)–(3.13) provided $\delta = 0$.

The second order term in the expansion (3.21) is given by

$$\begin{aligned}
\mathcal{D}^{(2)}_{fh} &= \frac{i\mathcal{F}^{(2)}_0}{6\Omega^{(0)}}, \\
\mathcal{F}^{(2)}_0 &= (((\Omega^{(0)})^3 + 2)\cos\Omega^{(0)} - 2)6\Omega^{(0)}\Omega^{(1)} + ((\Omega^{(0)})^2 - 2)3\cos\Omega^{(0)} \\
&+ ((\Omega^{(0)})^2 - 6(1+\epsilon^2))\Omega^{(0)}\sin\Omega^{(0)} + 6(\epsilon^2(\Omega^{(0)})^2 + 1).
\end{aligned} \tag{3.26}$$

The equation $\mathcal{D}^{(2)}_{fh} = 0$ provides the following second order corrections for cut-off frequencies

$$\Omega^{(2)}_{1fh} = -\frac{1+2\,\epsilon^2}{4\pi\, n}, \tag{3.27}$$

$$\Omega^{(2)}_{2fh} = \frac{1+6\,\epsilon^2}{6\Omega^{(0)}}, \tag{3.28}$$

which agree with the general formulations (3.14)–(3.15) when $\delta = 0$.

Hence for the scaled frequency Ω in the vicinity of the cut-off frequencies $\Omega^{(0)}_{ifh}$ the following second order approximation is valid

$$\Omega = \Omega^{(0)}_{ifh} + \Omega^{(2)}_{ifh}\eta^2 + O(\eta^4), \tag{3.29}$$

where $i = 1, 2$. The cut-off frequencies $\Omega^{(0)}_{ifh}$ are given by equations (3.24)–(3.25) and the second order corrections $\Omega^{(2)}_{ifh}$ expressed in relations (3.27)–(3.28). The third order term in the expansion (3.21) is given by

$$
\begin{aligned}
\mathcal{D}_{fh}^{(4)} &= \frac{i\mathcal{F}_0^{(4)}}{24(\Omega^{(0)})^3},\\
\mathcal{F}_0^{(4)} &= \left((\Omega^{(0)}(\Omega^{(0)}+2)\cos\Omega^{(0)}-2\right)24(\Omega^{(0)})^3\Omega^{(4)}\\
&+\left(2\Omega^{(0)}\cos\Omega^{(0)}-((\Omega^{(0)})^2+2)\sin\Omega^{(0)}\right)12(\Omega^{(0)})^3(\Omega^{(2)})^2\\
&+\left(\Omega^{(0)}\left(6-(\Omega^{(0)})^2\right)\sin\Omega^{(0)}+\left(6-3(2\epsilon^2+1)(\Omega^{(0)})^2+(\Omega^{(0)})^4\right)\cos\Omega^{(0)}\right)4\Omega^{(0)}\Omega^{(2)}\\
&+((\Omega^{(0)})^2\epsilon^2-1)24\Omega^{(0)}\Omega^{(2)}+\left(\left(1+4\epsilon^2\right)(\Omega^{(0)})^2-2\right)3\cos\Omega^{(0)}\\
&+\left(\left(1-4\epsilon^2\right)(\Omega^{(0)})^2-6\right)\Omega^{(0)}\sin\Omega^{(0)}-2(\epsilon^4(\Omega^{(0)})^4+6\epsilon^2(\Omega^{(0)})^2-3).
\end{aligned}
\tag{3.30}
$$

With the help of relation $\mathcal{D}_{fh}^{(4)}=0$, we establish the following third order corrections for cut-off frequencies

$$
\Omega_{1fh}^{(4)} = \frac{8\,\epsilon^4(\pi^2n^2-3)+4\epsilon^2(4\pi^2n^2-12)+8\,\pi^2n^2-15}{192\pi^3n^3}, \tag{3.31}
$$

$$
\Omega_{2fh}^{(4)} = -\frac{6\epsilon^4(\Omega_0^2+16)+12\epsilon^2(\Omega_0^2+11)+2\Omega_0^2-3}{72\Omega_0^3}. \tag{3.32}
$$

Hence, for the each harmonic of the dispersion relation (3.4) the associated scaled frequency Ω in the vicinity of cut-off frequencies $\Omega_{ifh}^{(0)}$ is determined by the following third order approximation

$$
\Omega = \Omega_{ifh}^{(0)} + \Omega_{ifh}^{(2)}\eta^2 + \Omega_{ifh}^{(4)}\eta^4 + O(\eta^6), \tag{3.33}
$$

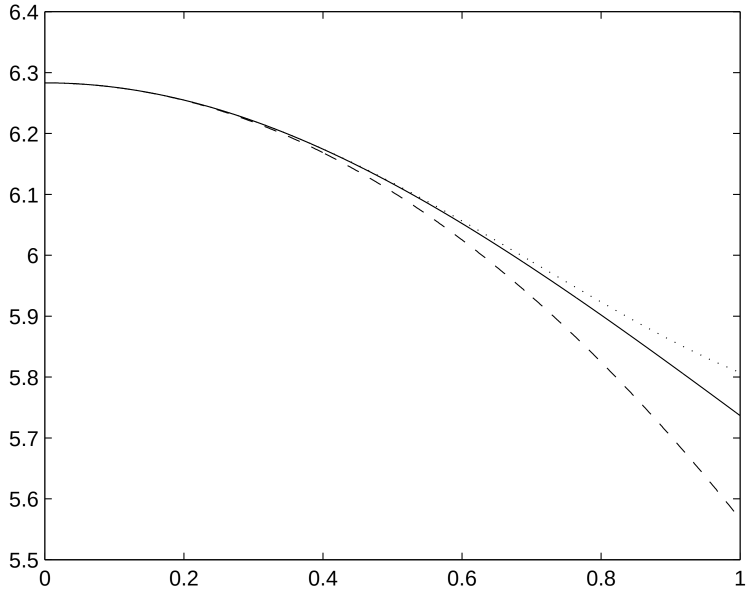
where $i=1,2$. The cut-off frequencies $\Omega_{ifh}^{(0)}$ are given by equations (3.24)–(3.25), the second order corrections $\Omega_{ifh}^{(2)}$ are expressed in relations (3.27)–(3.28) and the third order corrections $\Omega_{ifh}^{(4)}$ are represented by formulations (3.31)–(3.32).

We conclude this section with some illustrative numerical results, for which neo-Hookean strain-energy function (1.124) is employed. Figure 3.5 illustrates comparison of numerical and approximation solution for the first two harmonics in the vicinity of associated cut-off frequencies. Good agreement between the numerical solution, second order approximations (3.29) and third order approximations (3.33) is observed. The third order approximations are the closest to numerical curves, it is demonstrated on all the plots in the Figure 3.5.

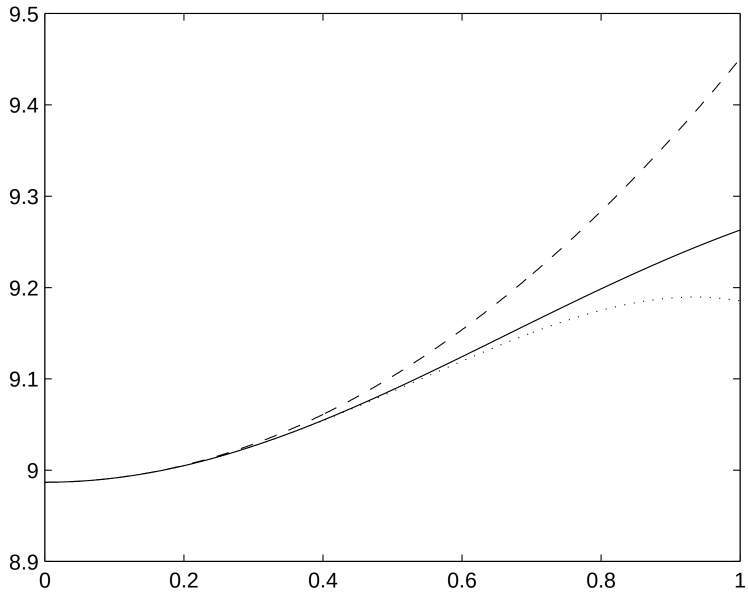
3.3.2 Layer with one free and one fixed faces

Leading order approximation

To obtain the leading order approximation of the dispersion relation (3.8) we employ expansions for the roots q_i up to $O(\eta^2)$ given by (2.24) and the approximations for the functions $f(q)$ and $g(q)$



(a)



(b)

Figure 3.5: *Layer with fixed faces: scaled frequency Ω (vertical axis) against scaled wave number η (horizontal axis) for long wave high frequency motion, numerical solution together with second order approximations (3.29) and third order approximations (3.33) for the neo-Hookean material (1.116) with ϵ=2, p=1; (a) first harmonic, (b) second harmonic. Dotted curves correspond to third order approximations (3.33), dashed curves to second order approximations (3.29) and solid curves to numerical results.*

given by relations (2.25) and (2.26), respectively. The dispersion relation (3.8) can be represent in a form

$$\mathcal{D}_m = \mathcal{D}_m^{(0)} + O(\eta^2), \tag{3.34}$$

where the leading order term is given by

$$\mathcal{D}_m^{(0)} = -\frac{2\,i\sqrt{1+\delta}\,\left(\lambda^2+1\right)\cos\left(\sqrt{1+\delta}\Omega\right)\Omega^5}{\lambda}. \tag{3.35}$$

The condition that $D_m^{(0)} = 0$ enables us to obtain the following cut-off frequencies

$$\Omega_m^{(0)} = \frac{(2n+1)\pi}{2\sqrt{1+\delta}}. \tag{3.36}$$

Long wave high frequency approximations in case $\delta = 0$

For a specific class of materials for which $\delta = 0$, we will derive long wave high frequency approximations of the simplified dispersion relation (3.9) up to $O(\eta^4)$. Motivated by previous results, we employ the third order expansion for the scaled frequency Ω given by (2.30). In addition, we use Taylor series expansions for the functions $\cosh(\eta)$, $\sinh(\eta)$, $\cos(\eta\epsilon)$ up to $O(\eta^5)$ and the expansions for κ, $\sinh(\eta\kappa)$ and $\cosh(\eta\kappa)$ expressed in (2.31). We are then able to produce the following approximation of the dispersion relation (3.9)

$$\mathcal{D}_{mh} = \mathcal{D}_{mh}^{(0)} + \mathcal{D}_{mh}^{(2)}\eta^2 + \mathcal{D}_{mh}^{(4)}\eta^4 + O(\eta^6). \tag{3.37}$$

The following leading order equation

$$\mathcal{D}_{mh}^{(0)} = 2i(\Omega^{(0)})^5\cos\Omega^{(0)} = 0, \tag{3.38}$$

provides cut-off frequencies, which may be expressed as

$$\Omega_{mh}^{(0)} = \frac{(2n+1)\pi}{2}. \tag{3.39}$$

These are observed to coincide with general result (3.36) provided $\delta = 0$.

Taking into account the cut-off frequencies (3.39), and the relation $\sin(\Omega_{mh}^{(0)}) = (-1)^n$, the second order term in the expansion (3.37) is given by

$$\mathcal{D}_{mh}^{(2)} = \frac{(i\pi^3(2n+1)^3}{32}\left(\pi\,(-1)^n\,(2n+1)((2n+1)\pi\Omega^{(2)}+1) - 8(p+1)\right). \tag{3.40}$$

The equation $\mathcal{D}_{mh}^{(2)} = 0$ enables us to establish the second order correction to the cut-off frequencies (3.39) as

$$\Omega_{mh}^{(2)} = \frac{8\,(-1)^n\,(p+1) - \pi(2n+1)}{\pi^2\,(2n+1)^2}. \tag{3.41}$$

For the scaled frequency Ω in the vicinity of cut-off frequencies $\Omega_{mh}^{(0)}$ the following second order approximation is valid

$$\Omega = \Omega_{mh}^{(0)} + \Omega_{mh}^{(2)}\eta^2 + O(\eta^4), \tag{3.42}$$

where the cut-off frequencies $\Omega_{mh}^{(0)}$ are given by equation (3.39) and second order corrections $\Omega_{mh}^{(2)}$ are expressed in relation (3.41).

Using the representations for the cut-off frequencies (3.39), the second order corrections (3.41) and noting that $\sin(\Omega_{mh}^{(0)}) = (-1)^n$, the third order term in (3.37) may be expressed as

$$\begin{aligned}
&\mathcal{D}_{mh}^{(4)} = \frac{i\mathcal{D}_1^{(4)}}{96},\\
&\mathcal{D}_1^{(4)} = 3(-1)^n\pi^5(7n^2(4n^3+5) + 70n^4 + 70n^3 + 10n + 1)\Omega^{(4)}\\
&+ 24((-1)^n(32 - \pi^2 + 4\pi^2 n(n-1)) - 4\pi(n+1))p^2\\
&+ 12\pi(8n^3\pi^2 + 12n^3\pi^2 + 2n(3\pi^2 - 8) - \pi^2 - 6)\epsilon^2 p +\\
&12(8(2(-1)^n(8 - \pi^2 n(n+1) - \pi) + n\pi(n\pi - 7)) + \pi(\pi^2(8n - 12n^2 + 1) - 28)p\\
&+ 12\pi(2\pi(-1)^n(4n\pi(n+1) + 1) + 8\pi^2 n^3 + 12\pi^2 n^2 + 2n(3\pi^2 - 16) + \pi^2 - 16)\epsilon^2\\
&- (-1)^n(32\pi^4 n^4 + 64\pi^4 n^3 + 12\pi^2 n^2(3 + 4\pi^2) + 4\pi^4 n(6 + 4\pi) + 9\pi^2 + 2\pi^4 - 768)\\
&+ 12\pi(12\pi^2 n^2 + 8\pi^2 n^3 + 2n(3\pi^2 - 20) + \pi^3 - 20).
\end{aligned} \tag{3.43}$$

The equation $\mathcal{D}_{mh}^{(4)} = 0$ yields the following third order correction to cut-off frequencies (3.39)

$$\begin{aligned}
&\Omega_{mh}^{(4)} = \frac{c_2^{(4)}\epsilon^2 + c_0^{(4)}}{3\pi^5(1+2n)^5},\\
&c_2^{(4)} = 12(-1)^n\pi(8n^3 - 12n^2 + 10 + 7)p - 96\pi^2 n(n+1)\\
&- 12(-1)^n\pi(8n^3\pi^2 + \pi^2 + 2\pi - 16) - 24(-1)^n\pi n(3\pi^2 + 6\pi^2 n - 16),\\
&c_0^{(4)} = 24(\pi^2(4n^2 + 20n + 3) + (-1)^n\pi(4\pi^2 n^3 + n(8 - 3\pi^2) + 4 + 14) - 32)p\\
&+ 12((-1)^n(56n - \pi^2 - 12\pi^2 n^2) - 128)p\\
&+ 12(-1)^n\pi(8\pi^2 n^3 - 12\pi^2 n^2 + n(40 - 6\pi^2) + 20 - \pi^2)\\
&+ 2\pi^4(16n^4 + 32n^3 + 24n^2 + 8n + 1) + 9\pi^2(2n+1)^2 - 768.
\end{aligned} \tag{3.44}$$

Hence, the scaled frequency Ω in the vicinity of cut-off frequencies $\Omega_{mh}^{(0)}$ can be expressed as

$$\Omega = \Omega_{mh}^{(0)} + \Omega_{mh}^{(2)}\eta^2 + \Omega_{mh}^{(4)}\eta^4 + O(\eta^6), \tag{3.45}$$

where the cut-off frequencies $\Omega_{mh}^{(0)}$, second order corrections $\Omega_{mh}^{(2)}$ and third order corrections $\Omega_{mh}^{(4)}$ are given by (3.39), (3.41) and (3.44), respectively.

We conclude this section by demonstrating some illustrative numerical results produced for a neo-Hookean material. The numerical solutions of the dispersion relation (3.9), second order approximations (3.42) and third order approximations (3.45) are presented for some harmonics in the vicinity of the first two cut-off frequencies in the Figure 3.6. All these plots show excellent agreement over a relatively large wave number region.

3.4 Relative asymptotic orders of displacements and incremental pressure

Our aim is now to establish the relative asymptotic orders of the displacement components u_1, u_2 and incremental pressure p_t. For long wave high frequency motion the displacement components u_1, u_2 and incremental pressure p_t can be expressed in the form (2.41) with the unknown constants $A_i, i = 1, 2, 3, 4$.

3.4.1 Layer with fixed faces

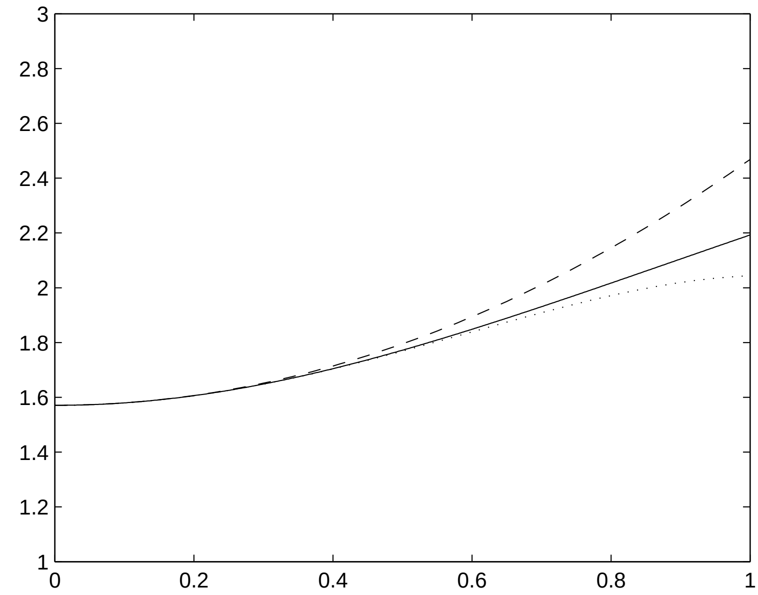
Considering the fixed face boundary conditions (3.1) the coefficients $A_i, i = 1, 2, 3, 4$ may be shown to satisfy the following homogeneous system

$$
\begin{aligned}
&A_1 + A_2 + A_3 + A_4 = 0,\\
&q_1 A_1 + q_2 A_2 + q_3 A_3 + q_4 A_4 = 0,\\
&e^{-iq_1\eta} A_1 + e^{-iq_2\eta} A_2 + e^{-iq_3\eta} A_3 + e^{-iq_4\eta} A_4 = 0,\\
&q_1 e^{-iq_1\eta} A_1 + q_2 e^{-iq_2\eta} A_2 + q_3 e^{-iq_3\eta} A_3 + q_4 e^{-iq_4\eta} A_4 = 0.
\end{aligned}
\tag{3.46}
$$

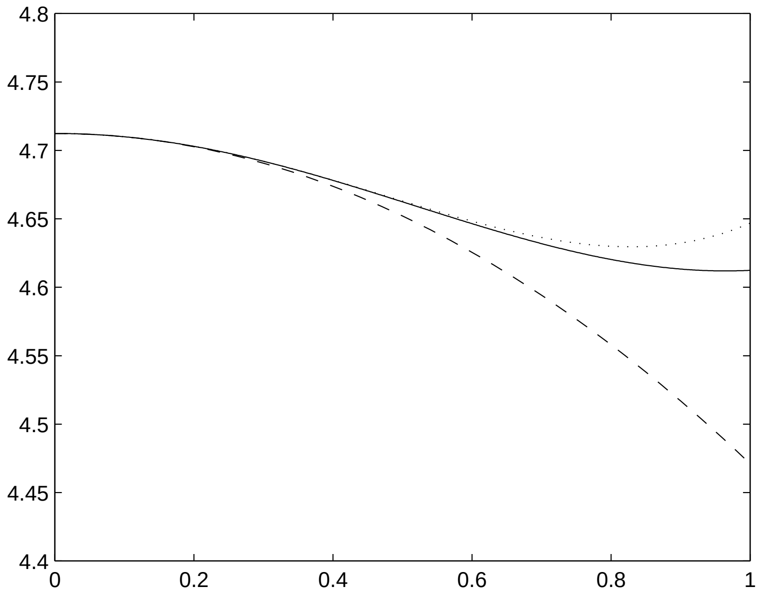
From this system it is possible to express each A_i in terms of one arbitrary constant $\tilde{U}$.

We remark that a different set of coefficients A_i corresponds to each of two families of cut-off frequencies. By inserting A_i into the relations (2.41) and calculating the order of all functions occurring in relations (2.41) and (3.46), we are able to compare the relative asymptotic orders of the displacement components u_1, u_2 and the incremental pressure p_t for each of the two families of cut-off frequencies.

Employing the expansions (2.24) and the representations of the cut-off frequencies (3.12) and (3.13), we establish the approximation for the exponential functions, up to and including $O(\eta)$ and then insert these into the relations (2.41). Then, we obtain the following leading order



(a)



(b)

Figure 3.6: *Layer with one fixed one free faces: scaled frequency Ω (vertical axis) against scaled wave number η (horizontal axis), numerical solution and approximations (3.42) and (3.45), for the neo-Hookean material (1.116) with ϵ=1, p=0.5; (a) first harmonic, (b) second harmonic. Dotted curves correspond to third order approximations (3.45), dashed curves to second order approximations (3.42) and solid curves to numerical results.*

approximations for the displacement components u_1, u_2 and incremental pressure p_t

$$u_1 \approx \mathcal{C}(\sum_{j=1}^{2} q_j A_j + \sum_{j=3}^{4} q_j A_j E_j^{(m)}), \quad u_2 \approx \mathcal{C}(\sum_{j=1}^{2} A_j + \sum_{j=3}^{4} A_j E_j^{(m)}),$$
$$p_t \approx \mathcal{C}k(\sum_{j=1}^{2} \mathcal{P}(q_j) A_j + \sum_{j=3}^{4} A_j \mathcal{P}(q_j) E_j^{(m)}), \tag{3.47}$$

where $\mathcal{C}$ denotes an arbitrary constant and the quantities $E_j^{(m)}, m = 1,2; j = 3,4$ are given by

$$\Omega = \frac{2\pi n}{\sqrt{1+\delta}}: \quad E_3^{(1)} = e^{(i\zeta 2n\pi)}, \quad E_4^{(1)} = e^{(-i\zeta 2n\pi)},$$
$$\Omega = \Omega_{2fh}^{(0)}: \quad E_3^{(2)} = e^{(i\zeta\sqrt{1+\delta}\Omega_{2fh}^{(0)})}, \quad E_4^{(2)} = e^{(-i\zeta\sqrt{1+\delta}\Omega_{2fh}^{(0)})}. \tag{3.48}$$

With the help of the expansions (2.24), we establish the following approximations for the exponential functions up to $O(\eta^2)$

$$e^{(-iq_1\eta)} = 1 + \eta + O(\eta^2), \quad e^{(-iq_2\eta)} = 1 - \eta + O(\eta^2),$$
$$e^{(-iq_3\eta)} = E_0(1 - i\eta\epsilon) + O(\eta^2), \quad e^{(-iq_4\eta)} = E_0^{-1}(1 - i\eta\epsilon) + O(\eta^2),$$
$$\Omega = \frac{2\pi n}{\sqrt{1+\delta}}: \quad E_0^{(1)} = 1, \quad \Omega = \Omega_{2fh}^{(0)} : E_0^{(2)} = e^{(-i\sqrt{1+\delta}\Omega_{2fh}^{(0)})}. \tag{3.49}$$

We are now in a position to insert expansions (3.49) into (3.46) to obtain the dependance of the coefficients A_i on scaled wave number η. With the help of representation (3.47) we are able to establish in complete generality the relative asymptotic orders of displacement components u_1, u_2 and incremental pressure p_t for each of two families of cut-off frequencies (3.12) and (3.13).

For the first family of cut-off frequencies (3.12) the relative asymptotic orders of the displacement components and incremental pressure component are given by

$$p_t \sim \eta u_1 \sim \eta^2 u_2. \tag{3.50}$$

Therefore for the first family of cut-off frequencies (3.12) the incremental pressure p_t is asymptotically leading, with the in-plane displacement component u_1 asymptotically larger than its normal counterpart u_2.

In respect of the second family of cut-off frequencies (3.13) the analogue of (3.50) is the following

$$p_t \sim \eta^2 u_1 \sim \eta^3 u_2. \tag{3.51}$$

Similar to relation (3.50), the representation (3.51) indicates the asymptotically leading incremental pressure p_t, with the in-plane displacement component u_1 asymptotically larger than its normal counterpart u_2.

Considering the case of $\epsilon = 0$ for the first family of cut-off frequencies (3.12), the asymptotic orders are given by

$$p_t \sim u_1 \sim \eta u_2, \tag{3.52}$$

with those associated with the second family of cut-off frequencies (3.13) are expressed as

$$p_t \sim \eta^2 u_1 \sim \eta^3 u_2. \tag{3.53}$$

We note that the relative asymptotic orders of the displacement components u_1, u_2 and incremental pressure p_t obtained from the representations (3.52)–(3.53) for both families of cut-off frequencies coincide with previously published results, see Nolde and Rogerson (2002).

3.4.2 Layer with one free and one fixed faces

In respect of a layer with fixed upper boundary and traction free lower boundary (3.6) the coefficients $A_i, i = 1, 2, 3, 4$ are related through

$$\begin{aligned}
&A_1 + A_2 + A_3 + A_4 = 0,\\
&q_1 A_1 + q_2 A_2 + q_3 A_3 + q_4 A_4 = 0,\\
&f(q_1)e^{-iq_1\eta}A_1 + f(q_2)e^{-iq_2\eta}A_2 + f(q_3)e^{-iq_3\eta}A_3 + f(q_4)e^{-iq_4\eta}A_4 = 0,\\
&g(q_1)e^{-iq_1\eta}A_1 + g(q_2)e^{-iq_2\eta}A_2 + g(q_3)e^{-iq_3\eta}A_3 + g(q_4)e^{-iq_4\eta}A_4 = 0,
\end{aligned} \tag{3.54}$$

and can be obtained in terms of one arbitrary constant $\tilde{U}$. With the help of approximations (2.24), and taking into account the cut-off frequencies (3.36), we establish the expansions for exponential functions up to $O(\eta)$ and insert them into the relations (2.41). Then, at leading order the displacement components u_1, u_2 and the incremental pressure p_t can be expressed as

$$\begin{aligned}
u_1 &\approx \mathcal{C}(\sum_{j=1}^{2} q_j A_j + q_3 A_3 e^{(i\zeta(2n+1)\pi/2)} + q_4 A_4 e^{(-i\zeta(2n+1)\pi/2)}),\\
u_2 &\approx \mathcal{C}(\sum_{j=1}^{2} A_j + A_3 e^{(i\zeta(2n+1)\pi/2)} + A_4 e^{(-i\zeta(2n+1)\pi/2)}),\\
p_t &\approx \mathcal{C}k(\sum_{j=1}^{2} \mathcal{P}(q_j) A_j + P(q_3) A_3 e^{(i\zeta(2n+1)\pi/2)} + P(q_4) A_4 e^{(-i\zeta(2n+1)\pi/2)}),
\end{aligned} \tag{3.55}$$

where $\mathcal{C}$ denotes an arbitrary constant.

Using the expansions (2.24), together with cut-off frequencies (3.36), we introduce the following approximations for the exponential functions up to $O(\eta^2)$

$$\begin{aligned} &e^{(-i\eta q_1)} = 1 + \eta + O(\eta^2), \quad e^{(-i\eta q_2)} = 1 - \eta + O(\eta^2), \\ &e^{(-i\eta q_3)} = (-1)^n(i + \eta\epsilon) + O(\eta^2), \quad e^{(-i\eta q_4)} = (-1)^n(i + \eta\epsilon) + O(\eta^2). \end{aligned} \tag{3.56}$$

Substituting expansions (3.56) into relations (3.54), considering the modified expressions (3.55), and normalizing by η^6 we establish the following asymptotic orders of displacement components u_1, u_2 and incremental pressure p_t

$$u_1 = C_1^{(4)}\eta^2 + O(\eta^3), \quad u_2 = C_2^{(3)}\eta^3 + O(\eta^4), \quad p_t = C^{(6)} + O(\eta), \tag{3.57}$$

$$\begin{aligned} &C_1^{(4)} = \tilde{U}\mathcal{Q}_1\mathcal{Q}_2, \\ &\mathcal{Q}_1 = 2(16n^4 + 32n^3 + 24n^2 + 8n + 1)((p-1)\delta + p + 1) \\ &- \pi\,(-1)^n\,(32n^5 + 80n^4 + 80n^3 + 40n^2 + 10n + 1), \\ &\mathcal{Q}_2 = -\frac{\pi^4\left(1+\lambda^2\right)\sin\left(\zeta\,(2\,n+1)\,\pi/2\right)}{8\lambda\,(1+\delta)^2}, \end{aligned} \tag{3.58}$$

$$\begin{aligned} &C_2^{(3)} = \tilde{U}\mathcal{H}_1\mathcal{H}_2, \\ &\mathcal{H}_1 = -\frac{\left(\epsilon^3\sqrt{\epsilon^2+4} + \epsilon^4 + 5\,\epsilon^2 + 4 + 3\,\epsilon\,\sqrt{\epsilon^2+4}\right)(2\,n+1)^3\,\pi^3}{(1+\delta)^2\left(\epsilon + \sqrt{\epsilon^2+4}\right)^3}, \\ &\mathcal{H}_2 = ((-1)^n\pi(2n+1) - 2(p+1)(\delta+1))\cos\left(\zeta\,(2\,n+1)\,\pi/2\right) + 2\delta(1-p) - 2(p+1), \\ &C^{(6)} = k\tilde{U}\mathcal{R}_1\mathcal{R}_2, \\ &\mathcal{R}_1 = \frac{(2n+1)^5\,\pi^5\,((-1)^n\pi(2n+1) - 2(p+1)(\delta+1))\sin\left(\zeta\,(2n+1)\,\pi/2\right)}{16\lambda\,(1+\lambda^2)\,(1+\delta)^{5/2}}, \\ &\mathcal{R}_2 = \gamma\lambda^4(2+\lambda^2) + \lambda^2(1+\delta)((\mathcal{K}_1 + \mathcal{K}_2) - \gamma(\lambda^2+1)). \end{aligned} \tag{3.59}$$

From the representation (3.57) we deduce that the relative asymptotic orders of u_1, u_2 and p_t are given by

$$p_t \sim \eta^2 u_1 \sim \eta^3 u_2. \tag{3.60}$$

The common tendency with previously established relations (2.58), (3.50) and (3.51) is that the incremental pressure component p_t is asymptotically leading, with the in-plane displacement component u_1 asymptotically larger than its normal counterpart u_2. Taking into account the representations (3.57) and (3.58), the relative asymptotic orders of the displacement components u_1, u_2 and incremental pressure p_t in the case of $\epsilon = 0$ are given by

$$p_t \sim \eta u_1 \sim \eta^2 u_2. \tag{3.61}$$

Chapter 4

Asymptotically consistent model for a long wave low frequency motion in a layer with free faces

In this chapter we derive a simplified asymptotic model for two-dimensional long wave low frequency motion in a sheared pre-stressed incompressible elastic layer with traction free faces. First, we derive an asymptotically consistent dynamic model in respect to the most general strain-energy function and then discuss simplifications for the case of a layer composed of neo-Hookean material. Using the fact that the asymptotic orders of displacements and incremental pressure are equal we establish non-dimensional linearized equations of motion, incompressibility condition and incremental traction components. A hierarchial system of traction free boundary value problems at various orders is then obtained. The first three asymptotic orders of the problem, both for the general and neo-Hookean material, are considered. Asymptotic integration is employed in vicinity of the long wave low frequency limits. The presence of pre-stress in a form of a simple shear deformation results in no decomposition of motion into symmetric and anti-symmetric parts. This leads to a novel vector governing equation obtained only at the third order problem. A matrix representation of the governing equations is employed to confirm the existence of two fundamental modes and illustrate the asymptotic consistency of the derived simplified dynamic model.

4.1 Asymptotic model in respect of the most general strain-energy function

4.1.1 Asymptotic scaling and dimensionless equations

Our aim is to derive an asymptotic simplified dynamic model for the long wave low frequency motion in a finitely sheared incompressible elastic layer for the most general strain-energy function. To do so we generalize the asymptotic methods first developed and exploited in respect of dynamic problems involving linear isotropic thin-walled elastic bodies, see Kaplunov et al. (1998).

In the case of the most general strain energy function, the linearized equations of motion are given by equations (1.68), (1.69) and the incremental traction components are expressed in (1.95), (1.96). It was shown that in the long wave low frequency regime the relative asymptotic orders of displacements and incremental pressure components are equivalent, see relation (2.51). Therefore, appropriate non-dimensional displacement and incremental pressure components are taken as

$$u_1 = lu_1^*, \quad u_2 = lu_2^*, \quad p_t = \sqrt{\alpha\gamma}p_t^* = \lambda^2\gamma p_t^*, \tag{4.1}$$

where the superscript * indicates dimensionless quantities and we employ the relation $\alpha = \lambda^4\gamma$. In addition, the scaled spatial and time variables are given by

$$x_1 = l\xi, \quad x_2 = h\zeta, \quad t = l\sqrt{\frac{\rho}{\sqrt{\alpha\gamma}}}\tau = l\sqrt{\frac{\rho}{\lambda^2\gamma}}\tau. \tag{4.2}$$

To distinguish between differentiation with respect to scaled and original variables we use the notation $(.)_{,\xi}, (.)_{,\zeta}$ and $(.)_{,\tau}$ to indicate differentiation with respect to ξ, ζ and τ respectively. Employing the scalings (4.1) and (4.2) the equations of motion (1.68), (1.69) can be represented in non-dimensional form and expressed in powers of η as

$$M_1 = C_2^1\eta^2 + C_1^1\eta + C_0^1 = 0, \quad M_2 = C_2^2\eta^2 + C_1^2\eta + C_0^2 = 0, \tag{4.3}$$

$$\begin{aligned}
C_0^1 &= \left((\sigma_2 - 2\,\gamma)\,\lambda^2 + \sigma_2 - \gamma + B_{1122} - B_{1111} + B_{1221}\right) u_{2,\zeta\zeta}^* + \\
&\qquad + \left(B_{1221} - B_{1111} + B_{1122} - \lambda^2\gamma\right) \lambda u_{1,\zeta\zeta}^*, \\
C_0^2 &= \left((\sigma_2 - \gamma) - \gamma\,\lambda^4 + (B_{1122} - B_{2222} + B_{1221} + \sigma_2 - \gamma)\,\lambda^2\right) \lambda u_{2,\zeta\zeta}^* + \\
&\qquad + \left(\lambda^2\gamma - B_{1221} + B_{2222} - B_{1122}\right) \lambda^2 u_{1,\zeta\zeta}^*,
\end{aligned} \tag{4.4}$$

$$
\begin{aligned}
C_1^1 &= \big((2(B_{1221}+B_{1122}-B_{1111})+\sigma_2)\,\lambda^2+\sigma_2-\gamma\big)\,u^*_{1,\xi\zeta}+\big(\lambda^2+1\big)\,\lambda^2\gamma p^*_{t,\zeta}+\\
&\qquad +\big((\sigma_2-\gamma)\,\lambda^2\gamma+2(B_{1221}+B_{1122}-B_{1111})+\sigma_2+\gamma\big)\,\lambda u^*_{2,\xi\zeta},\\
C_1^2 &= \big((\sigma_2-\gamma)\,(\lambda^2+1)+2(B_{1122}+B_{1221}-B_{2222}+\gamma\,\lambda^4)\big)\,\lambda u^*_{1,\xi\zeta}+\\
&+\big(\lambda^2+1\big)\,\lambda^3\gamma p^*_{t,\zeta}+\big((\gamma-\sigma_2+2(B_{2222}-B_{1221}-B_{1122}))\,\lambda^2-\gamma(2\lambda^6-1)-\sigma_2\big)\,u^*_{2,\xi\zeta},
\end{aligned}
\tag{4.5}
$$

$$
\begin{aligned}
C_2^1 &= \big(\lambda^2+1\big)\,\lambda^2\gamma(u^*_{2,\tau\tau}+\lambda p^*_{t,\xi}+\lambda u^*_{1,\tau\tau})+\\
&\qquad +\big((B_{1221}-B_{1111}+B_{1122})\,\lambda^2-\gamma\big)\,u^*_{2,\xi\xi}+\\
&\qquad\qquad +\big((B_{1122}+B_{1221}-B_{1111}-\gamma+\sigma_2)\,\lambda^2+\sigma_2-2\,\gamma\big)\,\lambda u^*_{1,\xi\xi},\\
C_2^2 &= \big(\lambda^2+1\big)\,\lambda^2\gamma(\lambda u^*_{2,\tau\tau}-p^*_{t,\xi}-u^*_{1,\tau\tau})+\\
&\qquad +\big(B_{1221}-B_{2222}+B_{1122}-\lambda^6\gamma\big)\,\lambda u^*_{2,\xi\xi}+\\
&\qquad\qquad +\big(\gamma\,\lambda^6-B_{1122}-B_{1221}+B_{2222}+(\gamma-\sigma_2)\,(\lambda^2-1)\big)\,u^*_{1,\xi\xi}.
\end{aligned}
\tag{4.6}
$$

The above equations of motion must be solved in conjunction with the non-dimensional incompressibility condition

$$
u^*_{1,\xi}\eta+u^*_{2,\zeta}=0,
\tag{4.7}
$$

and subject to the non-dimensional traction free boundary conditions at the upper and lower surfaces on the layer, which take the form

$$
T_1=T_1^1\eta+T_0^1=0,\quad T_2=T_1^2\eta+T_0^2=0,\quad \text{at}\quad \zeta=0,-1,
\tag{4.8}
$$

$$
\begin{aligned}
T_0^1 &= \big((\sigma_2-B_{1122}-2\,\gamma)\,\lambda^2-B_{1111}-\gamma+B_{1221}+\sigma_2\big)\,u^*_{2,\zeta}+\\
&\qquad +\big(B_{1221}-B_{1111}+B_{1122}-\lambda^2\gamma\big)\,\lambda u^*_{1,\zeta},\\
T_0^2 &= \big(\sigma_2-B_{1122}-\gamma(\lambda^4+1)+(\sigma_2+B_{1221}-B_{2222}-\gamma)\,\lambda^2\big)\,\lambda u^*_{2,\zeta}+\\
&\qquad +\big(\lambda^2\gamma-B_{1122}+B_{2222}-B_{1221}\big)\,\lambda^2u^*_{1,\zeta},
\end{aligned}
\tag{4.9}
$$

$$
\begin{aligned}
T_1^1 &= \big((\gamma+B_{1221}-B_{1111})\,\lambda^2-B_{1122}\big)\,u^*_{1,\xi}+\\
&\qquad +\big((B_{1122}+\sigma_2-B_{1111}+B_{1221})+(\sigma_2-\gamma)\,\lambda^2\big)\,\lambda u^*_{2,\xi}+\big(\lambda^2+1\big)\,\lambda^2\gamma p^*_t,\\
T_2^1 &= \big(\lambda^4\gamma-B_{2222}+B_{1221}-B_{1122}\lambda^2\big)\,\lambda u^*_{1,\xi}+\\
&+\big((\gamma-\sigma_2)(\lambda^2+1)+(B_{2222}-B_{1122}-B_{1221})\,\lambda^2-\lambda^6\gamma\big)\,u^*_{2,\xi}+\big(\lambda^2+1\big)\,\lambda^3\gamma p^*_t.
\end{aligned}
\tag{4.10}
$$

Solutions of equations (4.3)–(4.7), subject to boundary conditions (4.8), are sought in the form of the series

$$(u_1^*, u_2^*, p_t^*) = \sum_{m=0}^{r} (u_1^{(m)}, u_2^{(m)}, p_t^{(m)})\eta^m + O(\eta^{r+1}). \tag{4.11}$$

The form (4.11) leads to the hierarchical system of boundary value problems at various orders. In the representation (4.11) the problems of various orders are obtained by putting $m = 0, 1, 2...$ with $m = 0$ corresponding to the leading order problem.

4.1.2 Leading order problem

At leading order the non-dimensional equations of motion and associated incompressibility condition are given by

$$\begin{aligned}
&\left((\sigma_2 - 2\,\gamma)\,\lambda^2 + B_{1122} - B_{1111} + B_{1221} - \gamma + \sigma_2\right) u_{2,\zeta\zeta}^{(0)} + \\
&\qquad + \left(B_{1221} - B_{1111} + B_{1122} - \lambda^2\gamma\right) \lambda u_{1,\zeta\zeta}^{(0)} = 0, \\
&\left(\sigma_2 - \gamma(\lambda^4 + 1) + (B_{1221} - B_{2222} + B_{1122} - \gamma + \sigma_2)\,\lambda^2\right) \lambda u_{2,\zeta\zeta}^{(0)} + \\
&\qquad + \left(\lambda^2\gamma - B_{1221} + B_{2222} - B_{1122}\right) \lambda^2 u_{1,\zeta\zeta}^{(0)} = 0, \\
&u_{2,\zeta}^{(0)} = 0.
\end{aligned} \tag{4.12}$$

and subject to the following traction free boundary conditions

$$\begin{aligned}
&\left((\sigma_2 - B_{1122} - 2\,\gamma)\,\lambda^2 - B_{1111} + B_{1221} + \sigma_2 - \gamma\right) u_{2,\zeta}^{(0)} + \\
&\qquad + \left(B_{1122} - \lambda^2\gamma + B_{1221} - B_{1111}\right) \lambda u_{1,\zeta}^{(0)} = 0, \\
&\left(\sigma_2 - B_{1122} - \gamma(\lambda^4 + 1) + (B_{1221} - B_{2222} + \sigma_2 - \gamma)\,\lambda^2\right) \lambda u_{2,\zeta}^{(0)} + \\
&\qquad + \left(\lambda^2\gamma - B_{1122} + B_{2222} - B_{1221}\right) \lambda^2 u_{1,\zeta}^{(0)} = 0, \quad \text{at} \quad \zeta = 0, -1.
\end{aligned} \tag{4.13}$$

It is readily deduced that the solutions of equations (4.12), subjected to boundary conditions (4.13), are of the following form

$$u_1^{(0)} = U^{(0,0)}(\xi, \tau), \quad u_2^{(0)} = V^{(0,0)}(\xi, \tau). \tag{4.14}$$

Hence, the solution of the leading order problem consists of two displacement components each independent of ζ. We note that it is not possible to determine $U^{(0,0)}(\xi, \tau)$ and $V^{(0,0)}(\xi, \tau)$ without considering higher order problems.

We remark that in the solutions of all asymptotic problems we will denote various functions by capital letters involving double superscripts. These functions are independent of ζ. Moreover within two indices of the superscript the right one indicates the asymptotic order, with the left denoting the degree of ζ.

4.1.3 Second order problem

In order to determine the unknowns from the leading order problem the second order problem must be considered. To establish the second order problem we substitute the general solution (4.11) with $m = 1$ into equations (4.3)–(4.7) subject to boundary conditions (4.8) and consider terms of $O(\eta)$. At second order, we obtain

$$\begin{aligned}
&\left(\lambda^2+1\right)\lambda^2\gamma p^{(0)}_{t,\zeta} + \left(B_{1122}+B_{1221}-B_{1111}-\lambda^2\gamma\right)\lambda u^{(1)}_{1,\zeta\zeta} + \\
&\qquad + \left(\left(\sigma_2-2\,\gamma\right)\lambda^2+\sigma_2-\gamma+B_{1122}-B_{1111}+B_{1221}\right)u^{(1)}_{2,\zeta\zeta}=0,\\
&\left(\lambda^2+1\right)\lambda^3\gamma p^{(0)}_{t,\zeta} + \left(B_{2222}-B_{1122}-B_{1221}+\lambda^2\gamma\right)\lambda^2 u^{(1)}_{1,\zeta\zeta} - \\
&\qquad - \left(\gamma(\lambda^5-1)+\sigma_2+\left(B_{1122}-B_{2222}+B_{1221}+\sigma_2-\gamma\right)\lambda^2\right)\lambda u^{(1)}_{2,\zeta\zeta}=0,\\
&u^{(1)}_{2,\zeta}=-u^{(0)}_{1,\xi},
\end{aligned} \tag{4.15}$$

subject to the following traction free boundary conditions

$$\begin{aligned}
&\left(\lambda^2+1\right)\lambda^2\gamma p^{(0)}_{t} + \left(B_{1221}-B_{1111}+B_{1122}-\lambda^2\gamma\right)\lambda u^{(1)}_{1,\zeta} + \\
&\qquad + \left(\left(\sigma_2-2\,\gamma-B_{1122}\right)\lambda^2-B_{1111}+B_{1221}+\sigma_2-\gamma\right)u^{(1)}_{2,\zeta}= \\
&\qquad = \left(\left(B_{1111}-B_{1221}-\gamma\right)\lambda^2-B_{1122}\right)U^{(0,0)}_{,\xi} - \\
&\qquad - \left(\left(\sigma_2-\gamma\right)\lambda^2+\sigma_2-B_{1111}+B_{1122}+B_{1221}\right)\lambda V^{(0,0)}_{,\xi}, \quad \text{at} \quad \zeta=0,-1,\\
&\left(\lambda^2+1\right)\lambda^3\gamma p^{(0)}_{t} + \left(\lambda^2\gamma-B_{1221}+B_{2222}-B_{1122}\right)\lambda^2 u^{(1)}_{1,\zeta} + \\
&\qquad + \left(\sigma_2-\gamma(\lambda^4+1)-B_{1122}+\left(B_{1221}-B_{2222}+\sigma_2-\gamma\right)\lambda^2\right)\lambda u^{(1)}_{2,\zeta}= \\
&\qquad = \left(B_{2222}+B_{1122}\lambda^2-B_{1221}-\gamma\,\lambda^4\right)\lambda U^{(0,0)}_{,\xi} - \\
&- \left(\left(B_{2222}-B_{1122}-B_{1221}+\gamma-\sigma_2\right)\lambda^2-\sigma_2+\gamma(1-\lambda^6)\right)V^{(0,0)}_{,\xi}, \quad \text{at} \quad \zeta=0,-1.
\end{aligned} \tag{4.16}$$

The incompressibility condition $(4.15)_3$ enables us to establish the following form for the normal displacement component

$$u^{(1)}_2 = -\zeta U^{(0,0)}_{,\xi} + U^{(0,1)}_2. \tag{4.17}$$

The equations of motion (4.15) can then be written as

$$\begin{aligned}
&(\lambda^2+1)\lambda\gamma p^{(0)}_{t,\zeta} - (B_{1111}-B_{1122}-B_{1221}+\lambda^2\gamma)u^{(1)}_{1,\zeta\zeta}=0,\\
&(\lambda^2+1)\lambda\gamma p^{(0)}_{t,\zeta} - (B_{1221}-B_{2222}+B_{1122}-\lambda^2\gamma)u^{(1)}_{1,\zeta\zeta}=0.
\end{aligned} \tag{4.18}$$

Taking into account strong ellipticity condition (1.56) and the fact that $\alpha = \lambda^4\gamma$ the determinant of the homogeneous system (4.18) is given by

$$det\begin{bmatrix} B_{1221} - B_{1111} + B_{1122} - \lambda^2\gamma & (\lambda^2+1)\lambda\gamma \\ \lambda^2\gamma - B_{1221} + B_{2222} - B_{1122} & (\lambda^2+1)\gamma\,\lambda \end{bmatrix} = -2\,\lambda\gamma\,\left(\lambda^2+1\right)\left(\beta+\lambda^2\gamma\right) < 0. \tag{4.19}$$

Therefore the only solutions for two variables in system (4.18) are the trivial ones, thus

$$p^{(0)}_{t,\zeta} = 0, \quad u^{(1)}_{1,\zeta\zeta} = 0. \tag{4.20}$$

Hence, at second order, the in-plane displacement component $u_1^{(1)}$ is a linear function of ζ, and the leading order pressure $p_t^{(0)}$ is independent of ζ. Hence

$$u_1^{(1)} = U_1^{(1,1)}\zeta + U_1^{(0,1)}. \tag{4.21}$$

Substituting $u_1^{(1)}, p_t^{(0)}$ given by (4.21) and $u_2^{(1)}$ expressed in (4.17) into the boundary conditions (4.16), it is possible to obtain the following representations for the functions $p_t^{(0)}$ and $U_1^{(1,1)}$

$$\begin{aligned} p_t^{(0)} &= \mathcal{C}_1 U^{(0,0)}_{,\xi} + \mathcal{C}_2 V^{(0,0)}_{,\xi}, \\ U_1^{(1,1)} &= \mathcal{C}_3 U^{(0,0)}_{,\xi} + \mathcal{C}_4 V^{(0,0)}_{,\xi}, \end{aligned} \tag{4.22}$$

where the coefficients $\mathcal{C}_i$, $i = 1, 2, 3, 4$ are given by

$$\begin{aligned} \mathcal{C}_1 &= \frac{\mathcal{C}_1^{(1)}}{(\lambda^4\gamma + 2\,\beta\,\lambda^2 + \alpha)\,\lambda^2\gamma}, \\ \mathcal{C}_1^{(1)} &= (\sigma_2 - \gamma + B_{1122} + B_{1221} - B_{2222})\,\gamma\lambda^4 \\ &\quad + 2\,(\beta(\sigma_2-\gamma) - \alpha\,\gamma)\,\lambda^2 + (\sigma_2 - \gamma + B_{1122} + B_{1221} - B_{1111})\,\alpha, \end{aligned} \tag{4.23}$$

$$\begin{aligned} \mathcal{C}_2 &= \frac{\mathcal{C}_2^{(1)}}{(\lambda^4\gamma + 2\,\beta\,\lambda^2 + \alpha)\,\lambda^2\gamma}, \\ \mathcal{C}_2^{(1)} &= ((\alpha + \sigma_2 - \gamma)\,\gamma + (B_{1122} + B_{1221} - B_{2222})\,\sigma_2)\,\lambda^3 + \\ &\quad + ((B_{1111} - B_{1221} - B_{1122})\,\sigma_2 + (B_{1221} - B_{1111} + B_{1122})\,\gamma)\,\lambda + \\ &\quad + ((B_{1111} - \sigma_2 - B_{1221} - B_{1122})\,\alpha)\,\lambda, \end{aligned} \tag{4.24}$$

$$\begin{aligned} \mathcal{C}_3 &= \frac{2\lambda\,\left((\gamma-\beta)\,\lambda^2 - \alpha + \beta\right)}{\lambda^4\gamma + 2\,\beta\,\lambda^2 + \alpha}, \\ \mathcal{C}_4 &= -\frac{(\gamma - \sigma_2)\,\lambda^4 + (\gamma - \alpha - 2(\sigma_2 - \beta))\,\lambda^2 - \sigma_2 + \gamma}{\lambda^4\gamma + 2\,\beta\,\lambda^2 + \alpha}. \end{aligned} \tag{4.25}$$

Hence the solutions of second order problem can be written as

$$
\begin{aligned}
u_1^{(1)} &= \{\mathcal{C}_3 U_{,\xi}^{(0,0)} + \mathcal{C}_4 V_{,\xi}^{(0,0)}\}\zeta + U_1^{(0,1)},\\
u_2^{(1)} &= -U_{,\xi}^{(0,0)}\zeta + U_2^{(0,1)},\\
p_t^{(0)} &= \mathcal{C}_1 U_{,\xi}^{(0,0)} + \mathcal{C}_2 V_{,\xi}^{(0,0)},
\end{aligned} \tag{4.26}
$$

where the coefficients $\mathcal{C}_i, i = 1,2,3,4$ are given by (4.23)- (4.25). We note that the new functions $U_2^{(0,1)}, U_1^{(0,1)}$, together with the leading order functions $U^{(0,0)}$ and $V^{(0,0)}$, remain undefined within the solution of the second order problem. To make further progress we must consider the third order problem. The fact that we need to go to higher order to solve the problem is the novel aspect in comparison to previously work by Kaplunov et al. (2000).

4.1.4 Third order problem

The third order problem can be written as

$$
\begin{aligned}
&\mathcal{M}_0^{(1)} u_{1,\zeta\zeta}^{(2)} + \mathcal{M}_1^{(1)} u_{2,\zeta\zeta}^{(2)} + \mathcal{M}_2^{(1)} p_{t,\zeta}^{(1)} = \mathcal{M}_3^{(1)} p_{t,\xi}^{(0)} +\\
&+ \mathcal{M}_4^{(1)} u_{1,\tau\tau}^{(0)} + \mathcal{M}_5^{(1)} u_{2,\tau\tau}^{(0)} + \mathcal{M}_6^{(1)} u_{1,\xi\xi}^{(0)} + \mathcal{M}_7^{(1)} u_{2,\xi\xi}^{(0)} + \mathcal{M}_8^{(1)} u_{1,\xi\zeta}^{(1)} + \mathcal{M}_9^{(1)} u_{2,\xi\zeta}^{(1)},\\
&\mathcal{M}_0^{(2)} u_{1,\zeta\zeta}^{(2)} + \mathcal{M}_1^{(2)} u_{2,\zeta\zeta}^{(2)} + \mathcal{M}_2^{(2)} p_{t,\zeta}^{(1)} = \mathcal{M}_3^{(2)} p_{t,\xi}^{(0)} +\\
&+ \mathcal{M}_4^{(2)} u_{1,\tau\tau}^{(0)} + \mathcal{M}_5^{(2)} u_{2,\tau\tau}^{(0)} + \mathcal{M}_6^{(2)} u_{1,\xi\xi}^{(0)} + \mathcal{M}_7^{(2)} u_{2,\xi\xi}^{(0)} + \mathcal{M}_8^{(2)} u_{1,\xi\zeta}^{(1)} + \mathcal{M}_9^{(2)} u_{2,\xi\zeta}^{(1)},\\
&u_{2,\zeta}^{(2)} = -u_{1,\xi}^{(1)};
\end{aligned} \tag{4.27}
$$

$$
\begin{aligned}
\mathcal{M}_0^{(1)} &= \lambda\left(B_{1111} - B_{1221} - B_{1122} + \gamma\,\lambda^2\right),\\
\mathcal{M}_1^{(1)} &= \gamma + B_{1111} - \sigma_2 - B_{1122} - B_{1221} - \lambda^2(\sigma_2 - 2\,\gamma),\\
\mathcal{M}_2^{(1)} &= -\mathcal{M}_5^{(1)} = -\gamma\,\lambda^2\left(\lambda^2+1\right),\\
\mathcal{M}_3^{(1)} &= \mathcal{M}_4^{(1)} = \gamma\,\lambda^3\left(\lambda^2+1\right),\\
\mathcal{M}_6^{(1)} &= \lambda\left(\lambda^2(B_{1122} + B_{1221} - B_{1111} + \sigma_2 - \gamma) + \sigma_2 - 2\,\gamma\right),\\
\mathcal{M}_7^{(1)} &= \lambda^2(B_{1221} + B_{1122} - B_{1111}) - \gamma,\\
\mathcal{M}_8^{(1)} &= 2\lambda^2(B_{1122} - B_{1111} + B_{1221}) + \sigma_2 - \gamma + \lambda^2(\gamma + \sigma_2),\\
\mathcal{M}_9^{(1)} &= \lambda\left(\lambda^2(\sigma_2 - \gamma) - 2(B_{1111} - B_{1122} - B_{1221}) + \gamma + \sigma_2\right),
\end{aligned} \tag{4.28}
$$

$$
\begin{aligned}
\mathcal{M}_0^{(2)} &= \lambda^2 \left(B_{1122} + B_{1221} - B_{2222} - \gamma\,\lambda^2\right), \\
\mathcal{M}_1^{(2)} &= \lambda \left(\lambda^2(B_{2222} - B_{1221} - B_{1122} - \sigma_2 + (\lambda^2+1)\gamma) + \gamma - \sigma_2\right), \\
\mathcal{M}_2^{(2)} &= -\mathcal{M}_5^{(2)} = -\gamma\,\lambda^3 \left(\lambda^2+1\right), \\
\mathcal{M}_3^{(2)} &= \mathcal{M}_4^{(2)} = -\gamma\,\lambda^2 \left(\lambda^2+1\right), \\
\mathcal{M}_6^{(2)} &= B_{2222} - B_{1122} - B_{1221} - \sigma_2 + \gamma + \lambda^2(\gamma - \sigma_2 + \gamma\,\lambda^4), \\
\mathcal{M}_7^{(2)} &= \lambda \left(B_{1221} - B_{2222} + B_{1122} - \gamma\,\lambda^6\right), \\
\mathcal{M}_8^{(2)} &= \lambda \left(\lambda^2(2\,\lambda^2\gamma - \gamma + \sigma_2) + \sigma_2 - \gamma + 2(B_{1221} + B_{1122} - B_{2222})\right), \\
\mathcal{M}_9^{(2)} &= \lambda^2(2(B_{2222} - B_{1122} - B_{1221}) + \gamma(1 - 2\lambda^4) - \sigma_2)) - \sigma_2 + \gamma. && (4.29)
\end{aligned}
$$

Equations (4.27) must be solved subject to the following traction free boundary conditions

$$
\begin{aligned}
&\mathcal{T}_1^{(1)} u_{1,\zeta}^{(2)} + \mathcal{T}_2^{(1)} u_{2,\zeta}^{(2)} + \mathcal{T}_3^{(1)} p_t^{(1)} = \mathcal{T}_4^{(1)} u_{1,\xi}^{(1)} + \mathcal{T}_5^{(1)} u_{2,\xi}^{(1)}, \\
&\mathcal{T}_1^{(2)} u_{1,\zeta}^{(2)} + \mathcal{T}_2^{(2)} u_{2,\zeta}^{(2)} + \mathcal{T}_3^{(2)} p_t^{(1)} = \mathcal{T}_4^{(2)} u_{1,\xi}^{(1)} + \mathcal{T}_5^{(2)} u_{2,\xi}^{(1)}, \quad \text{at} \quad \zeta = 0, -1, && (4.30)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{T}_1^{(1)} &= \lambda \left(B_{1111} - B_{1221} - B_{1122} + \gamma\,\lambda^2\right), \\
\mathcal{T}_2^{(1)} &= B_{1111} + \gamma - \lambda^2(\sigma_2 - 2\,\gamma - B_{1122}) - B_{1221} - \sigma_2, \\
\mathcal{T}_3^{(1)} &= -\gamma\,\lambda^2 \left(\lambda^2+1\right), \\
\mathcal{T}_4^{(1)} &= \lambda^2(B_{1221} - B_{1111} + \gamma) - B_{1122}, \\
\mathcal{T}_5^{(1)} &= \lambda \left(\lambda^2(\sigma_2 - \gamma) - B_{1111} + B_{1122} + B_{1221} + \sigma_2\right), \\
\mathcal{T}_1^{(2)} &= \lambda^2 \left(B_{1122} + B_{1221} - B_{2222} - \gamma\,\lambda^2\right), \\
\mathcal{T}_2^{(2)} &= \lambda \left(\lambda^2(B_{2222} - B_{1221}) + B_{1122} + \lambda^2(\gamma - \sigma_2 + \lambda^2\gamma) + \gamma - \sigma_2\right), \\
\mathcal{T}_3^{(2)} &= -\gamma\,\lambda^3 \left(\lambda^2+1\right), \\
\mathcal{T}_4^{(2)} &= \lambda \left(\lambda^2(\lambda^2\gamma - B_{1122}) - B_{2222} + B_{1221}\right), \\
\mathcal{T}_5^{(2)} &= \lambda^2(B_{2222} - B_{1221} - B_{1122} + (1 - \lambda^4)\gamma - \sigma_2) + \gamma - \sigma_2. && (4.31)
\end{aligned}
$$

Taking into account the leading order and second order solutions given by (4.14) and (4.26) respectively, the incompressibility condition $(4.27)_3$ may be used to obtain the following representation for the normal displacement component

$$
u_2^{(2)} = (\mathcal{C}_5 U_{\xi\xi}^{(0,0)} + \mathcal{C}_6 V_{\xi\xi}^{(0,0)})\zeta^2 - U_{1,\xi}^{(0,1)}\zeta + U_2^{(0,2)}, \qquad (4.32)
$$

$$
\begin{aligned}
\mathcal{C}_5 &= \frac{\gamma\lambda^4 + (\beta - \gamma)\,\lambda^2 - \beta}{\lambda\,(\beta + \lambda^2\gamma)}, \\
\mathcal{C}_6 &= \frac{((\gamma - \sigma_2)\lambda^2 + 2(\beta - \sigma_2) + \gamma(1 - \lambda^4))\lambda^2 + \gamma - \sigma_2}{2\lambda^2\,(\beta + \lambda^2\gamma)}. && (4.33)
\end{aligned}
$$

Using (4.32), we can now establish the third order solutions for $u_1^{(2)}$ and $p_t^{(1)}$ in the form

$$u_1^{(2)} = U_1^{(2,2)}\zeta^2 + U_1^{(1,2)}\zeta + U_1^{(0,2)}, \quad p_t^{(1)} = P_t^{(1,1)}\zeta + P_t^{(0,1)}, \tag{4.34}$$

where the functions $U_1^{(1,2)}$, $U_1^{(2,2)}$, $P_t^{(0,1)}$, $P_t^{(1,1)}$ can be determined at third order, with $U_1^{(0,2)}$, $U_2^{(0,2)}$ obtainable at higher order.

Taking into account the relations (4.14), (4.26), (4.32) and (4.34), the equations of motion and boundary conditions yield a system of six equations in the six unknowns $U^{(0,0)}$, $V^{(0,0)}$, $U_1^{(1,2)}$, $U_1^{(2,2)}$, $P_t^{(0,1)}$, $P_t^{(1,1)}$. This system will be used later on, however it is not worth writing all six equations explicitly. We will need to specify here two equations of motion as they will be used to produce resulting governing equations later

$$\begin{aligned}
\mathcal{R}_1^1 P_t^{(1,1)} + \mathcal{R}_2^1 U_1^{(2,2)} &= \mathcal{R}_3^1 U_{\tau\tau}^{(0,0)} + \mathcal{R}_4^1 V_{\tau\tau}^{(0,0)} + \mathcal{R}_5^1 U_{\xi\xi}^{(0,0)} + \mathcal{R}_6^1 V_{\xi\xi}^{(0,0)},\\
\mathcal{R}_1^2 P_t^{(1,1)} + \mathcal{R}_2^2 U_1^{(2,2)} &= \mathcal{R}_3^2 U_{\tau\tau}^{(0,0)} + \mathcal{R}_4^2 V_{\tau\tau}^{(0,0)} + \mathcal{R}_5^2 U_{\xi\xi}^{(0,0)} + \mathcal{R}_6^2 V_{\xi\xi}^{(0,0)}.
\end{aligned} \tag{4.35}$$

The coefficients in equations (4.35) are quite lengthy, however the their explicit expressions have been obtained using Maple (1996).

From (4.30) it is possible to show that both incremental traction components are linear functions of ζ. Therefore the in-plane traction component at the upper surface of the layer $\zeta = 0$ can be expressed as

$$\mathcal{N}_1^1 U_1^{(1,2)} + \mathcal{N}_2^1 P_t^{(0,1)} = \mathcal{N}_3^1 U_{1,\xi}^{(0,1)} + \mathcal{N}_4^1 U_{2,\xi}^{(0,1)}, \tag{4.36}$$

and at lower surface of the layer $\zeta = -1$

$$\mathcal{N}_5^1 U_1^{(2,2)} + \mathcal{N}_6^1 P_t^{(1,1)} = \mathcal{N}_7^1 U_{\xi\xi}^{(0,0)} + \mathcal{N}_8^1 V_{\xi\xi}^{(0,0)}. \tag{4.37}$$

Similarly, the normal traction component at upper surface of the layer is given by

$$\mathcal{N}_1^2 U_1^{(1,2)} + \mathcal{N}_2^2 P_t^{(0,1)} = \mathcal{N}_3^2 U_{1,\xi}^{(0,1)} + \mathcal{N}_4^2 U_{2,\xi}^{(0,1)}, \tag{4.38}$$

and at lower surface of the layer

$$\mathcal{N}_5^2 U_1^{(2,2)} + \mathcal{N}_6^2 P_t^{(1,1)} = \mathcal{N}_7^2 U_{\xi\xi}^{(0,0)} + \mathcal{N}_8^2 V_{\xi\xi}^{(0,0)}. \tag{4.39}$$

The coefficients in equations (4.36)–(4.39) are too unwieldy to be written explicitly here, however their explicit representations were obtained with the help of Maple (1996).

From the two boundary conditions at the upper face of the layer (4.36) and (4.38) we obtain

$$P_t^{(0,1)} = \mathcal{C}_7 U_{1,\xi}^{(0,1)} + \mathcal{C}_8 U_{2,\xi}^{(0,1)}, \quad U_1^{(1,2)} = \mathcal{C}_9 U_{1,\xi}^{(0,1)} + \mathcal{C}_{10} U_{2,\xi}^{(0,1)}, \tag{4.40}$$

$$
\begin{aligned}
\mathcal{C}_7 &= \frac{\sigma_2-\gamma(\lambda^2+1)}{\gamma\,\lambda^2}, \quad \mathcal{C}_8 = \frac{\mathcal{C}_8^{(1)}}{2\,(\beta+\gamma\,\lambda^2)\,\gamma\,\lambda^3},\\
\mathcal{C}_8^{(1)} &= \gamma^2\lambda^6 + (B_{1111}-B_{1122}-B_{1221}-\sigma_2)\,\gamma\,\lambda^4 + \\
&\quad + (\gamma(1-\gamma)+B_{1122}+B_{1221}-B_{2222})\,\sigma_2\lambda^2 + \\
&\quad + (B_{1111}-B_{1122}-B_{1221})\,\sigma_2 + (B_{1221}+B_{1122}-B_{1111})\,\gamma,\\
\mathcal{C}_9 &= -\frac{(\lambda-1)\,(\lambda+1)}{\lambda}, \quad \mathcal{C}_{10} = \frac{(\lambda^2+1)^2\sigma_2 + \left(\lambda^6-\lambda^4-\lambda^2-1\right)\gamma - 2\,\beta\,\lambda^2}{2\,(\beta+\gamma\,\lambda^2)\,\lambda^2}.
\end{aligned}
\tag{4.41}
$$

From the two boundary conditions at the lower surface of the layer (4.37) and (4.39) we establish that

$$
P_t^{(1,1)} = \mathcal{C}_{11}U_{\xi\xi}^{(0,0)} + \mathcal{C}_{12}V_{\xi\xi}^{(0,0)}, \quad U_1^{(2,2)} = \mathcal{C}_{13}U_{\xi\xi}^{(0,0)} + \mathcal{C}_{14}V_{\xi\xi}^{(0,0)},
\tag{4.42}
$$

$$
\begin{aligned}
\mathcal{C}_{11} &= \frac{\mathcal{C}_{11}^{(1)}}{2\lambda^3\,(\beta+\lambda^2\gamma)\,\gamma},\\
\mathcal{C}_{11}^{(1)} &= \gamma^2\lambda^6 + (B_{2222}-\sigma_2-B_{1221}-B_{1122})\,\gamma\,\lambda^4 + \\
&\quad + (\gamma(\sigma_2-\gamma)+(B_{1122}-B_{1111}+B_{1221})\,\sigma_2)\,\lambda^2 + \\
&\quad + (B_{2222}-B_{1221}-B_{1122})\,\sigma_2 + (B_{1122}+B_{1221}-B_{2222})\,\gamma,\\
\mathcal{C}_{12} &= \frac{\left(\sigma_2-\gamma(\lambda^2+1)\right)\left(\gamma\,\lambda^6+(\sigma_2-\gamma)\,\lambda^4+(2(\sigma_2-\beta)-\gamma)\,\lambda^2-\gamma+\sigma_2\right)}{2\lambda^4\,(\beta+\lambda^2\gamma)\,\gamma},\\
\mathcal{C}_{13} &= \frac{\gamma\,\lambda^6+(2\,\beta-\sigma_2-3\,\gamma)\,\lambda^4-(2\,\beta-3\,\gamma+2\,\sigma_2)\,\lambda^2+\gamma-\sigma_2+2\,\beta}{2\lambda^2\,(\lambda^2\gamma+\beta)},\\
\mathcal{C}^{14} &= \frac{\left(1-\lambda^2\right)\left(\gamma\,\lambda^6+(\sigma_2-\gamma)\,\lambda^4+(2(\sigma_2-\beta)-\gamma)\,\lambda^2-\gamma+\sigma_2\right)}{2\,(\beta+\lambda^2\gamma)\,\lambda^3}.
\end{aligned}
\tag{4.43}
$$

We conclude that $u_1^{(2)}$ and $u_2^{(2)}$ are the quadratic functions of ζ, $p_t^{(1)}$ is linear function of ζ given by

$$
\begin{aligned}
u_1^{(2)} &= \{\mathcal{C}_{13}U_{\xi\xi}^{(0,0)} + \mathcal{C}_{14}V_{\xi\xi}^{(0,0)}\}\zeta^2 + \{\mathcal{C}_9U_{1,\xi}^{(0,1)} + \mathcal{C}_{10}U_{2,\xi}^{(0,1)}\}\zeta + U_1^{(0,2)},\\
u_2^{(2)} &= \{\mathcal{C}_5U_{\xi\xi}^{(0,0)} + \mathcal{C}_6V_{\xi\xi}^{(0,0)}\}\zeta^2 - \zeta U_{1,\xi}^{(0,1)} + U_2^{(0,2)},\\
p_t^{(1)} &= \{\mathcal{C}_{11}U_{\xi\xi}^{(0,0)} + \mathcal{C}_{12}V_{\xi\xi}^{(0,0)}\}\zeta + \mathcal{C}_7U_{1,\xi}^{(0,1)} + \mathcal{C}_8U_{2,\xi}^{(0,1)},
\end{aligned}
\tag{4.44}
$$

where all the coefficients are given by (4.33), (4.41) and (4.43). In the solution (4.44) four functions remain unknown, namely $U_1^{(0,1)}, U_2^{(0,1)}, U_1^{(0,2)}$ and $U_2^{(0,2)}$. To determine $U_1^{(0,1)}$ and $U_2^{(0,1)}$ one has to consider the fourth order problem and for $U_1^{(0,2)}$ and $U_2^{(0,2)}$ the fifth order problem, respectively. The governing equation for the leading order functions $U^{(0,0)}$ and $V^{(0,0)}$ is obtained from the third order problem and established below.

Using the solutions (4.14), (4.26) and (4.44), the equations of motion (4.35) can be expressed as

$$\begin{aligned}
&\mathcal{G}_1^{(1)} U_{\tau\tau}^{(0,0)} + \mathcal{G}_2^{(1)} V_{\tau\tau}^{(0,0)} + \mathcal{G}_3^{(1)} U_{\xi\xi}^{(0,0)} + \mathcal{G}_4^{(1)} V_{\xi\xi}^{(0,0)} = 0,\\
&\mathcal{G}_1^{(2)} U_{\tau\tau}^{(0,0)} + \mathcal{G}_2^{(2)} V_{\tau\tau}^{(0,0)} + \mathcal{G}_3^{(2)} U_{\xi\xi}^{(0,0)} + \mathcal{G}_4^{(2)} V_{\xi\xi}^{(0,0)} = 0,
\end{aligned} \tag{4.45}$$

$$\mathcal{G}_1^{(1)} = \mathcal{G}_2^{(2)} = \lambda^3\gamma\left(\lambda^2+1\right), \quad \mathcal{G}_2^{(1)} = -\mathcal{G}_1^{(2)} = \lambda^2\gamma\left(\lambda^2+1\right), \tag{4.46}$$

$$\begin{aligned}
\mathcal{G}_3^{(1)} &= \frac{\left(\lambda^2+1\right)^2\left(\gamma(\lambda^2+1)-\sigma_2\right)}{\lambda},\\
\mathcal{G}_4^{(1)} &= \frac{\left(\lambda^2+1\right)^2\left(\gamma^2\lambda^4(\lambda^2-1)+\lambda^2\left(\sigma_2(4\beta+\sigma_2)-\gamma^2\right)+(\sigma_2-\gamma)^2\right)}{\lambda^2\left(2\,\beta+2\,\lambda^2\gamma\right)},
\end{aligned} \tag{4.47}$$

$$\begin{aligned}
&\mathcal{G}_3^{(2)} = \left(\lambda^2+1\right)^2\left(\gamma(\lambda^2+1)\sigma_2\right), \quad \mathcal{G}_4^{(2)} = -\frac{\left(\lambda^2+1\right)^2 G_{41}^2}{2\lambda\left(\beta+\lambda^2\gamma\right)},\\
&G_{41}^2 = \gamma^2\lambda^6 + \lambda^4\gamma\left(2(\beta-\sigma_2)+\gamma\right) -\\
&\qquad - \lambda^2\left(\gamma^2+\sigma_2(\sigma_2-2\gamma)\right) - 2(\gamma(\beta+\sigma_2)+\sigma_2\beta) - {\sigma_2}^2 - \gamma^2.
\end{aligned} \tag{4.48}$$

The governing equation for the functions $U^{(0,0)}$ and $V^{(0,0)}$ can, by using (4.45) be written in the following matrix form

$$\frac{\partial^2}{\partial\tau^2}\begin{pmatrix} U^{(0,0)} \\ V^{(0,0)} \end{pmatrix} + \boldsymbol{D}\frac{\partial^2}{\partial\zeta^2}\begin{pmatrix} U^{(0,0)} \\ V^{(0,0)} \end{pmatrix} = 0, \quad \boldsymbol{D} = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}, \tag{4.49}$$

$$\begin{aligned}
&D_{11} = \frac{2(\sigma_2-\gamma(\lambda^2+1))}{\gamma\,\lambda^2},\\
&D_{12} = D_{21} = \frac{\left(\lambda^2-1\right)\left(\gamma(\lambda^2+1)-\sigma_2\right)}{\lambda^3\gamma},\\
&D_{22} = -\frac{D_{22}^{(1)}}{\lambda^4\gamma\,\left(2\,\beta+2\,\lambda^2\gamma\right)},\\
&D_{22}^{(1)} = \gamma^2\lambda^8 + 2\lambda^6\gamma\left(\beta-\sigma_2\right) + \lambda^4\sigma_2\left(2\,\gamma-\sigma_2\right) +\\
&\qquad + 2\lambda^2\left(\beta(2\,\sigma_2-\gamma)+\sigma_2(\gamma-\sigma_2)\right) - (\sigma_2-\gamma)^2.
\end{aligned} \tag{4.50}$$

The solutions of equation (4.49) enables to determine displacement components in the leading order problem. A special type of one dimensional vector governing equation (4.49) exist due to the fact that the asymptotic orders of displacements and hydrostatic pressure (2.51) are equivalent. The equation (4.49) provide a significant simplification in comparison with the equations of motion (1.68) and (1.69), subject to (1.95), (1.96). It demonstrates that to describe the long wave low frequency motion one can use equation (4.49) with only two essential parameters - long wave amplitudes of in-plane and normal displacement components $U^{(0,0)}(\xi,\tau)$ and $V^{(0,0)(\xi,\tau)}$ respectively. We remark that the governing equation (4.49) is of novel matrix type in comparison to previously derived analogous equations in the asymptotic models by Kaplunov et al. (1998, 2000), Prikazchikova (2004).

The eigenvalues of the matrix $\mathbf{D}$ may be found from the following quadratic equation for non-dimensional squared wave speed $\hat{v}_0$

$$\hat{v}_0^2 + \mathcal{C}_{2f}\hat{v}_0 + \mathcal{C}_{0f} = 0, \tag{4.51}$$

with

$$\mathcal{C}_{2f} = -\frac{\left(\lambda^2+1\right)^2\left(\lambda^4\gamma^2 + 2\,\lambda^2\gamma(\beta+\gamma-\sigma_2) - (\sigma_2-\gamma)^2\right)}{2\gamma\,\left(\beta+\lambda^2\gamma\right)\lambda^4}, \tag{4.52}$$

$$\mathcal{C}_{0f} = \frac{\left(\lambda^2+1\right)^2\left(\gamma(1+\lambda^2)-\sigma_2\right)\left(\sigma_2-\gamma+\lambda^2\gamma\right)\left(\gamma+\beta-\sigma_2\right)}{\lambda^6\left(\beta+\lambda^2\gamma\right)\gamma^2}. \tag{4.53}$$

With the help of relations (1.36) the coefficients (4.52) can be represented as

$$\begin{aligned}\mathcal{C}_{2f}^{(s)} &= (\delta+1)(p-1)^2 - \epsilon^2 - 4,\\ \mathcal{C}_{0f}^{(s)} &= -\left(p^2-1\right)\left(2(p(\delta+1)-(\delta-1)+1)+\epsilon^2\right).\end{aligned} \tag{4.54}$$

We remark that the quadratic equation (4.51) coincides with equation (2.8) which is the leading order expansion of dispersion relation (2.4) in long wave low frequency regime. The solutions of equation (4.51) are therefore the long wave low frequency limits of the fundamental modes expressed in (2.9). Hence the asymptotic long wave low frequency model is asymptotically consistent with the analysis of dispersion relation (2.4).

To conclude this section we would like to comment on existence of quasi-fronts and the related question of constructing the higher order theories. There can be discontinuities in solution of associated with second order of matrix equation (4.49). In order to smooth these discontinuities the fourth-order correction terms can be obtained by constructing higher order theories. In the theory of thin structures this phenomenon is known as a quasi-front. The examples of higher order theories and associated quasi-fronts can be found in the papers by Kaplunov et al. (2000) and Pichugun and Rogerson (2002). In the considered system of asymptotically approximate equations the fourth-order governing equation can be obtained only at the stage of firth order asymptotic problem. Therefore due to the algebraic complexity the fourth-order corrections to second-order governing equation were not derived.

4.2 Asymptotic model in respect of neo-Hookean strain-energy function

4.2.1 Non-dimensional equations

Our intension now is to derive a simplified asymptotic dynamic model for long wave low frequency motion in a sheared pre-stressed incompressible elastic layer composed of neo-Hookean material. We remark that it is possible to obtain all the results for a neo-Hookean material model from the general formulas derived in the previous section. In order to do that one would require the form of material constants and elasticity tensor components given by (1.118), the relation between the parameters p and σ_2 (1.110). However, we consider this section specifically to give more details about the derivation of the dynamic asymptotic model for the certain type of material.

For the special case of a neo-Hookean strain energy function the equations of motion (1.120) and incremental traction components (1.121) are significantly simplified in comparison to the general formulas. It was shown that the relative asymptotic orders of displacements and incremental pressure components are equivalent (2.51). Taking into account relations (1.118) and (1.110) the scalings (4.1) and (4.2) enables us to establish non-dimensional governing equations in the form

$$\begin{aligned}
&m_1 = c_2^1\eta^2 + c_1^1\eta + c_0^1 = 0, \quad m_2 = c_2^2\eta^2 + c_1^2\eta + c_0^2 = 0,\\
&c_0^1 = -\lambda^3 u^*_{1,\zeta\zeta} - (p+1)\,\lambda^2 u^*_{2,\zeta\zeta}, \quad c_0^2 = \lambda^2 u^*_{1,\zeta\zeta} - (p+1)\,\lambda^3 u^*_{2,\zeta\zeta},\\
&c_1^1 = \lambda\left(2-(2+p)\,\lambda^2\right) u^*_{2,\xi\zeta} - \lambda^2\left(2(\lambda^2-1)+p)\right) u^*_{1,\xi\zeta} + p^*_{t,\zeta}\lambda^2,\\
&c_1^2 = \lambda^2\left((2+p)-2\lambda^2\right) u^*_{2,\xi\zeta} - \lambda\left(2-(2-p)\lambda^2\right) u^*_{1,\xi\zeta} + p^*_{t,\zeta}\lambda^3,\\
&c_2^1 = \left(\lambda^2(1-\lambda^2)-1\right) u^*_{2,\xi\xi} + p^*_{t,\xi}\lambda^3 + \lambda^3 u^*_{1,\tau\tau} + \lambda^2 u^*_{2,\tau\tau} - \lambda\left(\lambda^2\,(p-1)+1+\lambda^4\right) u^*_{1,\xi\xi},\\
&c_2^2 = \lambda\left(\lambda^2(1-\lambda^2)-1\right) u^*_{2,\xi\xi} - p^*_{t,\xi}\lambda^2 - \lambda^2 u^*_{1,\tau\tau} + \lambda^3 u^*_{2,\tau\tau} + \left(\lambda^2\,(p-1)+1+\lambda^4\right) u^*_{1,\xi\xi},\\
&u^*_{1,\xi}\eta + u^*_{2,\zeta} = 0.
\end{aligned}\tag{4.55}$$

The above equations must be solved subject to traction free boundary conditions, which employing representations (1.121) can be expressed as

$$\begin{aligned}
&t_1 = t_1^1\eta + t_0^1 = 0, \quad t_2 = t_1^2\eta + t_0^2 = 0, \quad \text{at} \quad \zeta = 0, -1,\\
&t_0^1 = -\lambda\,(1+p)\,u^*_{2,\zeta} - \lambda^2 u^*_{1,\zeta}, \quad t_0^2 = \lambda\,u^*_{1,\zeta} - (p+1)\,\lambda^2 u^*_{2,\zeta},\\
&t_1^1 = \lambda\left(1-\lambda^2\right) u^*_{1,\xi} - \left(\lambda^2(p+1)-1\right) u^*_{2,\xi} + \lambda p^*_t,\\
&t_1^2 = \left(\lambda^2-1\right) u^*_{1,\xi} - \lambda\left(\lambda^2-(p+1)\right) u^*_{2,\xi} + \lambda^2 p^*_t.
\end{aligned}\tag{4.56}$$

In a similar way to the general case, the solutions of equations (4.55), subject to boundary conditions (4.56), are sought in the form of series (4.11), leading to a hierarchical system of boundary value problems at various orders.

4.2.2 Leading order problem

The leading order the equations of motion and incompressibility condition are given by

$$\lambda u_{1,\zeta\zeta}^{(0)} + (p+1)\, u_{2,\zeta\zeta}^{(0)} = 0, \quad u_{1,\zeta\zeta}^{(0)} - (p+1)\, \lambda u_{2,\zeta\zeta}^{(0)} = 0, \quad u_{2,\zeta}^{(0)} = 0, \tag{4.57}$$

these being subject to the following traction free boundary conditions

$$(1+p)\, u_{2,\zeta}^{(0)} + \lambda u_{1,\zeta}^{(0)} = 0, \quad u_{1,\zeta}^{(0)} - (p+1)\, \lambda u_{2,\zeta}^{(0)} = 0, \ \text{ at } \ \zeta = 0, -1. \tag{4.58}$$

It is easy to see that the solutions of the equations (4.57), subjected to boundary conditions (4.58), are of the form

$$u_1^{(0)} = U^{(0,0)}(\xi, \tau), \quad u_2^{(0)} = V^{(0,0)}(\xi, \tau). \tag{4.59}$$

4.2.3 Second order problem

The non-dimensional equations of motion and associated incompressibility condition at second order are given by

$$\begin{aligned}
&\lambda^2 u_{1,\zeta\zeta}^{(1)} + \lambda\, (p+1)\, u_{2,\zeta\zeta}^{(1)} - \lambda p_{t,\zeta}^{(0)} = \lambda(2(\lambda^2+1) - p) u_{1,\xi\zeta}^{(0)} + (2 - \lambda^2(p+2)) u_{2,\xi\zeta}^{(0)}, \\
&\lambda u_{1,\zeta\zeta}^{(1)} - \lambda^2\, (1+p)\, u_{2,\zeta\zeta}^{(1)} + \lambda^2 p_{t,\zeta}^{(0)} = (2 + \lambda^2(p-2)) u_{1,\xi\zeta}^{(0)} + \lambda(2(\lambda^2-1) - p) u_{2,\xi\zeta}^{(0)}, \\
&u_{2,\zeta}^{(1)} = -u_{1,\xi}^{(0)},
\end{aligned} \tag{4.60}$$

with the appropriate traction free boundary conditions taking the form

$$\begin{aligned}
&\lambda^2 u_{1,\zeta}^{(1)} + \lambda\, (p+1)\, u_{2,\zeta}^{(1)} - \lambda p_t^{(0)} = \lambda \left(1 - \lambda^2\right) u_{1,\xi}^{(0)} + \left(1 - \lambda^2 (p+1)\right) u_{2,\xi}^{(0)}, \quad \text{at} \quad \zeta = 0, -1, \\
&\lambda\, u_{1,\zeta}^{(1)} - \lambda^2\, (p+1)\, u_{2,\zeta}^{(1)} + \lambda^2 p_t^{(0)} = \left(1 - \lambda^2\right) u_{1,\xi}^{(0)} + \lambda \left(\lambda^2 - (p+1)\right) u_{2,\xi}^{(0)}, \quad \text{at} \quad \zeta = 0, -1.
\end{aligned} \tag{4.61}$$

Taking into account the solutions of the leading order problem (4.59), the equations of motion and incompressibility condition can be expressed in the following simplified form

$$\lambda u_{1,\zeta\zeta}^{(1)} + (1+p)\, u_{2,\zeta\zeta}^{(1)} - p_{t,\zeta}^{(0)} = 0, \quad u_{1,\zeta\zeta}^{(1)} - \lambda\, (1+p)\, u_{2,\zeta\zeta}^{(1)} + \lambda p_{t,\zeta}^{(0)} = 0, \quad u_{1,\xi}^{(0)} + u_{2,\zeta}^{(1)} = 0, \tag{4.62}$$

with the associated boundary conditions (4.61) yielding

$$\lambda^2 u^{(1)}_{1,\zeta} + \lambda\,(p+1)\,u^{(1)}_{2,\zeta} - \lambda p^{(0)}_t = \lambda\left(1-\lambda^2\right) U^{(0,0)}_{,\xi} - \left(\lambda^2(1+p)-1\right) V^{(0,0)}_{,\xi}, \quad \text{at} \quad \zeta = 0,-1,$$
$$\lambda^2\,(p+1)\,u^{(1)}_{2,\zeta} - \lambda\, u^{(1)}_{1,\zeta} - \lambda^2 p^{(0)}_t = \left(\lambda^2-1\right) U^{(0,0)}_{,\xi} + \lambda\left(1-\lambda^2+p\right) V^{(0,0)}_{,\xi}, \quad \text{at} \quad \zeta = 0,-1. \tag{4.63}$$

The incompressibility condition $(4.62)_3$ yields the following representation for the normal displacement component

$$u^{(1)}_2 = -\zeta U^{(0,0)}_{,\xi} + U^{(0,1)}_2. \tag{4.64}$$

With the help of relation (4.64), the equations of motion can be simplified to

$$\lambda u^{(1)}_{1,\zeta\zeta} - p^{(0)}_{t,\zeta} = 0, \quad u^{(1)}_{1,\zeta\zeta} + \lambda p^{(0)}_{t,\zeta} = 0. \tag{4.65}$$

The determinant of the coefficients of the system (4.65) is not equal to zero, hence the only solutions for two variables are the trivial solutions, thus

$$p^{(0)}_{t,\zeta} = 0, \quad u^{(1)}_{1,\zeta\zeta} = 0. \tag{4.66}$$

We are now able to deduce that at second order the in-plane displacement component $u^{(1)}_1$ is a linear function of ζ and the leading order incremental pressure $p^{(0)}_t$ is independent of ζ, thus

$$u^{(1)}_1 = U^{(1,1)}_1 \zeta + U^{(0,1)}_1. \tag{4.67}$$

Inserting the previously obtained representations (4.67) and (4.64) for $u^{(1)}_1$, $u^{(1)}_2$ and $p^{(0)}_t$ into the boundary conditions (4.63) we obtain a system of two equations in $U^{(1,1)}_1$ and $p^{(0)}_t$, from which we obtain the following relations

$$p^{(0)}_t = -\,(p+1)\,U^{(0,0)}_{\xi} + \left(\lambda - \lambda^{-1}\right) V^{(0,0)}_{\xi}, \quad U^{(1,1)}_1 = \left(\lambda^{-1} - \lambda\right) U^{(0,0)}_{\xi} - pV^{(0,0)}_{\xi}. \tag{4.68}$$

To sum up, in the solution of the second order problem both displacement components are the following linear functions of ζ, with the leading order incremental pressure independent of ζ, their form being expressible as

$$u^{(1)}_1 = \{\left(\lambda^{-1}-\lambda\right) U^{(0,0)}_{,\xi} - pV^{(0,0)}_{,\xi}\}\zeta + U^{(0,1)}_1,$$
$$u^{(1)}_2 = -U^{(0,0)}_{,\xi}\zeta + U^{(0,1)}_2,$$
$$p^{(0)}_t = -\,(p+1)\,U^{(0,0)}_{,\xi} + \left(\lambda - \lambda^{-1}\right) V^{(0,0)}_{,\xi}. \tag{4.69}$$

We note that the functions $U^{(0,1)}_2$, $U^{(0,1)}_1$ in (4.69) remain undefined at second order, to obtain further information about them we have to consider higher order problems.

4.2.4 Third order problem

At third order the non-dimensional equations of motion and incompressibility condition are given by

$$
\begin{aligned}
&\lambda^3 u^{(2)}_{1,\zeta\zeta} + \lambda^2 (p+1)\, u^{(2)}_{2,\zeta\zeta} - \lambda^2 p^{(1)}_{t,\zeta} = \lambda(\lambda^2(p+1) - (\lambda^4+1))u^{(0)}_{1,\xi\xi} + (\lambda^2(1-\lambda^2)-1)u^{(0)}_{2,\xi\xi} \\
&+ \lambda^2(2(1-\lambda^2)-p)u^{(1)}_{1,\xi\zeta} + \lambda(2-\lambda^2(p-2))u^{(1)}_{2,\xi\zeta} + \lambda^2 u^{(0)}_{2,\tau\tau} + \lambda^3 u^{(0)}_{1,\tau\tau} + \lambda^3 p^{(0)}_{t,\xi}, \\
&\lambda^3 (1+p)\, u^{(2)}_{2,\zeta\zeta} - \lambda^2 u^{(2)}_{1,\zeta\zeta} - \lambda^3 p^{(1)}_{t,\zeta} = (\lambda^2(p-1) + \lambda^4 + 1)u^{(0)}_{1,\xi\xi} + \lambda(\lambda^2(1-\lambda^2)-1)u^{(0)}_{2,\xi\xi} \\
&+ \lambda(\lambda^2(2-p)-2)u^{(1)}_{1,\xi\zeta} + \lambda^2(p+2(1-\lambda^2))u^{(1)}_{2,\xi\zeta} - \lambda^2 u^{(0)}_{1,\tau\tau} + \lambda^3 u^{(0)}_{2,\tau\tau} - \lambda^2 p^{(0)}_{t,\xi}, \\
&u^{(2)}_{2,\zeta} = -u^{(1)}_{1,\xi},
\end{aligned}
\tag{4.70}
$$

and subject to the appropriate traction free boundary conditions, namely

$$
\begin{aligned}
&\lambda^2 u^{(2)}_{1,\zeta} + \lambda (p+1)\, u^{(2)}_{2,\zeta} - \lambda p^{(1)}_t = \lambda \left(1-\lambda^2\right) u^{(1)}_{1,\xi} + \left(1 - \lambda^2(p+1)\right) u^{(1)}_{2,\xi}, \quad \text{at} \quad \zeta = 0, -1, \\
&\lambda^2 (p+1)\, u^{(2)}_{2,\zeta} - \lambda\, u^{(2)}_{1,\zeta} - \lambda^2 p^{(1)}_t = \left(\lambda^2 - 1\right) u^{(1)}_{1,\xi} + \lambda \left(p + 1 - \lambda^2\right) u^{(1)}_{2,\xi}, \quad \text{at} \quad \zeta = 0, -1.
\end{aligned}
\tag{4.71}
$$

Taking into account the solutions of the leading order problem (4.59) and second order problem (4.69) the incompressibility condition is employed to obtain the following representation for the normal displacement component

$$
u^{(2)}_2 = (\left(\lambda - \lambda^{-1}\right) U^{(0,0)}_{\xi\xi} + pV^{(0,0)}_{\xi\xi})\zeta^2 - \zeta U^{(0,1)}_{1,\xi} + U^{(0,2)}_2. \tag{4.72}
$$

From (4.72) one can see that the normal displacement is a quadratic function of ζ. Taking into account the leading order solutions (4.59) and second order solutions (4.69) we introduce the following representations for the in-plane displacement component and incremental pressure

$$
u^{(2)}_1 = U^{(2,2)}_1 \zeta^2 + U^{(1,2)}_1 \zeta + U^{(0,2)}_1 \quad p^{(1)}_t = P^{(1,1)}_t \zeta + P^{(0,1)}_t. \tag{4.73}
$$

In the third order solution (4.72)–(4.73) there are four unknown functions $U^{(2,2)}_1$, $U^{(1,2)}_1$, $P^{(1,1)}_t$, $P^{(0,1)}_t$, which it is possible to obtain from third order equations. The remaining two unknown functions $U^{(0,2)}_1$, $U^{(0,2)}_2$ can be determined only by considering higher order problems. Substituting the relations (4.72)–(4.73) into the two equations of motion and the boundary conditions a system of six equations in the six unknowns $U^{(0,0)}$, $V^{(0,0)}$, $U^{(1,2)}_1$, $U^{(2,2)}_1$, $P^{(0,1)}_t$, $P^{(1,1)}_t$ may be obtained. In this system the equations of motion provide the source of a matrix representation of the resulting governing equations. To obtain this we note that the equations of motion may be now written in the form

$$
\begin{aligned}
&\lambda^3 U_1^{(2,2)} - \lambda^2 P_t^{(1,1)} = \\
&\qquad = \lambda^3 U_{\tau\tau}^{(0,0)} - \lambda^3 \left(3 - \lambda^2 + p\right) U_{\xi\xi}^{(0,0)} + \left(p\lambda^2(2\lambda^2 - 3) - 1\right) V_{\xi\xi}^{(0,0)} + \lambda^2 V_{\tau\tau}^{(0,0)}, \\
&\lambda^2 U_1^{(2,2)} + \lambda^3 P_t^{(1,1)} = \\
&\qquad = \lambda^2 U_{\tau\tau}^{(0,0)} - \left(\lambda^2(p+3) - 1\right) U_{\xi\xi}^{(0,0)} - \lambda \left(p(2 - 3\lambda^2) - \lambda^4\right) V_{\xi\xi}^{(0,0)} - \lambda^3 V_{\tau\tau}^{(0,0)}.
\end{aligned}
\tag{4.74}
$$

Using (4.71), we deduce that both incremental traction components are linear functions of ζ. Hence, the in-plane traction component boundary condition at the upper surface of layer can be expressed as

$$
\lambda^2 U_1^{(1,2)} - \lambda P_t^{(0,1)} = \left(1 - \lambda^2(p+1)\right) U_{2,\xi}^{(0,1)} + \lambda \left(p + 2 - \lambda^2\right) U_{1,\xi}^{(0,1)}, \tag{4.75}
$$

with its the lower surface counterpart given by

$$
\lambda^2 U_1^{(2,2)} - \lambda P_t^{(1,2)} = \left(p + (\lambda^2 - 1)^2\right) U_{\xi\xi}^{(0,0)} - \lambda p \left(2 + p + \lambda^2\right) V_{\xi\xi}^{(0,0)}. \tag{4.76}
$$

Similarly, the normal traction component has the following form at the upper surface

$$
\lambda U_1^{(1,2)} + \lambda^2 P_t^{(0,1)} = \left(1 - \lambda^2(p+2)\right) U_{1,\xi}^{(0,1)} - \lambda \left(p + 1 - \lambda^2\right) U_{2,\xi}^{(0,1)}, \tag{4.77}
$$

and at the lower surface yields the boundary condition is the following

$$
\lambda^2 P_t^{(1,1)} + \lambda U_1^{(2,2)} = \left((p+1)\lambda^3 + \lambda^{-1} - 2\lambda\right) U_{\xi\xi}^{(0,0)} + p\left((p+2)\lambda^2 - 1\right) V_{\xi\xi}^{(0,0)}. \tag{4.78}
$$

With the help of the two boundary conditions at the upper surface of the layer (4.75), (4.77) we obtain the following representations for the functions $P_t^{(0,1)}$ and $U_1^{(1,2)}$

$$
\begin{aligned}
P_t^{(0,1)} &= -(p+1) U_{1,\xi}^{(0,1)} + \left(\lambda - \lambda^{-1}\right) U_{2,\xi}^{(0,1)}, \\
U_1^{(1,2)} &= \left(-\lambda + \lambda^{-1}\right) U_{1,\xi}^{(0,1)} - p U_{2,\xi}^{(0,1)}.
\end{aligned}
\tag{4.79}
$$

Employing the other two boundary conditions at the lower surface of layer (4.76), (4.78), the functions $P_t^{(1,1)}$ and $U_1^{(2,2)}$ can be expressed as

$$
\begin{aligned}
P_t^{(1,1)} &= \left(\lambda - \lambda^{-1}\right) p U_{\xi\xi}^{(0,0)} + p\,(p+1) V_{\xi\xi}^{(0,0)}, \\
U_1^{(2,2)} &= \left(p + \lambda^2 - 2 + \lambda^{-2}\right) U_{\xi\xi}^{(0,0)} + \left(\lambda - \lambda^{-1}\right) p V_{\xi\xi}^{(0,0)}.
\end{aligned}
\tag{4.80}
$$

Hence, the solutions of the third order problem indicate that as both displacement component are quadratic functions of ζ, with the second order incremental pressure a linear function of ζ, these being given by

$$
\begin{aligned}
u_1^{(2)} &= \{\left(p+\lambda^2-2+\lambda^{-2}\right) U_{\xi\xi}^{(0,0)} + \left(\lambda-\lambda^{-1}\right) p V_{\xi\xi}^{(0,0)}\}\zeta^2 + \\
&\qquad + \{\left(-\lambda+\lambda^{-1}\right) U_{1,\xi}^{(0,1)} - p U_{2,\xi}^{(0,1)}\}\zeta + U_1^{(0,2)}, \\
u_2^{(2)} &= \{\left(\lambda-\lambda^{-1}\right) U_{\xi\xi}^{(0,0)} + p V_{\xi\xi}^{(0,0)}\}\zeta^2 - \zeta U_{1,\xi}^{(0,1)} + U_2^{(0,2)}, \\
p_t^{(1)} &= \{\left(\lambda-\lambda^{-1}\right) p U_{\xi\xi}^{(0,0)} + p\left(p+1\right) V_{\xi\xi}^{(0,0)}\}\zeta - \left(p+1\right) U_{1,\xi}^{(0,1)} + \left(\lambda-\lambda^{-1}\right) U_{2,\xi}^{(0,1)}.
\end{aligned}
\tag{4.81}
$$

We note that to determine the unknown functions $U_1^{(0,1)}, U_2^{(0,1)}$ one has to consider the fourth order problem and to define the unknown functions $U_1^{(0,2)}, U_2^{(0,2)}$ one has to examine fifth order problem. However the governing equations relating $U^{(0,0)}$ and $V^{(0,0)}$ may be established at the third order problem.

With the help of the relations (4.81), the equations of motion (4.74) take the following form

$$
\begin{aligned}
&\lambda^3 U_{\tau\tau}^{(0,0)} + \lambda^2 V_{\tau\tau}^{(0,0)} - \lambda\left(\lambda^2+1\right)(p+1) U_{\xi\xi}^{(0,0)} + \left(\lambda^2(p^2+p(\lambda^2-1))-1\right) V_{\xi\xi}^{(0,0)} = 0, \\
&-\lambda^2 U_{\tau\tau}^{(0,0)} + \lambda^3 V_{\tau\tau}^{(0,0)} + \lambda^2\left(\lambda^2+1\right)(p+1) U_{\xi\xi}^{(0,0)} + \lambda\left(p^2\lambda^2+(1-\lambda^2)p-\lambda^4\right) V_{\xi\xi}^{(0,0)} = 0.
\end{aligned}
\tag{4.82}
$$

The governing equations (4.82) may be written in the following matrix form

$$
\frac{\partial^2}{\partial\tau^2}\begin{pmatrix} U^{(0,0)} \\ V^{(0,0)} \end{pmatrix} + \boldsymbol{D}\frac{\partial^2}{\partial\zeta^2}\begin{pmatrix} U^{(0,0)} \\ V^{(0,0)} \end{pmatrix} = 0, \quad \boldsymbol{D} = \frac{1}{\lambda}\begin{pmatrix} -2(p+1)\lambda & \left(\lambda^2-1\right)(1+p) \\ \left(\lambda^2-1\right)(1+p) & \lambda^2(1+p)-\lambda^4-1 \end{pmatrix}.
\tag{4.83}
$$

We remark that the eigenvalues of $\mathbf{D}$ may be shown to be the scaled squared phase speeds limits (2.16), confirming asymptotic consistency of the established model.

Chapter 5

Asymptotically consistent model for a long wave high frequency motion in a layer with free faces

In this chapter we derive a one-dimensional asymptotically consistent model for two-dimensional long wave high frequency motion in an incompressible elastic layer subject to a primary simple shear deformation. We focus our attention to motion in a layer composed of the neo-Hookean material with traction free boundary conditions. The relative asymptotic orders of displacements and incremental pressure are used to obtain non-dimensional governing equations. Then a hierarchy of traction free boundary value asymptotic problems at various orders is established. The first three asymptotic orders are considered and asymptotic integration is carried out in the vicinity of cut-off frequencies. A one-dimensional governing equation for the long wave amplitude is derived only at the stage of the third order problem, this fact is a novel aspect of the model. To conclude, the governing equation is employed to demonstrate the consistency of derived model with asymptotic analysis of corresponding dispersion relation.

5.1 Asymptotic scaling and dimensionless equations

Taking into account the algebraic complexity of the general equations, we consider the simplified case of the layer composed of neo-Hookean material to illustrate the main principles in the derivation of a one-dimensional asymptotically consistent model for the long wave high frequency motion. The novel aspect in the derivation of asymptotic long wave high frequency dynamic model is that pre-stress in a form of a simple shear deformation does not allow decomposition of motion into symmetric and anti-symmetric parts.

For the case of neo-Hookean strain energy function the linearized two-dimensional equations of motion and incremental traction components are given by (1.120) and (1.121), respectively. To obtain non-dimensional equations we use the previously established relative asymptotic orders of both displacements and pressure component (2.58). The appropriate non-dimensional displacement components and incremental pressure component are given by

$$u_1 = l u_1^*, \quad u_2 = l\eta u_2^*, \quad p_t = \mu p_t^* \eta^{-1}, \tag{5.1}$$

where the superscript * indicates dimensionless quantities. Also the corresponding non-dimensional scaled spatial and time variables are introduced by

$$x_1 = l\xi, \quad x_2 = l\eta\zeta, \quad t = l\eta\sqrt{\frac{\rho}{\mu}}\tau, \tag{5.2}$$

where $\zeta = x_2/h$. To distinguish between differentiation with respect to scaled and original variables we use the notation $(.)_{,\xi}, (.)_{,\zeta}$ and $(.)_{,\tau}$ to indicate differentiation with respect to ξ, ζ and τ respectively. Hence the equations of motion (1.120) can be represented in non-dimensional form and expressed as an expansion of a small parameter η. Then we take into account that for the high frequency motion in the vicinity of cut-off frequencies Ω the following relations are valid

$$u_{1,\tau\tau} + \Omega^2 u_1 \sim \eta^2 u_1, \quad u_{2,\tau\tau} + \Omega^2 u_2 \sim \eta^2 u_2, \quad p_{t,\tau\tau} + \Omega^2 p_t \sim \eta^2 p_t, \tag{5.3}$$

which can be verified using approximation (2.40). To facilitate the representation for the non-dimensional equations of motion, taking into account relation (5.3), we introduce the following notations W_1, W_2 and W_p

$$u_{1,\tau\tau} + \Omega^2 u_1 = \eta^2 W_1, \quad u_{2,\tau\tau} + \Omega^2 u_2 = \eta^2 W_2, \quad p_{t,\tau\tau} + \Omega^2 p_t = \eta^2 W_p. \tag{5.4}$$

Then we exclude the quantities $u_{1,\tau\tau}, u_{2,\tau\tau}$ from equations of motion in favor of W_1 and W_2. After that the non-dimensional equations of motion (1.120) can be expressed as the following expansions

$$
\begin{aligned}
& m_1 = c_3^1\eta^3 + c_2^1\eta^2 + c_1^1\eta + c_0^1 = 0, \quad m_2 = c_3^2\eta^3 + c_2^2\eta^2 + c_1^2\eta + c_0^2 = 0,\\
& c_0^1 = \lambda^2(p^*_{t,\zeta} - \lambda u^*_{1,\zeta\zeta} - \lambda\Omega^2 u_1^*), \quad c_0^2 = \lambda^2(u^*_{1,\zeta\zeta} + p^*_{t,\zeta}\lambda + \Omega^2 u_1^*),\\
& c_1^1 = p^*_{t,\xi}\lambda^3 - (p+1)\,\lambda^2 u^*_{2,\zeta\zeta} - \big(2(\,\lambda^2-1)+p\big)\,\lambda^2 u^*_{1,\xi\zeta} - \lambda^2\Omega^2 u_2^*,\\
& c_1^2 = -p^*_{t,\xi}\lambda^2 - (p+1)\,\lambda^3 u^*_{2,\zeta\zeta} + \big((2-p)\,\lambda^2 - 2\big)\,\lambda u^*_{1,\xi\zeta} - \lambda^3\Omega^2 u_2^*,\\
& c_2^1 = \big((1-p)\,\lambda^2 - \lambda^4 - 1\big)\,\lambda u^*_{1,\xi\xi} + \big(2-(2+p)\,\lambda^2\big)\,\lambda u^*_{2,\xi\zeta} + \lambda^3 W_1^*,\\
& c_2^2 = \big(\lambda^4 + \lambda^2\,(p-1)+1\big)\,u^*_{1,\xi\xi} + \big(p + 2(1-\lambda^2)\big)\,\lambda^2 u^*_{2,\xi\zeta} - \lambda^2 W_1^*,\\
& c_3^1 = \big(\lambda^2(1-\lambda^2)-1\big)\,u^*_{2,\xi\xi} + \lambda^2 W_2^*, \quad c_3^2 = \big(\lambda^2(1-\lambda^2)-1\big)\,\lambda u^*_{2,\xi\xi} + \lambda^3 W_2^*,
\end{aligned}
\tag{5.5}
$$

where the cut-off frequencies Ω are given by (2.34) . Equations (5.5) must be solved in conjunction with the non-dimensional incompressibility condition

$$
u^*_{1,\xi} + u^*_{2,\zeta} = 0, \tag{5.6}
$$

and subject to non-dimensional traction free boundary conditions at upper and lower surfaces on the layer, which are obtained from (1.121) and expressed as

$$
\begin{aligned}
& \mathcal{T}_1 = t_2^1\eta^2 + t_1^1\eta + t_0^1 = 0, \quad \mathcal{T}_2 = t_2^2\eta^2 + t_1^2\eta + t_0^2 = 0, \quad \text{at} \quad \zeta = 0, -1,\\
& t_0^1 = \lambda(p_t^* - \lambda u^*_{1,\zeta}), \quad t_0^2 = \lambda(\lambda p_t^* + u^*_{1,\zeta}),\\
& t_1^1 = \big(1-\lambda^2\big)\,\lambda u^*_{1,\xi} - (p+1)\,\lambda u^*_{2,\zeta}, \quad t_1^2 = \big(\lambda^2-1\big)\,u^*_{1,\xi} - (p+1)\,\lambda^2 u^*_{2,\zeta},\\
& t_2^1 = \big(1-(p+1)\,\lambda^2\big)\,u^*_{2,\xi}, \quad t_2^2 = \big(1-\lambda^2+p\big)\,\lambda u^*_{2,\xi}.
\end{aligned}
\tag{5.7}
$$

Solutions of equations (5.5)–(5.6) subject to boundary conditions (5.7) are sought in the form of series (4.11).

5.2 Leading order problem

The leading order problem consists of the non-dimensional equations of motion and associated incompressibility condition which taken into account cut-off frequencies $\Omega = n\pi$ (2.34) are given by

$$
p^{(0)}_{t,\zeta} - \lambda\, u^{(0)}_{1,\zeta\zeta} - \lambda n^2\pi^2 u_1^{(0)} = 0, \quad p^{(0)}_{t,\zeta}\lambda + u^{(0)}_{1,\zeta\zeta} + n^2\pi^2 u_1^{(0)} = 0, \quad u^{(0)}_{1,\xi} + u^{(0)}_{2,\zeta} = 0, \tag{5.8}
$$

and subjected to the following traction free boundary conditions

$$
p_t^{(0)} - \lambda u^{(0)}_{1,\zeta} = 0, \quad \lambda p_t^{(0)} + u^{(0)}_{1,\zeta} = 0, \quad \text{at} \quad \zeta = 0, -1. \tag{5.9}
$$

We remark that specific values of ζ coordinate in (5.9) were taken to demonstrate consistency in derivation of dynamic models with boundary conditions (2.1) and analysis of dispersion relation (2.2). From the first equation of motion $(5.8)_1$ we deduce the following representation for $p_\zeta^{(0)}$

$$p_{t,\zeta}^{(0)} = \left(u_{1,\zeta\zeta}^{(0)} + n^2\pi^2 u_1^{(0)}\right)\lambda. \tag{5.10}$$

The second equation of motion $(5.8)_2$ then becomes

$$u_{1,\zeta\zeta}^{(0)} + n^2\pi^2 u_1^{(0)} = 0. \tag{5.11}$$

Therefore, we establish that at leading order the incremental pressure is independent of ζ

$$p_{t,\zeta}^{(0)} = 0, \quad p_t^{(0)} = p_t^{(0)}(\xi,\tau). \tag{5.12}$$

Hence from the equation (5.11) the in-plane displacement component can be expressed as

$$u_1^{(0)} = U_{1c}^{(0,0)}\cos(n\pi\zeta) + U_{1s}^{(0,0)}\sin(n\pi\zeta), \tag{5.13}$$

where the unknown long wave amplitudes $U_{1c}^{(0,0)}(\xi,\tau), U_{1s}^{(0,0)}(\xi,\tau)$ is independent of ζ. From the incompressibility condition $(5.8)_3$ we obtain the following representation for the normal displacement component

$$u_2^{(0)} = \frac{U_{1s,\xi}^{(0,0)}}{n\pi}\cos(n\pi\zeta) - \frac{U_{1c,\xi}^{(0,0)}}{n\pi}\sin(n\pi\zeta) + v_2^{(0,0)}, \tag{5.14}$$

here the function $v_2^{(0,0)}(\xi,\tau)$ does not depend on ζ.

The traction free boundary conditions (5.9) yield a homogeneous system of two equations in the two unknowns $p_t^{(0)}$, $u_{1,\zeta}^{(0)}$ with non-zero determinant, hence the only solution of this system is the trivial solution

$$p_t^{(0)} = 0, \quad u_{1,\zeta}^{(0)} = 0. \tag{5.15}$$

It is interesting to note that formula (5.15) indicate the symmetry in solutions of in-plane displacement component $u_1^{(0)}$.

From the boundary conditions (5.9), taking into account solution (5.15) and the form of the function $u_1^{(0)}$ given by (5.13), we obtain

$$u_{1,\zeta}^{(0)} = n\pi U_{1s}^{(0,0)}\cos(n\pi\zeta) - n\pi U_{1c}^{(0,0)}\sin(n\pi\zeta) = 0, \quad \text{at} \quad \zeta = 0, -1, \tag{5.16}$$

which yields the relation $U_{1s}^{(0,0)} = 0$.

To sum up, at leading order both displacement component are the following trigonometrical functions of ζ and the incremental pressure is equal to zero

$$u_1^{(0)} = U_{1c}^{(0,0)} \cos(n\pi\zeta), \quad u_2^{(0)} = -\frac{U_{1c,\xi}^{(0,0)}}{n\pi} \sin(n\pi\zeta) + v_2^{(0,0)}, \quad p_t^{(0)} = 0. \tag{5.17}$$

The leading order solution represents two displacement components and the incremental pressure in terms of long-wave amplitude $U_{1c}^{(0,0)}$. As a result of relations (5.15) the form of solutions (5.17) have similarities with the corresponding problem for extensional waves in case of pure homogeneous strain, see Kaplunov et al. (2002). We remark that in relations (5.17) two functions $U_{1c}^{(0,0)}$ and $v_2^{(0,0)}$ remain undefined, it is not possible to determine these without resorting to higher order problem.

5.3 Second order problem

The second order problem can be written as

$$\begin{aligned}
&\lambda n^2\pi^2 u_1^{(1)} + \lambda u_{1,\zeta\zeta}^{(1)} - p_{t,\zeta}^{(1)} = \\
&\qquad = \lambda p_{t,\xi}^{(0)} - n^2\pi^2 u_2^{(0)} + \left(2(1-\lambda^2) - p\right) u_{1,\xi\zeta}^{(0)} - (p+1)\, u_{2,\zeta\zeta}^{(0)}, \\
&\lambda^2 p_{t,\zeta}^{(1)} + \lambda n^2\pi^2 u_1^{(1)} + \lambda u_{1,\zeta\zeta}^{(1)} = \\
&\qquad = \lambda p_{t,\xi}^{(0)} + \lambda^2 n^2\pi^2 u_2^{(0)} + \left(p\lambda^2 + 2(1-\lambda^2)\right) u_{1,\xi\zeta}^{(0)} + (p+1)\,\lambda^2 u_{2,\zeta\zeta}^{(0)}, \\
&u_{1,\xi}^{(1)} + u_{2,\zeta}^{(1)} = 0.
\end{aligned} \tag{5.18}$$

The above equations (5.18) are subject to the traction free boundary conditions

$$\begin{aligned}
&\lambda\, u_{1,\zeta}^{(1)} - p_t^{(1)} = \left(1-\lambda^2\right) u_{1,\xi}^{(0)} - (p+1)\, u_{2,\zeta}^{(0)}, \quad \text{at} \quad \zeta = 0, -1, \\
&\lambda\, u_{1,\zeta}^{(1)} + \lambda^2 p_t^{(1)} = \left(1-\lambda^2\right) u_{1,\xi}^{(0)} + \lambda^2\,(p+1)\, u_{2,\zeta}^{(0)}, \quad \text{at} \quad \zeta = 0, -1.
\end{aligned} \tag{5.19}$$

Employing the leading order solution (5.17), the equation of motion $(5.18)_1$ becomes

$$\lambda n^2\pi^2 u_1^{(1)} + \lambda u_{1,\zeta\zeta}^{(1)} - p_{t,\zeta}^{(1)} = 2n\pi U_{1c,\xi}^{(0,0)} \left(\lambda^2 - 1\right) \sin(n\pi\zeta) - n^2\pi^2 v_2^{(0,0)}, \tag{5.20}$$

with the equation $(5.18)_2$ now given by

$$\lambda^2 p_{t,\zeta}^{(1)} + n^2\pi^2\lambda u_1^{(1)} + \lambda u_{1,\zeta\zeta}^{(1)} = 2n\pi U_{1c,\xi}^{(0,0)} \left(\lambda^2 - 1\right) \sin(n\pi\zeta) + \lambda^2 n^2\pi^2 v_2^{(0,0)}. \tag{5.21}$$

Combining equations (5.20) and (5.21) the following relation can be obtained

$$p_{t,\zeta}^{(1)} = n^2\pi^2 v_2^{(0,0)} \tag{5.22}$$

Hence, at second order the incremental pressure can be expressed in the form

$$p_t^{(1)} = n^2\pi^2 v_2^{(0,0)}\zeta + P_t^{(0,1)}. \tag{5.23}$$

Equations (5.20) and (5.21) may now be used to establish that

$$n^2\pi^2\lambda\, u_1^{(1)} + \lambda\, u_{1,\zeta\zeta}^{(1)} = 2n\pi\left(\lambda^2 - 1\right) U_{1c,\xi}^{(0,0)} \sin\left(n\pi\zeta\right), \tag{5.24}$$

from which the in-plane displacement component can be represented in the following form

$$u_1^{(1)} = U_{1c}^{(0,1)} \cos(n\pi\zeta) + U_{1s}^{(0,1)} \sin(n\pi\zeta) - \frac{(\lambda^2 - 1)U_{1c,\xi}^{(0,0)}}{\lambda}\zeta\cos(n\pi\zeta). \tag{5.25}$$

With the help of the incompressibility condition $(5.18)_3$ the normal displacement component can be expressed as

$$\begin{aligned} u_2^{(1)} = \frac{U_{1s,\xi}^{(0,1)}}{n\pi}\cos(n\pi\zeta) - \frac{U_{1c,\xi}^{(0,1)}}{n\pi}\sin(n\pi\zeta) \\ + \frac{(\lambda^2-1)U_{1c,\xi\xi}^{(0,0)}}{\lambda n\pi}\zeta\sin(n\pi\zeta) + \frac{(\lambda^2-1)U_{1c,\xi\xi}^{(0,0)}}{\lambda n^2\pi^2}\cos(n\pi\zeta) + v_2^{(0,1)}. \end{aligned} \tag{5.26}$$

Employing the leading order solution (5.17) the following simplifications now become possible for the boundary conditions

$$\begin{aligned} p_t^{(1)} - \lambda\, u_{1,\zeta}^{(1)} &= U_{1c,\xi}^{(0,0)} \cos\left(n\pi\zeta\right)\left(\lambda^2 - (2+p)\right), \quad \text{at} \quad \zeta = 0, -1, \\ \lambda\, u_{1,\zeta}^{(1)} + \lambda^2 p_t^{(1)} &= U_{1c,\xi}^{(0,0)} \cos\left(n\pi\zeta\right)\left(1 - \lambda^2(2+p)\right), \quad \text{at} \quad \zeta = 0, -1. \end{aligned} \tag{5.27}$$

We consider the boundary conditions at the upper surface of the layer $\zeta = 0$ in relations (5.27) and use the representation for incremental pressure (5.23) to produce the following relations

$$\begin{aligned} U_{1c,\xi}^{(0,0)}\left(2 + p - \lambda^2 + (-1)^n(\lambda^2 - 1)\right) - \lambda n\pi U_{1s}^{(0,1)} + P_t^{(0,1)} = 0, \\ U_{1c,\xi}^{(0,0)}\left(\lambda^2(2+p) - 1 - (-1)^n(\lambda^2 - 1)\right) + \lambda n\pi U_{1s}^{(0,1)} + \lambda^2 P_t^{(0,1)} = 0. \end{aligned} \tag{5.28}$$

From equations (5.28) we obtain the two following functions

$$U_{1s}^{(0,1)} = \frac{(\lambda^2 - 1)((-1)^n - 1)U_{1c,\xi}^{(0,0)}}{\lambda n\pi}, \quad P_t^{(0,1)} = -(p+1)U_{1c,\xi}^{(0,0)}. \tag{5.29}$$

Similarly, we employ the boundary conditions at the lower surface of the layer $\zeta = -1$ in equations (5.27) together with (5.23), and take into account $\cos(n\pi) = (-1)^n$ to establish the following relation

$$v_2^{(0,0)} = \frac{(p+1)((-1)^n - 1)U_{1c,\xi}^{(0,0)}}{n^2\pi^2}. \tag{5.30}$$

To conclude this subsection, at second order the two displacement components and incremental pressure can be expressed as

$$\begin{aligned}
u_1^{(1)} &= U_{1c}^{(0,1)} \cos(n\pi\zeta) + U_{1s}^{(0,1)} \sin(n\pi\zeta) + U_{1c}^{(1,1)} \zeta \cos(n\pi\zeta), \\
u_2^{(1)} &= U_{2c}^{(0,1)} \cos(n\pi\zeta) + U_{2s}^{(0,1)} \sin(n\pi\zeta) + U_{2s}^{(1,1)} \zeta \sin(n\pi\zeta) + v_2^{(0,1)}, \\
p_t^{(1)} &= P_t^{(0,1)} + P_t^{(1,1)} \zeta,
\end{aligned} \tag{5.31}$$

where non-zero coefficients are given by

$$\begin{aligned}
U_{1c}^{(1,1)} &= \frac{(1-\lambda^2)U_{1c,\xi}^{(0,0)}}{\lambda}, \quad U_{1s}^{(0,1)} = \frac{(\lambda^2-1)((-1)^n - 1)U_{1c,\xi}^{(0,0)}}{\lambda n\pi}, \\
U_{2s}^{(0,1)} &= -\frac{U_{1c,\xi}^{(0,1)}}{n\pi}, \quad U_{2c}^{(0,1)} = \frac{(\lambda^2-1)U_{1c,\xi\xi}^{(0,0)}}{\lambda n^2\pi^2}, \quad U_{2s}^{(1,1)} = \frac{(\lambda^2-1)U_{1c,\xi\xi}^{(0,0)}}{\lambda n\pi}, \\
P_t^{(0,1)} &= -(p+1)U_{1c,\xi}^{(0,0)}, \quad P_t^{(1,1)} = (p+1)((-1)^n - 1)U_{1c,\xi}^{(0,0)}.
\end{aligned} \tag{5.32}$$

In addition, the second order problem yields the following connection between the two leading order functions

$$v_2^{(0,0)} = \frac{(p+1)((-1)^n - 1)U_{1c,\xi}^{(0,0)}}{n^2\pi^2}. \tag{5.33}$$

We note that the three following functions $U_{1c}^{(0,1)}$, $U_{1c}^{(0,0)}$, $v_2^{(0,1)}$ remain unknown within the second order solution (5.31), to determine these one have to resort to higher order problems. We also remark that the governing equation for long wave amplitude $U_{1c}^{(0,0)}$ cannot be derived at the stage of the second order problem. This is the novel aspect of the asymptotic model in comparison to previously derived models by Kaplunov et al. (2002) and Prikazchikova (2004).

5.4 Third order problem

The third order problem is given by

$$\begin{aligned}
&\lambda^2 W_1^{(0)} + \lambda p_{t,\zeta}^{(2)} - \lambda^2 u_{1,\zeta\zeta}^{(2)} - \lambda^2 n^2\pi^2 u_1^{(2)} = \lambda\, \left(2(\lambda^2-1)+p\right) u_{1,\xi\zeta}^{(1)} + \lambda\, (1+p)\, u_{2,\zeta\zeta}^{(1)} \\
&+ \left(\lambda^2(p-1)+1+\lambda^4\right) u_{1,\xi\xi}^{(0)} + \left(\lambda^2(p+2)-2\right) u_{2,\xi\zeta}^{(0)} - \lambda^2 p_{t,\xi}^{(1)} + \lambda n^2\pi^2 u_2^{(1)}, \\
&\lambda^2 W_1^{(0)} - \lambda^3 p_{t,\zeta}^{(2)} - \lambda^2 u_{1,\zeta\zeta}^{(2)} - \lambda^2 n^2\pi^2 u_1^{(2)} = \lambda\, \left(\lambda^2(2-p)-2\right) u_{1,\xi\zeta}^{(1)} - \lambda^3\,(1+p)\, u_{2,\zeta\zeta}^{(1)} \\
&+ \left(\lambda^2(p-1)+1+\lambda^4\right) u_{1,\xi\xi}^{(0)} + \left(2(1-\lambda^2)+p\right) \lambda^2 u_{2,\xi\zeta}^{(0)} - \lambda^2 p_{t,\xi}^{(1)} - \lambda^3 n^2\pi^2 u_2^{(1)}, \\
&u_{1,\xi}^{(2)} + u_{2,\zeta}^{(2)} = 0.
\end{aligned} \tag{5.34}$$

Equations (5.34) must be solved subject to the appropriate third order traction free boundary conditions

$$\begin{aligned}
&\lambda^2 u_{1,\zeta}^{(2)} - \lambda p_t^{(2)} = -\lambda\,(p+1)\, u_{2,\zeta}^{(1)} + \left(1-(p+1)\lambda^2\right) u_{2,\xi}^{(0)} + \lambda\left(1-\lambda^2\right) u_{1,\xi}^{(1)}, \quad \text{at} \quad \zeta = 0, -1, \\
&\lambda u_{1,\zeta}^{(2)} + \lambda^2 p_t^{(2)} = \lambda^2\,(p+1)\, u_{2,\zeta}^{(1)} - \lambda\left(1+p-\lambda^2\right) u_{2,\xi}^{(0)} + \left(1-\lambda^2\right) u_{1,\xi}^{(1)}, \quad \text{at} \quad \zeta = 0, -1.
\end{aligned} \tag{5.35}$$

From equations (5.34) the following relation can be derived

$$\lambda p_{t,\zeta}^{(2)} = 2\left(\lambda^2-1\right) u_{2,\xi\zeta}^{(0)} + \lambda\,(p+1)\, u_{2,\zeta\zeta}^{(1)} + \lambda p u_{1,\xi\zeta}^{(1)} + \lambda n^2\pi^2 u_2^{(1)}. \tag{5.36}$$

Inserting solutions (5.17) and (5.31) into representation (5.36) at third order the incremental pressure can be expressed as

$$p_t^{(2)} = n^2\pi^2 v_2^{(0,1)}\zeta - pU_{1c,\xi}^{(0,1)}\cos(n\pi\zeta) + P_t^{(2,0)}. \tag{5.37}$$

We note that the solutions of the leading order (5.17) and second order (5.31) problem were obtained without any assumptions about its forms. It is quite possible to integrate the system of equations (5.34) subject to boundary conditions (5.35) in a similar manner to that employed at leading and second order problems. However, to facilitate the analysis we note that at third order system of equations (5.34) subject to boundary conditions (5.35) has the following general solution

$$\begin{aligned}
u_1^{(2)} &= U_{1c}^{(0,2)}\cos(\Omega\zeta) + U_{1s}^{(0,2)}\sin(\Omega\zeta) + U_{1c}^{(1,2)}\zeta\cos(\Omega\zeta) + U_{1s}^{(1,2)}\zeta\sin(\Omega\zeta) \\
&\quad + U_{1c}^{(2,2)}\zeta^2\cos(\Omega\zeta) + U_{1s}^{(2,2)}\zeta^2\sin(\Omega\zeta) + v_1^{(0,2)} + v_1^{(1,2)}\zeta + v_1^{(2,2)}\zeta^2, \\
u_2^{(2)} &= U_{2c}^{(0,2)}\cos(\Omega\zeta) + U_{2s}^{(0,2)}\sin(\Omega\zeta) + U_{2c}^{(1,2)}\zeta\cos(\Omega\zeta) + U_{2s}^{(1,2)}\zeta\sin(\Omega\zeta) \\
&\quad + U_{2c}^{(2,2)}\zeta^2\cos(\Omega\zeta) + U_{2s}^{(2,2)}\zeta^2\sin(\Omega\zeta) + v_2^{(0,2)} + v_2^{(1,2)}\zeta + v_2^{(2,2)}\zeta^2,
\end{aligned} \tag{5.38}$$

where in this chapter we assume that $\Omega = n\pi$. Employing the forms of displacement components (5.38) and incremental pressure (5.37) each of the three equations (5.34) can be written in a similar general form, given by

$$(A^{(1)}\zeta^2 + B^{(1)}\zeta + C^{(1)})\sin(\Omega\zeta) + (D^{(1)}\zeta^2 + E^{(1)}\zeta + F^{(1)})\cos(\Omega\zeta) + \\ + K^{(1)}\zeta^2 + L^{(1)}\zeta + M^{(1)} = 0, \tag{5.39}$$

providing for each particular equation the following system

$$A^{(1)} = 0, \quad B^{(1)} = 0, \quad C^{(1)} = 0, \quad D^{(1)} = 0, \quad E^{(1)}, \quad F^{(1)} = 0, \\ K^{(1)} = 0, \quad L^{(1)} = 0, \quad M^{(1)} = 0. \tag{5.40}$$

At this point we remark that within the third order problem there are twenty three unknowns: nineteen unknown functions in third order solutions (5.38) with (5.37), two second order functions $U_{1c}^{(0,1)}, v_2^{(0,1)}$ and two leading order functions $W_1^{(0)}, U_{1c}^{(0,0)}$. We now insert the general representations for displacement components (5.38) with the incremental pressure (5.37) into the equation of motion $(5.34)_1$ and then represent it in a form (5.39). Then we obtain a system of equations to determine unknown functions in (5.38). The equations connected to the term $\sin(n\pi\zeta)$ are the following

$$\left(\lambda^2 - 1\right)^2 U_{1c,\xi\xi}^{(0,0)} - 2\,\lambda^2 U_{1c}^{(2,2)} = 0, \tag{5.41}$$

$$2\lambda n\pi U_{1c}^{(1,2)} - 2\lambda U_{1s}^{(2,2)} + p\,(1 - pn\pi)\,U_{1c,\xi}^{(0,1)} = 0, \tag{5.42}$$

equations coming from $\cos(n\pi\zeta)$ are given by

$$U_{1s}^{(2,2)} = 0, \tag{5.43}$$

$$\left(\lambda^4 - 3\,\lambda^2 + 1\right) U_{1c,\xi\xi}^{(0,0)} + \lambda^2 W_1^{(0)} - 2\,\lambda^2 U_{1s}^{(1,2)} n\pi - 2\,\lambda^2 U_{1c}^{(2,2)} = 0, \tag{5.44}$$

with the remaining relations arising from equation (5.34) given by

$$v_1^{(2,2)} = 0, \tag{5.45}$$

$$\lambda^2 n^2 \pi^2 v_1^{(1,2)} - \lambda^2 U_{1c,\xi\xi}^{(0,0)}\,(p+1)\left((-1)^n - 1\right) = 0, \tag{5.46}$$

$$n^2\pi^2 v_1^{(0,2)} + (p+1)\,U_{1c,\xi\xi}^{(0,0)} + 2v_1^{(2,2)} = 0. \tag{5.47}$$

Then we apply a similar procedure to the equation of motion $(5.34)_2$, from which with the help of representation (5.39), and the consequence (5.40), we deduce two additional equations to determine unknown functions in (5.38). The following equation is connected to the term $\sin(n\pi\zeta)$

$$\lambda p\,(n\pi - 1)\,U_{1c,\xi}^{(0,1)} - 2U_{1s}^{(2,2)} + 2U_{1c}^{(1,2)} n\pi = 0, \tag{5.48}$$

with the term $\cos(n\pi\zeta)$ providing

$$4\lambda^2 n\pi U_{1s}^{(2,2)} = 0. \tag{5.49}$$

Finally the equivalent method applied to incompressibility condition $(5.34)_3$ yields additional equations to determine other unknown functions in (5.38). The equations connected to the term $\sin(n\pi\zeta)$ are the following

$$U_{1s,\xi}^{(2,2)} - n\pi U_{2c}^{(2,2)} = 0, \tag{5.50}$$

$$2U_{2s}^{(2,2)} + U_{1s,\xi}^{(1,2)} - n\pi U_{2c}^{(1,2)} = 0, \tag{5.51}$$

$$U_{1s,\xi}^{(0,2)} - n\pi U_{2c}^{(0,2)} + U_{2s}^{(1,2)} = 0, \tag{5.52}$$

with the term $\cos(n\pi\zeta)$ yielding

$$U_{1c,\xi}^{(2,2)} + n\pi U_{2s}^{(2,2)} = 0, \tag{5.53}$$

$$n\pi U_{2s}^{(1,2)} + 2U_{2c}^{(2,2)} + U_{1c,\xi}^{(1,2)} = 0, \tag{5.54}$$

$$U_{1c,\xi}^{(0,2)} + n\pi U_{2s}^{(0,2)} + U_{2c}^{(1,2)} = 0, \tag{5.55}$$

and the remaining equations arising from $(5.34)_3$ can be expressed as

$$2v_2^{(2,2)} + v_{1,\xi}^{(1,2)} = 0, \tag{5.56}$$

$$v_{1,\xi}^{(0,2)} + v_2^{(1,2)} = 0. \tag{5.57}$$

Then the traction free boundary conditions at the upper and lower surfaces of the layer are considered. We substitute the general solution for displacement components (5.38), taking into account the form for incremental pressure (5.37), into the in-plane traction component $(5.35)_1$ at the upper surface of the layer $\zeta = 0$, it takes the following form

$$\begin{aligned}(p+1)\left((1+p)\lambda^2 - 1\right)\left((-1)^n - 1\right)U_{1c,\xi\xi}^{(0,0)} + \left(\lambda^2 - 2\right)\lambda n^2\pi^2 U_{1c,\xi}^{(0,1)} +& \\ + \lambda^2 n^3\pi^3 U_{1s}^{(0,2)} + \lambda^2 n^2\pi^2 U_{1c}^{(1,2)} - \lambda n^2\pi^2 P_t^{(0,2)} + \lambda^2 n^2\pi^2 v_1^{(1,2)} &= 0.\end{aligned} \tag{5.58}$$

Similarly, the in-plane traction component $(5.35)_1$ at the lower surface of the layer $\zeta = -1$, taking into account solutions (5.38), (5.37), can be expressed as

$$\begin{aligned}&C_1^{(t)} U_{1c,\xi\xi}^{(0,0)} + \lambda n^4\pi^4 v_2^{(0,1)} + \lambda^2 U_{1s}^{(2,2)} n^3\pi^3 (-1)^n \\ &- 2(-1)^n \lambda^2 U_{1c}^{(2,2)} n^2\pi^2 - 2\lambda^2 n^2\pi^2 v_1^{(2,2)} + (-1)^n\left(\lambda^2 - 2\right)\lambda U_{1c,\xi}^{(0,1)} n^2\pi^2 - \lambda P_t^{(0,2)} n^2\pi^2 \\ &+ \lambda^2 n^2\pi^2 v_1^{(1,2)} + \lambda^2 n^3\pi^3 (-1)^n U_{1s}^{(0,2)} + \lambda^2 (-1)^n U_{1c}^{(1,2)} n^2\pi^2 - \lambda^2 n^3\pi^3 (-1)^n U_{1s}^{(1,2)} = 0, \\ &C_1^{(t)} = \left((-1)^n - 1\right)\lambda^2 p^2 + \left(n^2\pi^2 (-1)^n \left(-\lambda^2 + 1\right) + \left((-1)^n - 1\right)\left(2\lambda^2 - 1\right)\right)p \\ &\qquad + \left(n^2\pi^2\left(\lambda^4 - 3\lambda^2 + 2\right) - \lambda^2 - 1\right)(-1)^n - \lambda^2.\end{aligned} \tag{5.59}$$

The boundary condition for the normal traction component $(5.35)_2$ the upper surface of the layer $\zeta = 0$ taking into account solutions (5.38), (5.37) is given by

$$\left(1-(-1)^n\right)\left(p^2+\left(2-\lambda^2\right)p+1-\lambda^2\right)\lambda U_{1c,\xi\xi}^{(0,0)}+n^2\pi^2\left(1-2\,\lambda^2\right)U_{1c,\xi}^{(0,1)}- \\ -\lambda n^3\pi^3U_{1s}^{(0,2)}-\lambda n^2\pi^2U_{1c}^{(1,2)}-\lambda^2n^2\pi^2P_t^{(0,2)}-\lambda n^2\pi^2v_1^{(1,2)}=0. \tag{5.60}$$

Finally, we insert solutions (5.38), (5.37) to the normal traction component $(5.35)_2$ at the lower surface of the layer $\zeta = -1$ to obtain the following equation

$$\begin{aligned} &C_2^{(t)}U_{1c,\xi\xi}^{(0,0)}-\lambda^2\left(-1\right)^nU_{1c}^{(1,2)}n^2\pi^2+\lambda^2n^3\pi^3\left(-1\right)^nU_{1s,\xi}^{(1,2)} \\ &+2\lambda^2\left(-1\right)^nU_{1c,\xi}^{(2,2)}n^2\pi^2-\lambda^3P_t^{(0,2)}n^2\pi^2-\lambda^2n^3\pi^3\left(-1\right)^nU_{1s}^{(0,2)}+2\lambda^2n^2\pi^2v_1^{(2,2)} \\ &-\lambda^2n^2\pi^2v_1^{(1,2)}+\left(-1\right)^n\left(1-2\lambda^2\right)\lambda n^2\pi^2U_{1c,\xi}^{(0,1)}-\lambda^2U_{1s}^{(2,2)}n^3\pi^3\left(-1\right)^n+\lambda^3n^4\pi^4v_2^{(0,1)}=0, \\ &C_2^{(t)}=\lambda^2\left(1-(-1)^n\right)p^2-\lambda^2\left(\left(-1\right)^nn^2\pi^2\left(\lambda^2-1\right)+\left(\left(-1\right)^n-1\right)\left(2-\lambda^2\right)\right)p \\ &\qquad+n^2\pi^2\left(-1\right)^n\left(3\,\lambda^2-2\,\lambda^4-1\right)-\lambda^2\left(\left(-1\right)^n-1\right)-\lambda^4. \end{aligned} \tag{5.61}$$

We can solve the homogeneous system of equations (5.41)–(5.61) with the help of Maple (1996) to determine unknown functions in representations (5.38) and (5.37). To sum up, at third order the displacement components and incremental pressure can be expressed in the following form

$$\begin{aligned} u_1^{(2)}&=U_{1c}^{(0,2)}\cos(n\pi\zeta)+U_{1s}^{(0,2)}\sin(n\pi\zeta) \\ &\quad+U_{1s}^{(1,2)}\zeta\sin(n\pi\zeta)+U_{1c}^{(2,2)}\zeta^2\cos(n\pi\zeta)+v_1^{(0,2)}+v_1^{(1,2)}\zeta, \\ u_2^{(2)}&=U_{2c}^{(0,2)}\cos(n\pi\zeta)+U_{2s}^{(0,2)}\sin(n\pi\zeta) \\ &\quad+U_{2c}^{(1,2)}\zeta\cos(n\pi\zeta)+U_{2s}^{(2,2)}\zeta^2\sin(n\pi\zeta)+v_2^{(0,2)}+v_2^{(1,2)}\zeta+v_2^{(2,2)}\zeta^2, \\ p_t^{(2)}&=P_t^{(1,2)}\zeta+P_t^{(0,2)}, \end{aligned} \tag{5.62}$$

where

$$U_{1s}^{(0,2)} = \frac{U_{1c,\xi\xi}^{(0,0)}\,(p+1)^2\,(1-(-1)^n)}{n^3\pi^3}, \quad U_{1s}^{(1,2)} = \frac{2U_{1c,\xi\xi}^{(0,0)}\,(1-(-1)^n)\,(p+1)^2}{n^3\pi^3},$$
$$U_{1c}^{(2,2)} = \frac{U_{1c,\xi\xi}^{(0,0)}\left(\lambda^4-1\right)}{2\lambda^2}, \quad v_1^{(0,2)} = -\frac{U_{1c,\xi\xi}^{(0,0)}\,(p+1)}{n^2\pi^2},$$
$$v_1^{(1,2)} = \frac{U_{1c,\xi\xi}^{(0,0)}\,(p+1)\,((-1)^n-1)}{n^2\pi^2}, \quad U_{2c}^{(0,2)} = \frac{U_{1c,\xi\xi\xi}^{(0,0)}\,(p+1)^2\,(1-(-1)^n)}{n^4\pi^4},$$
$$U_{2s}^{(0,2)} = \frac{U_{1c,\xi\xi\xi}^{(0,0)}\left(2\,\lambda^2\,(p+1)^2\,((-1)^n-1)+\left(\lambda^2-1\right)^2 n^2\pi^2\right)}{\lambda^2 n^5\pi^5} - \frac{U_{1c,\xi}^{(0,2)}}{n\pi},$$
$$U_{2c}^{(1,2)} = \frac{U_{1c,\xi\xi\xi}^{(0,0)}\left(2\,\lambda^2\,(p+1)^2\,(1-(-1)^n)-\left(\lambda^2-1\right)^2 n^2\pi^2\right)}{\lambda^2 n^4\pi^4},$$
$$U_{2s}^{(2,2)} = \frac{U_{1c,\xi\xi\xi}^{(0,0)}\left(1-\lambda^4\right)}{2\lambda^2 n\pi}, \quad v_2^{(1,2)} = \frac{U_{1c,\xi\xi\xi}^{(0,0)}\,(p+1)}{n^2\pi^2}, \quad v_2^{(2,2)} = \frac{U_{1c,\xi\xi\xi}^{(0,0)}\,(p+1)\,(1-(-1)^n)}{2n^2\pi^2},$$
$$P_t^{(0,2)} = \frac{U_{1c,\xi\xi}^{(0,0)}((-1)^n-1)(p+1)\left(\lambda^2-1\right)}{\lambda n^2\pi^2}, \quad P_t^{(1,2)} = \frac{U_{1c,\xi\xi}^{(0,0)}\,(-1)^n\left(\lambda^2-1\right)(p+1)}{\lambda}. \tag{5.63}$$

We remark that the functions $U_{1c}^{(0,2)}$, $U_{1c}^{(0,1)}(\tau)$ and $v_2^{(0,2)}$ remains undefined in the solution of third order problem (5.62). However at third order we determine the following second order functions

$$v_2^{(0,1)} = \frac{U_{1c,\xi\xi}^{(0,0)}\,(-1)^n\left(\lambda^2-1\right)(p+1)}{\lambda n^2\pi^2}, \quad U_{1c,\xi}^{(0,1)} = 0. \tag{5.64}$$

Therefore, taking into account (5.64) the updated solution for the second order displacement components (5.31) with the coefficients (5.32) can be expressed as

$$\begin{aligned} u_1^{(1)} &= U_{1c}^{(0,1)}\cos(n\pi\zeta) + U_{1c}^{(1,1)}\zeta\cos(n\pi\zeta), \\ u_2^{(1)} &= U_{2c}^{(0,1)}\cos(n\pi\zeta) + U_{2s}^{(1,1)}\zeta\sin(n\pi\zeta) + v_2^{(0,1)}, \end{aligned} \tag{5.65}$$

$$U_{1c}^{(0,1)} = U_{1c}^{(0,1)}(\tau), \quad U_{1c,\xi}^{(0,1)} = 0, \quad U_{1c}^{(1,1)} = \frac{(1-\lambda^2)U_{1c,\xi}^{(0,0)}}{\lambda},$$
$$U_{2c}^{(0,1)} = \frac{(\lambda^2-1)U_{1c,\xi\xi}^{(0,0)}}{\lambda n^2\pi^2}, \quad U_{2s}^{(1,1)} = \frac{(\lambda^2-1)U_{1c,\xi\xi}^{(0,0)}}{\lambda n\pi}, \quad v_2^{(0,1)} = \frac{U_{1c,\xi\xi}^{(0,0)}\,(-1)^n\left(\lambda^2-1\right)(p+1)}{\lambda n^2\pi^2}, \tag{5.66}$$

At third order we are able to derive a governing equation for the long wave amplitude $U_{1c}^{(0,0)}$ which is used to represent the leading order displacement components (5.17). We employ the following representation for the function $W_1^{(0)}$

$$W_1^{(0)} = \eta^{-2}(u_{1,\tau\tau}^{(0)} + n^2\pi^2 u_1^{(0)}), \tag{5.67}$$

in order to establish the governing equation for the long wave amplitude $U_{1c}^{(0,0)}$ in the following form

$$\eta^{-2}\left(U_{1c,\tau\tau}^{(0,0)} + n^2\pi^2 U_{1c}^{(0,0)}\right) - \left(\frac{4\,(p+1)^2\,(1-(-1)^n) + n^2\pi^2}{n^2\pi^2}\right) U_{1c,\xi\xi}^{(0,0)} = 0. \qquad (5.68)$$

Once the long wave amplitude $U_{1c}^{(0,0)}$ is determined from equation (5.68) the leading order displacement components are obtained. In addition, equation (5.68) enables us to demonstrate asymptotic consistency of the derived model and obtain the second order asymptotic expansion for non-dimensional frequency (2.37). For that purpose we insert non-dimensional squared frequency

$$\Omega^2 = \frac{\rho k^2 h^2 v^2}{\mu}, \qquad (5.69)$$

obtained from characteristic equation (1.77) into the governing equation (5.68) represented in non-scaled variables t and x_1 as the following

$$\frac{\rho h^2}{\mu}\frac{\partial^2 U_{1c}^{(0,0)}}{\partial^2 t^2} + n^2\pi^2 U_{1c}^{(0,0)} - h^2\left(\frac{4\,(p+1)^2\,(1-(-1)^n) + n^2\pi^2}{n^2\pi^2}\right)\frac{\partial^2 U_{1c}^{(0,0)}}{\partial^2 x_1^2} = 0. \qquad (5.70)$$

We remark here that the governing equation (5.70) is of hyperbolic type due to the following quantity being positive for all n

$$\mathcal{A}_e = \frac{4\,(p+1)^2\,(1-(-1)^n) + n^2\pi^2}{n^2\pi^2} > 0, \qquad (5.71)$$

If we specify $U_{1c}^{(0,0)}$ in the form (1.72) and substitute it into the governing equation (5.70) we obtain the following second order asymptotic expansion for squared non-dimensional frequency

$$\Omega^2 = n^2\pi^2 + \mathcal{A}_e\eta^2 + O(\eta^4). \qquad (5.72)$$

The representation (2.37) can be expressed as

$$\Omega^2 = n^2\pi^2 + 2n\pi\Omega^{(2)}\eta^2 + O(\eta^4), \qquad (5.73)$$

where $\Omega^{(2)}$ is given by (2.36). The fact that the expansions (5.72) and (5.73) coincide proves the consistency of derived one-dimensional model with asymptotic analysis of dispersion relation (2.5).

Chapter 6

Asymptotically consistent models for a long wave high frequency motion in a layer subject to some non-classical boundary conditions

In this chapter we derive one-dimensional asymptotic models for two-dimensional long wave high frequency motion in a sheared pre-stressed incompressible elastic layer with some non-classical boundary conditions. To facilitate this analysis, and illustrate the main features of dynamic models, we consider motion in a layer composed of the neo-Hookean material. Direct asymptotic integration in the vicinity of the cut-off frequencies is employed to construct the models. Appropriate asymptotically approximate equations are derived and integrated through a systematic perturbation process considering three orders of the problem. In the asymptotic model for motion in the layer with fixed faces, the cut-off frequencies are given by transcendental equation and the essential parameter is the incremental pressure. In contrast, in all the other models considered in this thesis the essential parameter is the long wave amplitude. We remark that the governing equations in all considered models are obtained at the stage of the third order problem. Finally, the governing equations are employed to demonstrate the asymptotic consistency of the derived dynamic models.

6.1 Asymptotically approximate equations for the layer with fixed faces: first family of cut-off frequencies

6.1.1 Asymptotic scaling and dimensionless equations

Throughout this chapter, we consider a layer composed of neo-Hookean material to illustrate the main principles in derivation of asymptotically approximate equations. The results are simplified one-dimensional models for long wave high frequency motion in a sheared pre-stressed incompressible elastic layer with non-classical boundary conditions.

In this section we focus attention on a layer with fixed faces. In order to derive a dynamic model we perform asymptotic integration in the vicinity of the cut-off frequencies (3.24). We remark that the associated relative asymptotic orders of displacements and incremental pressure are given by (3.50) and hence equivalent to relations (2.58) in respect to free faces problem. Therefore, the non-dimensional equations of motion and associated incompressibility condition are given by (5.5) and (5.6), respectively, taking into account the cut-off frequencies $\Omega = 2n\pi$. These are subjected to the fixed faces boundary conditions

$$u_1^* = 0, \quad u_2^* = 0, \quad \text{at} \quad \zeta = 0, -1. \tag{6.1}$$

The solutions of equations (5.5)–(5.6), subject to boundary conditions (6.1), are sought in the form of the series (4.11).

6.1.2 Leading order problem

At leading order, the non-dimensional equations of motion and associated incompressibility condition are given by

$$p_{t,\zeta}^{(0)} - \lambda\, u_{1,\zeta\zeta}^{(0)} - \lambda\, 4n^2\pi^2 u_1^{(0)} = 0, \quad p_{t,\zeta}^{(0)}\lambda + u_{1,\zeta\zeta}^{(0)} + 4n^2\pi^2 u_1^{(0)} = 0, \quad u_{1,\xi}^{(0)} + u_{2,\zeta}^{(0)} = 0, \tag{6.2}$$

subject to the following boundary conditions

$$u_1^{(0)} = 0, \quad u_2^{(0)} = 0, \quad \text{at} \quad \zeta = 0, -1. \tag{6.3}$$

We remark that specific values of ζ coordinate in (6.3) were taken to demonstrate consistency in derivation of dynamic models with boundary conditions (3.1) and analysis of dispersion relation (3.2). From the equation $(6.2)_1$ we obtain

$$p_{t,\zeta}^{(0)} = \left(u_{1,\zeta\zeta}^{(0)} + 4n^2\pi^2 u_1^{(0)}\right)\lambda, \tag{6.4}$$

with the equation $(6.2)_2$ immediately providing

$$u^{(0)}_{1,\zeta\zeta} + 4n^2\pi^2 u^{(0)}_1 = 0. \tag{6.5}$$

Hence, at leading order, the incremental pressure does not depend on ζ, thus

$$p^{(0)}_{t,\zeta} = 0, \quad p^{(0)}_t = p^{(0)}_t(\xi,\tau), \tag{6.6}$$

The solution of equation $(6.2)_2$ is of the form

$$u^{(0)}_1 = U^{(0,0)}_{1c}\cos(2n\pi\zeta) + U^{(0,0)}_{1s}\sin(2n\pi\zeta), \tag{6.7}$$

with incompressibility condition $(6.2)_3$ yielding

$$u^{(0)}_2 = \frac{U^{(0,0)}_{1s,\xi}}{2n\pi}\cos(2n\pi\zeta) - \frac{U^{(0,0)}_{1c,\xi}}{2n\pi}\sin(2n\pi\zeta) + v^{(0,0)}_2. \tag{6.8}$$

From the boundary conditions (6.3) we now obtain the following relations

$$U^{(0,0)}_{1c} = 0, \quad v^{(0,0)}_2 = -\frac{U^{(0,0)}_{1s,\xi}}{2n\pi}, \tag{6.9}$$

hence at leading order the incremental pressure $p^{(0)}_t$ remains undefined. To sum up, at leading order the displacement components are given by

$$u^{(0)}_1 = U^{(0,0)}_{1s}\sin(2n\pi\zeta), \quad u^{(0)}_2 = \frac{U^{(0,0)}_{1s,\xi}}{2n\pi}\cos(2n\pi\zeta) - \frac{U^{(0,0)}_{1s,\xi}}{2n\pi}. \tag{6.10}$$

The form of leading order solution (6.10) have similarities with the corresponding problem for flexural waves in case of pure homogeneous strain, see Nolde and Rogerson (2002).

6.1.3 Second order problem

At second order, the non-dimensional equations of motion and associated incompressibility condition are given by

$$\begin{aligned}
&p^{(1)}_{t,\zeta} - \lambda 4n^2\pi^2 u^{(1)}_1 - \lambda u^{(1)}_{1,\zeta\zeta} + \lambda p^{(0)}_{t,\xi} = \\
&\qquad = 4n^2\pi^2 u^{(0)}_2 + \left(p + 2(\lambda^2-1)\right) u^{(0)}_{1,\xi\zeta} + (p+1)\, u^{(0)}_{2,\zeta\zeta},\\
&\lambda^2 p^{(1)}_{t,\zeta} + \lambda 4n^2\pi^2 u^{(1)}_1 + \lambda u^{(1)}_{1,\zeta\zeta} - \lambda p^{(0)}_{t,\xi} = \\
&\qquad = \lambda^2 4n^2\pi^2 u^{(0)}_2 + \left(p\lambda^2 + 2(1-\lambda^2)\right) u^{(0)}_{1,\xi\zeta} + (p+1)\,\lambda^2 u^{(0)}_{2,\zeta\zeta},\\
&u^{(1)}_{1,\xi} + u^{(1)}_{2,\zeta} = 0,
\end{aligned} \tag{6.11}$$

subject to the boundary conditions

$$u_1^{(1)} = 0 \quad u_2^{(1)} = 0, \quad \text{at} \quad \zeta = 0, -1. \tag{6.12}$$

Taking into account the leading order solution (6.10), the equations of motion (6.11) can be expressed as

$$\begin{aligned}
&\lambda 4n^2\pi^2 u_1^{(1)} + \lambda u_{1,\zeta\zeta}^{(1)} - \lambda p_{t,\xi}^{(0)} - p_{t,\zeta}^{(1)} = 4\left(1-\lambda^2\right) k\pi U_{1s,\xi}^{(0,0)} \cos\left(2n\pi\zeta\right) + 2n\pi U_{1s,\xi}^{(0,0)},\\
&4n^2\pi^2\lambda u_1^{(1)} + \lambda u_{1,\zeta\zeta}^{(1)} - \lambda p_{t,\xi}^{(0)} + \lambda^2 p_{t,\zeta}^{(1)} = 4\left(1-\lambda^2\right) n\pi U_{1s,\xi}^{(0,0)} \cos\left(n\pi\zeta\right) - 2\lambda^2 n\pi U_{1s,\xi}^{(0,0)}.
\end{aligned} \tag{6.13}$$

From equations (6.13) we obtain the following relation

$$p_t^{(1)} = -\ 2n\pi U_{1s,\xi}^{(0,0)}\zeta + P_t^{(0,1)}. \tag{6.14}$$

A combination of equations $(6.13)_1$ and $(6.13)_2$ allows us to establish that

$$4\lambda n^2\pi^2 u_1^{(1)} + \lambda\, u_{1,\zeta\zeta}^{(1)} - \lambda p_{t,\xi}^{(0)} = 4\left(1-\lambda^2\right) U_{1s,\xi}^{(0,0)} \cos\left(2n\pi\,\zeta\right) k\pi. \tag{6.15}$$

Equation (6.15) enables us to obtain the following form for the in-plane displacement component and leading order incremental pressure

$$\begin{aligned}
&u_1^{(1)} = U_{1c}^{(0,1)} \cos(2n\pi\zeta) + U_{1s}^{(0,1)} \sin(2n\pi\zeta) - \frac{(\lambda^2-1)U_{1s,\xi}^{(0,0)}}{\lambda}\zeta \sin(2n\pi\zeta) + v_1^{(0,1)},\\
&p_{t,\xi}^{(0)} = 4n^2\pi^2 v_1^{(0,1)}.
\end{aligned} \tag{6.16}$$

The incompressibility condition $(6.13)_3$ provides the following representation for the normal displacement component

$$\begin{aligned}
u_2^{(1)} = {}& \frac{U_{1s,\xi}^{(0,1)}}{2n\pi} \cos(2n\pi\zeta) - \frac{U_{1c,\xi}^{(0,1)}}{2n\pi} \sin(2n\pi\zeta)\\
&- \frac{(\lambda^2-1)U_{1s,\xi\xi}^{(0,0)}}{2\lambda n\pi}\zeta \cos(2n\pi\zeta) + \frac{(\lambda^2-1)U_{1s,\xi\xi}^{(0,0)}}{2\lambda n\pi} \sin(2n\pi\zeta) + v_2^{(1,1)}\zeta + v_2^{(0,1)}.
\end{aligned} \tag{6.17}$$

Finally, the zero displacements boundary conditions (6.12) are employed to derive the following relations

$$\begin{aligned}
&p_t^{(0)} = \frac{(1-\lambda^2)2n\pi U_{1s}^{(0,0)}}{\lambda}, \quad v_1^{(0,1)} = \frac{P_{t,\xi}^{(0,0)}}{4n^2\pi^2} = \frac{(1-\lambda^2)U_{1s,\xi}^{(0,0)}}{2\lambda n\pi}, \quad v_2^{(0,1)} = -\frac{U_{1s,\xi}^{(0,1)}}{2n\pi},\\
&U_{1c}^{(0,1)} = -\frac{P_{t,\xi}^{(0,0)}}{4n^2\pi^2} = \frac{(\lambda^2-1)U_{1s,\xi}^{(0,0)}}{2\lambda n\pi}, \quad v_2^{(1,1)} = -\frac{P_{t,\xi\xi}^{(0,0)}}{4n^2\pi^2} = \frac{(\lambda^2-1)U_{1s,\xi\xi}^{(0,0)}}{2\lambda n\pi}.
\end{aligned} \tag{6.18}$$

To conclude, the solution of the second order problem is given by

$$
\begin{aligned}
&u_1^{(1)} = U_{1c}^{(0,1)} \cos(2n\pi\zeta) + U_{1s}^{(0,1)} \sin(2n\pi\zeta) + U_{1s}^{(1,1)} \zeta \sin(2n\pi\zeta) + v_1^{(0,1)},\\
&u_2^{(1)} = U_{2c}^{(0,1)} \cos(2n\pi\zeta) + U_{2c}^{(1,1)} \zeta \cos(2n\pi\zeta) + v_2^{(1,1)} \zeta + v_2^{(0,1)},\\
&p_t^{(1)} = P_t^{(0,1)} + P_t^{(1,1)} \zeta,
\end{aligned} \tag{6.19}
$$

where the non-zero coefficients are given by

$$
\begin{aligned}
&U_{1c}^{(0,1)} = \frac{U_{1s,\xi}^{(0,0)}(\lambda^2-1)}{2\lambda n\pi}, \quad U_{1s}^{(1,1)} = \frac{U_{1s,\xi}^{(0,0)}(1-\lambda^2)}{\lambda}, \quad v_1^{(0,1)} = \frac{U_{1s,\xi}^{(0,0)}(1-\lambda^2)}{\lambda 2n\pi},\\
&U_{2c}^{(0,1)} = \frac{U_{1s,\xi}^{(0,1)}}{2n\pi}, \quad U_{2c}^{(1,1)} = \frac{U_{1s,\xi\xi}^{(0,0)}(1-\lambda^2)}{2\lambda n\pi}, \quad v_2^{(1,1)} = \frac{U_{1s,\xi\xi}^{(0,0)}(\lambda^2-1)}{2\lambda n\pi},\\
&v_2^{(0,1)} = -\frac{U_{1s,\xi}^{(0,1)}}{2n\pi}, \quad P_t^{(1,1)} = -2n\pi U_{1s,\xi}^{(0,1)}.
\end{aligned} \tag{6.20}
$$

We note that in the solution (6.19), the functions $U_{1s}^{(0,0)}$, $U_{1s}^{(0,1)}$ and $P_t^{(0,1)}$ remain undefined and the following relation between the two leading order functions is valid

$$
p_t^{(0)} = \frac{U_{1s}^{(0,0)} 2(1-\lambda^2) n\pi}{\lambda}. \tag{6.21}
$$

6.1.4 Third order problem

At third order, the non-dimensional equations of motion and associated incompressibility condition are given by

$$
\begin{aligned}
&\lambda^2 W_1^{(0)} + \lambda p_{t,\zeta}^{(2)} - \lambda^2 u_{1,\zeta\zeta}^{(2)} - 4n^2\pi^2\lambda^2 u_1^{(2)} + \lambda^2 p_{t,\xi}^{(1)} = \lambda\left(2(\lambda^2-1)+p\right) u_{1,\xi\zeta}^{(1)} + \lambda\,(1+p)\, u_{2,\zeta\zeta}^{(1)}\\
&+\left((p-1)\lambda^2+1+\lambda^4\right) u_{1,\xi\xi}^{(0)} + \left((p+2)\lambda^2-2\right) u_{2,\xi\zeta}^{(0)} + 4n^2\pi^2\lambda u_2^{(1)},\\
&\lambda^2 W_1^{(0)} - \lambda^3 p_{t,\zeta}^{(2)} - \lambda^2 u_{1,\zeta\zeta}^{(2)} - 4n^2\pi^2\lambda^2 u_1^{(2)} + \lambda^2 p_{t,\xi}^{(1)} = \lambda\left(\lambda^2(2-p)-2\right) u_{1,\xi\zeta}^{(1)} -\\
&-\lambda^3\,(1+p)\, u_{2,\zeta\zeta}^{(1)} + \left((p-1)\lambda^2+1+\lambda^4\right) u_{1,\xi\xi}^{(0)} + \left(p\lambda^2 - 2\,\lambda^2(1-\lambda^2)\right) u_{2,\xi\zeta}^{(0)} - \lambda^3 4n^2\pi^2 u_2^{(1)},\\
&u_{1,\xi}^{(2)} + u_{2,\zeta}^{(2)} = 0,
\end{aligned} \tag{6.22}
$$

subject to the zero displacement boundary conditions

$$
u_1^{(2)} = 0 \quad u_2^{(2)} = 0, \quad \text{at} \quad \zeta = 0, -1. \tag{6.23}
$$

From equations $(6.22)_1$ and $(6.22)_2$ we deduce that

$$
\begin{aligned}
\left(\lambda^2+1\right)\lambda^2 p_{t,\zeta}^{(2)} &= \left(1+\lambda^2\right)\lambda^2 4n^2\pi^2 u_2^{(1)} + \left(1+\lambda^2\right) p\lambda^2 u_{1,\xi\zeta}^{(1)} +\\
&+ (1+p)\,(\lambda^2+1)\lambda^2 u_{2,\zeta\zeta}^{(1)} + \left(\lambda^4-1\right) 2\,\lambda u_{2,\xi\zeta}^{(0)},
\end{aligned} \tag{6.24}
$$

from which, with the help of leading order solutions (6.10) and second order solutions (6.19), we derive the following representation for the incremental pressure

$$p_t^{(2)} = \frac{n\pi\left(\lambda^2-1\right)}{\lambda} U_{1s,\xi\xi}^{(0,0)} \zeta^2 - 2n\pi U_{1s,\xi}^{(0,1)} \zeta + P_t^{(0,2)}. \tag{6.25}$$

Using the form of the leading order solution (6.10) and second order solution (6.19) we are able to establish that at third order the general form of in-plane and normal displacement components are given by (5.38) with the cut-off frequencies $\Omega = 2n\pi$. Our aim is to determine the unknown coefficients in (5.38) and hence establish a solution of the third order problem. Employing the forms of displacement components (5.38), together with the form of incremental pressure (6.25), each of equations (6.22) can be expressed in the form (5.39). We note that in the associated system (5.40) some of the equations are identically zero. There are twenty three unknown functions within the third order problem: nineteen unknown functions arises at third order, two undefined functions from second order $U_{1s}^{(0,1)}, P_t^{(0,1)}$ and two functions connected with the leading order $W_1^{(0)}, U_{1s}^{(0,0)}$.

We now insert the general representations for the displacement components (5.38), with the incremental pressure given by (6.25), into the first equation of motion $(6.22)_1$ and then represent it in the form (5.39). The relation (5.38) yields a system of equations to determine the unknown functions in (5.38). Then from equation $(6.22)_1$ we note that the term in $\sin(2n\pi\zeta)$ yields

$$U_{1c}^{(2,2)} = 0, \tag{6.26}$$

$$\lambda^2 W_1^{(0)} + (6\lambda^2 - 7)\lambda^2 U_{1s,\xi\xi}^{(0,0)} - 2\,\lambda^2 U_{1s}^{(2,2)} + 4\lambda^2 n\pi U_{1c}^{(1,2)} = 0, \tag{6.27}$$

with the term in $\cos(2n\pi\zeta)$ providing

$$(\lambda^2-1)^2 U_{1s,\xi\xi}^{(0,0)} - 2\,\lambda^2 U_{1s}^{(2,2)} = 0, \tag{6.28}$$

$$\lambda 2n\pi U_{1s}^{(1,2)} + \lambda U_{1c}^{(2,2)} + 2n\pi(\lambda^2-1) U_{1s,\xi}^{(0,1)} = 0, \tag{6.29}$$

the remaining relations, arising within equation $(6.22)_1$, being given by

$$v_1^{(2,2)} = 0, \tag{6.30}$$

$$U_{1s,\xi\xi}^{(0,0)} + 2n\pi v_1^{(1,2)} = 0, \tag{6.31}$$

$$2v_1^{(2,2)} + 4n^2\pi^2 v_1^{(0,2)} - P_{t,\xi}^{(0,1)} = 0. \tag{6.32}$$

We remark that the equation of motion $(6.22)_2$ gives no additional equations to determine other unknowns in (5.38). The incompressibility condition $(6.22)_3$ yields the following relations from the $\sin(2n\pi\zeta)$ term

$$U_{1s,\xi}^{(2,2)} - 2n\pi U_{2c}^{(2,2)} = 0, \tag{6.33}$$

$$U_{1s,\xi}^{(1,2)} - 2n\pi U_{2c}^{(1,2)} + 2U_{2s}^{(2,2)} = 0, \tag{6.34}$$

$$U_{1s,\xi}^{(0,2)} + U_{2s}^{(1,2)} - 2n\pi U_{2c}^{(0,2)} = 0, \tag{6.35}$$

the equations connected with the $\cos(2n\pi\zeta)$ term provide

$$2n\pi U_{2s}^{(2,2)} + U_{1c,\xi}^{(2,2)} = 0, \tag{6.36}$$

$$2n\pi U_{2s}^{(1,2)} + 2U_{2c}^{(2,2)} + U_{1c,\xi}^{(1,2)} = 0, \tag{6.37}$$

$$U_{1c,\xi}^{(0,2)} + 2n\pi U_{2s}^{(0,2)} + U_{2c}^{(1,2)} = 0, \tag{6.38}$$

with the remaining equations arising from $(6.22)_3$ are given by

$$2v_2^{(2,2)} + v_{1,\xi}^{(1,2)} = 0, \tag{6.39}$$

$$v_{1,\xi}^{(0,2)} + v_2^{(1,2)} = 0. \tag{6.40}$$

The zero displacement boundary conditions (6.23) at the upper and lower surfaces of the layer are now employed. We substitute the general solution for the displacement components (5.38), taking into account the form of incremental pressure (6.25), into the in-plane displacement component $(6.23)_1$, vanishing of which at the upper surface of the layer $\zeta = 0$ yielding

$$U_{1c}^{(0,2)} + v_1^{(0,2)} = 0. \tag{6.41}$$

Similarly, the in-plane displacement component $(6.23)_1$ is also zero at the lower surface of the layer $\zeta = -1$, providing the following relation

$$U_{1c}^{(0,2)} - U_{1c}^{(1,2)} + U_{1c}^{(2,2)} + v_1^{(0,2)} - v_1^{(1,2)} + v_1^{(2,2)} = 0. \tag{6.42}$$

Within the same spirit, the normal displacement component $(6.23)_2$ vanishes at the upper surface of the layer $\zeta = 0$, yielding the following relation

$$U_{2c}^{(0,2)} + v_2^{(0,2)} = 0, \tag{6.43}$$

with the similar condition at $\zeta = -1$ providing

$$U_{2c}^{(0,2)} - U_{2c}^{(1,2)} + U_{2c}^{(2,2)} + v_2^{(0,2)} - v_2^{(1,2)} + v_2^{(2,2)} = 0. \tag{6.44}$$

We make use of Maple (1996) to obtain the solution of the homogeneous system of equations (6.26)–(6.44) to determine the unknown functions in the representations (5.38) and (6.25). At third order the displacement components and incremental pressure can now be expressed as

$$
\begin{aligned}
u_1^{(2)} &= U_{1c}^{(0,2)}\cos(2n\pi\zeta) + U_{1s}^{(0,2)}\sin(2n\pi\zeta) + U_{1c}^{(1,2)}\zeta\cos(2n\pi\zeta) + U_{1s}^{(1,2)}\zeta\sin(2n\pi\zeta) \\
&\qquad + U_{1s}^{(2,2)}\zeta^2\sin(2n\pi\zeta) + v_1^{(0,2)} + v_1^{(1,2)}\zeta, \\
u_2^{(2)} &= U_{2c}^{(0,2)}\cos(2n\pi\zeta) + U_{2s}^{(0,2)}\sin(2n\pi\zeta) + U_{2c}^{(1,2)}\zeta\cos(2n\pi\zeta) + U_{2s}^{(1,2)}\zeta\sin(2n\pi\zeta) \\
&\qquad + U_{2c}^{(2,2)}\zeta^2\cos(2n\pi\zeta) + v_2^{(0,2)} + v_2^{(1,2)}\zeta + v_2^{(2,2)}\zeta^2, \\
p_t^{(2)} &= P_t^{(0,2)} + P_t^{(1,2)}\zeta + P_t^{(2,2)}\zeta^2,
\end{aligned}
\tag{6.45}
$$

where the non-zero coefficients are given by

$$
\begin{aligned}
U_{1c}^{(0,2)} &= \frac{2U_{1s,\xi}^{(0,1)}\lambda\left(\lambda^2-1\right) + U_{1s,\xi\xi}^{(0,0)}\left(\lambda^2(\lambda^2-1)+1\right)}{4\lambda^2 n\pi}, \quad U_{1c}^{(1,2)} = \frac{U_{1s,\xi\xi}^{(0,0)}}{2n\pi}, \\
U_{1s}^{(1,2)} &= \frac{U_{1s,\xi}^{(0,1)}\left(1-\lambda^2\right)}{\lambda}, \quad U_{1s}^{(2,2)} = \frac{U_{1s,\xi\xi}^{(0,0)}\left(\lambda^2-1\right)^2}{2\lambda^2}, \\
v_1^{(0,2)} &= \frac{U_{1s,\xi}^{(0,1)}\lambda\left(1-\lambda^2\right) - U_{1s,\xi\xi}^{(0,0)}\left(\lambda^2(\lambda^2-1)+1\right)}{2\lambda^2 n\pi}, \quad v_1^{(1,2)} = -\frac{U_{1s,\xi\xi}^{(0,0)}}{2n\pi},
\end{aligned}
\tag{6.46}
$$

$$
\begin{aligned}
U_{2c}^{(0,2)} &= \frac{U_{1s,\xi}^{(0,2)}\lambda^2 4n^2\pi^2 - U_{1s,\xi\xi\xi}^{(0,0)}\left(\lambda^2(\lambda^2-1)+1\right)}{8\lambda^2 n^3\pi^3}, \quad U_{2s}^{(0,2)} = \frac{U_{1s,\xi\xi\xi}^{(0,0)}\left(\lambda^2(1-\lambda^2)-1\right)}{8\lambda^2 n^2\pi^2}, \\
U_{2c}^{(1,2)} &= \frac{U_{1s,\xi\xi}^{(0,1)}\left(1-\lambda^2\right)}{2\lambda n\pi}, \quad U_{2s}^{(1,2)} = \frac{U_{1s,\xi\xi\xi}^{(0,0)}\left(\lambda^2(1-\lambda^2)-1\right)}{4\lambda^2 n^2\pi^2}, \\
U_{2c}^{(2,2)} &= \frac{U_{1s,\xi\xi\xi}^{(0,0)}\left(\lambda-1\right)^2\left(\lambda+1\right)^2}{4\lambda^2 n\pi}, \quad v_2^{(0,2)} = \frac{U_{1s,\xi\xi\xi}^{(0,0)}\left(\lambda^2(\lambda^2-1)+1\right) - U_{1s,\xi}^{(0,2)}\lambda^2 4n^2\pi^2}{8\lambda^2 n^3\pi^3}, \\
v_2^{(1,2)} &= \frac{2U_{1s,\xi\xi}^{(0,1)}\left(\lambda^2-1\right)\lambda + U_{1s,\xi\xi\xi}^{(0,0)}\left(\lambda^2(\lambda^2-1)+1\right)}{4\lambda^2 n\pi}, \quad v_2^{(2,2)} = \frac{U_{1s,\xi\xi\xi}^{(0,0)}}{4n\pi}, \\
P_t^{(2,2)} &= \frac{U_{1s,\xi\xi}^{(0,0)} n\pi\left(\lambda^2-1\right)}{\lambda}, \quad P_t^{(1,2)} = -\,2n\pi U_{1s,\xi}^{(0,1)}.
\end{aligned}
\tag{6.47}
$$

We note that at third order the functions $U_{1s}^{(0,1)}$, $U_{1s}^{(0,2)}$ and $P_t^{(0,2)}$ remain undefined. However, the function $P_t^{(0,1)}$ in the solution (6.19) can be expressed as

$$
P_t^{(0,1)} = \frac{n\pi\left(U_{1s,\xi}^{(0,0)}\left(\lambda^2(1-\lambda^2)-1\right) - 2\,U_{1s}^{(0,1)}\lambda\left(\lambda^2+1\right)\right)}{\lambda^2}.
\tag{6.48}
$$

Now we are ready to derive the governing equation for the long wave amplitude $U_{1s}^{(0,0)}$. This governing equation can be obtained only at the third order problem. This fact is the novel aspect in comparison to previously derived dynamic models, see for example Nolde and Rogerson

(2002), Prikazchikova (2004). To begin with, we employ the following representation for the function $W_1^{(0)}$

$$W_1^{(0)} = \frac{U_{1s,\xi\xi}^{(0,0)}\left(3\,\lambda^2 - 2(\lambda^4+1)\right)}{\lambda^2}, \tag{6.49}$$

which yields the following non-dimensional governing equation for the long wave amplitude $U_{1s}^{(0,0)}$

$$\eta^{-2}\left(U_{1s,\tau\tau}^{(0,0)} + 4n^2\pi^2 U_{1s}^{(0,0)}\right) + \frac{\left(2(\lambda^4+1) - 3\,\lambda^2\right)}{\lambda^2} U_{1s,\xi\xi}^{(0,0)} = 0. \tag{6.50}$$

Employing the representation for the dimensionless frequency (5.69) the governing equation (6.50) can be represented in original variables x_1 and t in the following form

$$\frac{\rho h^2}{\mu}\frac{\partial^2 U_{1s}^{(0,0)}}{\partial^2 t^2} + 4n^2\pi^2 U_{1s}^{(0,0)} + h^2\left(\frac{\left(2(\lambda^4+1) - 3\,\lambda^2\right)}{\lambda^2}\right)\frac{\partial^2 U_{1s}^{(0,0)}}{\partial^2 x_1^2} = 0. \tag{6.51}$$

We remark here that the governing equation (6.51) is elliptic due to the following quantity being positive for all λ

$$\mathcal{B}_e^{(1)} = \frac{\left(2(\lambda^4+1) - 3\,\lambda^2\right)}{\lambda^2} > 0. \tag{6.52}$$

The existence of negative group velocity in dispersive curves, see Figure 3.1, for the long wave high frequency motion in the vicinity of cut-off frequencies, can be associated with the equation (6.51) being elliptic. In addition, the function $\mathcal{B}_e^{(1)}$ is consistent with the corresponding coefficient in the model derived by Nolde and Rogerson (2002) when $\epsilon = 0$.

If now we specify the function $U_{1s}^{(0,0)}$ in the form (1.72) and substitute it into the governing equation (6.51). Then taking into account (5.69) we obtain the following asymptotic expansion for squared non-dimensional frequency

$$\Omega^2 = 4n^2\pi^2 - \mathcal{B}_e^{(1)}\eta^2 + O(\eta^4). \tag{6.53}$$

The representation (3.29) yields

$$\Omega^2 = 4n^2\pi^2 + 4n\pi\Omega_{1fh}^{(2)}\eta^2 + O(\eta^4), \tag{6.54}$$

where $\Omega_{1fh}^{(2)}$ is given by (3.27). With the help of relation $\epsilon = (\lambda^2-1)/\lambda$ one can see that expansions (6.53) and (6.54) coincide. This fact demonstrates the consistency of the derived dynamic one-dimensional model with asymptotic analysis of dispersion relation (3.5) when the cut-off frequencies are given by (3.24).

6.2 Asymptotically approximate equations for the layer with fixed faces: second family of cut-off frequencies

6.2.1 Asymptotic scaling and dimensionless equations

In this section we focus attention to a layer with fixed faces, in order to obtain a simplified dynamic model, we perform asymptotic integration in the vicinity of the cut-off frequencies given by the transcendental equation (3.13), with the associated relative asymptotic orders of displacements and incremental pressure are expressed in (3.51). Therefore, the appropriate scalings are given by

$$\begin{aligned}
&x_1 = l\xi, \quad x_2 = l\eta\zeta, \\
&u_1 = lu_1^*, \quad u_2 = l\eta u_2^*, \quad p_t = \mu p_t^* \eta^{-2}, \quad t = l\eta\sqrt{\frac{\rho}{\mu}}\tau,
\end{aligned} \tag{6.55}$$

where the superscript * indicates dimensionless quantities. Hence the equations of motion (1.120) can be represented in a non-dimensional form. In addition, we employ relations (5.4), valid for the high frequency motion in the vicinity of cut-off frequencies. The non-dimensional equations of motion can now be expressed as

$$\begin{aligned}
&M^(t)_1 = c_4^1\eta^4 + c_3^1\eta^3 + c_2^1\eta^2 + c_1^1\eta + c_0^1 = 0, \quad M^(t)_2 = c_4^2\eta^4 + c_3^2\eta^3 + c_2^2\eta^2 + c_1^2\eta + c_0^2 = 0, \\
&c_0^1 = p_{t,\zeta}^*\lambda^2, \quad c_0^2 = p_{t,\zeta}^*\lambda^3, \\
&c_1^1 = \lambda^3(p_{t,\xi}^* - \Omega^2 u_1^* - u_{1,\zeta\zeta}^*), \quad c_1^2 = \lambda^2(\Omega^2 u_1^* + u_{1,\zeta\zeta}^* - p_{t,\xi}^*), \\
&c_2^1 = \left(2(1-\lambda^2) - p\right)\lambda^2 u_{1,\xi\zeta}^* - \lambda^2\Omega^2 u_2^* - (p+1)\lambda^2 u_{2,\zeta\zeta}^*, \\
&c_2^2 = \left((2-p)\lambda^2 - 2\right)\lambda u_{1,\xi\zeta}^* - \lambda^3\Omega^2 u_2^* - (p+1)\lambda^3 u_{2,\zeta\zeta}^*, \\
&c_3^1 = \left((1-p)\lambda^2 - \lambda^4 - 1\right)\lambda u_{1,\xi\xi}^* + \lambda^3 W_1^* + \left(2 - (2+p)\lambda^2\right)\lambda u_{2,\xi\zeta}^*, \\
&c_3^2 = \left(\lambda^4 + (p-1)\lambda^2 + 1\right)u_{1,\xi\xi}^* - \lambda^2 W_1^* + \left(2(1-\lambda^2) + p\right)\lambda^2 u_{2,\xi\zeta}^*, \\
&c_4^1 = \left(\lambda^2(1-\lambda^2) - 1\right)u_{2,\xi\xi}^* + \lambda^2 W_2^*, \quad c_4^2 = \left(\lambda^2(1-\lambda^2) - 1\right)\lambda u_{2,\xi\xi}^* + \lambda^3 W_2^*,
\end{aligned} \tag{6.56}$$

where Ω is the solution of the transcendental equation (3.13). Equations (6.56) must be solved in conjunction with the non-dimensional incompressibility condition (5.6) and subject to the zero displacement boundary conditions (6.1). The solutions of equations (6.56),(5.6), subject to boundary conditions (6.1), are sought in the form of series (4.11).

6.2.2 Leading order problem

At leading order, the non-dimensional equations of motion, and associated incompressibility condition, are given by

$$p^{(0)}_{t,\zeta} = 0, \quad {\Omega_0}^2 u^{(0)}_1 + u^{(0)}_{1,\zeta\zeta} - p^{(0)}_{t,\xi} = 0, \quad u^{(0)}_{1,\xi} + u^{(0)}_{2,\zeta} = 0, \tag{6.57}$$

subject to the boundary conditions (6.3). Equations $(6.57)_1$ and $(6.57)_2$ enable us to obtain the following relations

$$p^{(0)}_t = P^{(0,0)}_t, \quad u^{(0)}_1 = U^{(0,0)}_{1c}\cos(\Omega_0\zeta) + U^{(0,0)}_{1s}\sin(\Omega_0\zeta) + \frac{P^{(0,0)}_{t,\xi}}{\Omega_0^2}. \tag{6.58}$$

Using the incompressibility condition $(6.57)_3$ we can represent the normal displacement component in the following form

$$u^{(0)}_2 = \frac{U^{(0,0)}_{1s,\xi}}{\Omega_0}\cos(\Omega_0\zeta) - \frac{U^{(0,0)}_{1c,\xi}}{\Omega_0}\sin(\Omega_0\zeta) - \frac{P^{(0,0)}_{t,\xi\xi}}{\Omega_0^2}\zeta + v^{(0,0)}_2. \tag{6.59}$$

The boundary conditions (6.3) at the upper and lower surfaces of the layer, together with relations

$$\sin(\Omega_0) = 4\Omega_0/(4+\Omega_0^2), \quad \cos(\Omega_0) = (4-\Omega_0^2)/(4+\Omega_0^2), \tag{6.60}$$

providing the following representations

$$U^{(0,0)}_{1c,\xi} = -\frac{P^{(0,0)}_t}{\Omega_0^2}, \quad U^{(0,0)}_{1s,\xi} = -\frac{P^{(0,0)}_t}{2\Omega_0}, \quad v_2 = -\frac{P^{(0,0)}_{t,\xi\xi}}{2\Omega_0^2}. \tag{6.61}$$

Hence, the solution of the leading order problem is given by

$$\begin{aligned}
u^{(0)}_1 &= -\frac{P^{(0,0)}_{t,\xi}\cos(\Omega_0\,\zeta)}{{\Omega_0}^2} + \frac{P^{(0,0)}_{t,\xi}\sin(\Omega_0\,\zeta)}{2\Omega_0} + \frac{P^{(0,0)}_{t,\xi}}{{\Omega_0}^2},\\
u^{(0)}_2 &= \frac{P^{(0,0)}_{t,\xi\xi}\cos(\Omega_0\,\zeta)}{2{\Omega_0}^2} + \frac{P^{(0,0)}_{t,\xi\xi}\sin(\Omega_0\,\zeta)}{{\Omega_0}^3} - \frac{P^{(0,0)}_{t,\xi\xi}}{2{\Omega_0}^2} - \frac{P^{(0,0)}_{t,\xi\xi}\zeta}{{\Omega_0}^2},\\
p^{(0)}_t &= P^{(0,0)}_t.
\end{aligned} \tag{6.62}$$

The representation (6.62) indicate that the incremental pressure $P^{(0,0)}_t$ is the essential parameter in the model, this fact is consistent with the results obtained by Nolde and Rogerson (2002). We remark that in all the other models considered in this thesis the essential parameter is the long wave amplitude.

6.2.3 Second order problem

At second order, the non-dimensional equations of motion and associated incompressibility condition are given by

$$p^{(1)}_{t,\zeta} = 0, \quad \lambda\Omega_0{}^2 u^{(1)}_1 - \lambda p^{(1)}_{t,\xi} + \lambda u^{(1)}_{1,\zeta\zeta} = 2\left(1-\lambda^2\right) u^{(0)}_{1,\xi\zeta}, \quad u^{(1)}_{1,\xi} + u^{(1)}_{2,\zeta} = 0, \tag{6.63}$$

and subject to the boundary conditions (6.12). From equation $(6.63)_1$ we deduce that the second order incremental pressure does not depend on ζ, thus $p^{(1)}_t = P^{(0,1)}_t$.

It is possible to integrate the system of equations (6.63), subject to boundary conditions (6.12), in a similar manner to that employed for the leading order problem. However, to facilitate the analysis we establish that the general solution of the equations (6.63), subject to the boundary conditions (6.12), can be expressed as

$$\begin{aligned} u^{(1)}_1 &= U^{(0,1)}_{1c}\cos(\Omega\zeta) + U^{(0,1)}_{1s}\sin(\Omega\zeta) + U^{(1,1)}_{1c}\zeta\cos(\Omega\zeta) + U^{(1,1)}_{1s}\zeta\sin(\Omega\zeta) \\ &\qquad + v^{(0,1)}_1 + v^{(1,1)}_1\zeta, \\ u^{(1)}_2 &= U^{(0,1)}_{2c}\cos(\Omega\zeta) + U^{(0,1)}_{2s}\sin(\Omega\zeta) + U^{(1,1)}_{2c}\zeta\cos(\Omega\zeta) + U^{(1,1)}_{2s}\zeta\sin(\Omega\zeta) \\ &\qquad + v^{(0,1)}_2 + v^{(1,1)}_2\zeta + v^{(2,1)}_2\zeta^2, \end{aligned} \tag{6.64}$$

where cut-off frequencies $\Omega = \Omega_0$ are given by transcendental equation (3.13). Employing the representation (6.64), each of the equation of motion $(6.63)_1$ and $(6.63)_2$, incompressibility $(6.63)_3$ and boundary conditions (6.12) can be written in a similar general form, given by

$$(a^{(1)}\zeta + b^{(1)})\sin(\Omega\zeta) + (b^{(1)}\zeta + d^{(1)})\cos(\Omega\zeta) + e^{(1)}\zeta + f^{(1)} = 0, \tag{6.65}$$

providing for each particular equation

$$a^{(1)} = 0, \quad b^{(1)} = 0, \quad c^{(1)} = 0, \quad d^{(1)} = 0, \quad e^{(1)}, \quad f^{(1)} = 0, \tag{6.66}$$

The equations of motion $(6.63)_1$ and $(6.63)_2$ yielding the following relations

$$P^{(0,0)}_{t,\xi\xi}\left(\lambda^2+1\right) - \lambda\Omega_0{}^2 U^{(1,1)}_{1c} = 0, \tag{6.67}$$

$$2\lambda\Omega_0 U^{(1,1)}_{1s} + P^{(0,0)}_{t,\xi\xi}\left(\lambda^2-1\right) = 0, \tag{6.68}$$

$$v^{(1,1)}_1 = 0, \tag{6.69}$$

$$\Omega_0{}^2 v^{(0,1)}_1 - P^{(0,1)}_{t,\xi} = 0, \tag{6.70}$$

and incompressibility condition $(6.63)_3$ provides

$$\Omega_0 U_{2c}^{(1,1)} - U_{1s,\xi}^{(1,1)} = 0, \tag{6.71}$$

$$U_{1s,\xi}^{(0,1)} + U_{2s}^{(1,1)} - \Omega_0 U_{2c}^{(0,1)} = 0, \tag{6.72}$$

$$\Omega_0 U_{2s}^{(1,1)} + U_{1c,\xi}^{(1,1)} = 0, \tag{6.73}$$

$$U_{2c}^{(1,1)} + U_{1c,\xi}^{(0,1)} + \Omega_0 U_{2s}^{(0,1)} = 0, \tag{6.74}$$

$$v_{1,\xi}^{(1,1)} + 2v_2^{(2,1)} = 0, \tag{6.75}$$

$$v_{1,\xi}^{(0,1)} + v_2^{(1,1)} = 0. \tag{6.76}$$

From the zero displacement boundary conditions (6.12) we obtain the following relations

$$U_{1c}^{(0,1)} + v_1^{(0,1)} = 0, \tag{6.77}$$

$$\begin{aligned}&(4-{\Omega_0}^2)U_{1c}^{(0,1)} - 4U_{1s}^{(0,1)}\Omega_0 - (4-{\Omega_0}^2)U_{1c}^{(1,1)}\\ &+ 4\Omega_0 U_{1s}^{(1,1)} + (4+{\Omega_0}^2)v_1^{(0,1)} - (4-{\Omega_0}^2)v_1^{(1,1)} = 0,\end{aligned} \tag{6.78}$$

$$U_{2c}^{(0,1)} + v_2^{(0,1)} = 0, \tag{6.79}$$

$$\begin{aligned}&(4-{\Omega_0}^2)U_{2c}^{(0,1)} - 4\Omega_0 U_{2s}^{(0,1)} - (4-{\Omega_0}^2)U_{2c}^{(1,1)} + 4\Omega_0 U_{2s}^{(1,1)}\\ &+ (4+{\Omega_0}^2)v_2^{(0,1)} - (4+{\Omega_0}^2)v_2^{(1,1)} + (4+{\Omega_0}^2)v_2^{(2,1)} = 0.\end{aligned} \tag{6.80}$$

We solve the homogeneous system of equations (6.67)–(6.80) with the help of Maple (1996). To summarize, at second order the displacement components and incremental pressure may be expressed in the following form

$$\begin{aligned}u_1^{(1)} &= U_{1c}^{(0,1)}\cos(\Omega_0\zeta) + U_{1s}^{(0,1)}\sin(\Omega_0\zeta) + U_{1c}^{(1,1)}\zeta\cos(\Omega_0\zeta) + U_{1s}^{(1,1)}\zeta\sin(\Omega_0\zeta) + v_1^{(0,1)},\\ u_2^{(1)} &= U_{2c}^{(0,1)}\cos(\Omega_0\zeta) + U_{2s}^{(0,1)}\sin(\Omega_0\zeta) + U_{2c}^{(1,1)}\zeta\cos(\Omega_0\zeta) + U_{2s}^{(1,1)}\zeta\sin(\Omega_0\zeta)+\\ &+ v_2^{(0,1)} + v_2^{(1,1)}\zeta, \quad p_t^{(1)} = P_t^{(0,1)},\end{aligned} \tag{6.81}$$

where the non-zero coefficients are represented as

$$\begin{aligned}U_{1c}^{(0,1)} &= -\frac{P_{t,\xi}^{(0,1)}}{{\Omega_0}^2}, \quad U_{1s}^{(0,1)} = \frac{P_{t,\xi\xi}^{(0,0)}\left(4+{\Omega_0}^2\right)\left(1-\lambda^2\right) + 2\,\lambda P_{t,\xi}^{(0,1)}{\Omega_0}^2}{4{\Omega_0}^3\lambda},\\ U_{1c}^{(1,1)} &= \frac{P_{t,\xi\xi}^{(0,0)}\left(\lambda^2-1\right)}{{\Omega_0}^2\lambda}, \quad U_{1s}^{(1,1)} = \frac{P_{t,\xi\xi}^{(0,0)}\left(1-\lambda^2\right)}{2\Omega_0\,\lambda}, \quad v_1^{(0,1)} = \frac{P_{t,\xi}^{(0,1)}}{{\Omega_0}^2},\end{aligned} \tag{6.82}$$

$$U_{2c}^{(0,1)} = \frac{2\,\lambda P_{t,\xi\xi}^{(0,1)}{\Omega_0}^2 - P_{t,\xi\xi\xi}^{(0,0)}\left(8+{\Omega_0}^2\right)\left(\lambda^2-1\right)}{4{\Omega_0}^4\lambda}, \quad U_{2s}^{(0,1)} = \frac{2\,\lambda P_{t,\xi\xi}^{(0,1)} + P_{t,\xi\xi\xi}^{(0,0)}\left(\lambda^2-1\right)}{2{\Omega_0}^3\lambda},$$
$$U_{2c}^{(1,1)} = \frac{P_{t,\xi\xi\xi}^{(0,0)}\left(1-\lambda^2\right)}{2{\Omega_0}^2\lambda}, \quad U_{2s}^{(1,1)} = \frac{P_{t,\xi\xi\xi}^{(0,0)}\left(1-\lambda^2\right)}{{\Omega_0}^3\lambda}, \quad v_2^{(1,1)} = -\frac{P_{t,\xi\xi}^{(0,1)}}{{\Omega_0}^2},$$
$$v_2^{(0,1)} = \frac{P_{t,\xi\xi\xi}^{(0,0)}\left(8+{\Omega_0}^2\right)\left(\lambda^2-1\right) - 2\,\lambda P_{t,\xi\xi}^{(0,1)}{\Omega_0}^2}{4{\Omega_0}^4\lambda}. \tag{6.83}$$

We remark that the functions $P_t^{(0,0)}$ and $P_t^{(0,1)}$ remain undefined within the second order solution (6.81).

6.2.4 Third order problem

At third order, the non-dimensional equations of motion and associated incompressibility condition are given by

$$\begin{aligned}
&p_{t,\zeta}^{(2)} = {\Omega_0}^2 u_2^{(0)} + \left(p + 2(\lambda^2-1)\right) u_{1,\xi\zeta}^{(0)} + (p+1)\, u_{2,\zeta\zeta}^{(0)} + \lambda\,{\Omega_0}^2 u_1^{(1)} - \lambda p_{t,\xi}^{(1)} + \lambda\, u_{1,\zeta\zeta}^{(1)},\\
&\lambda^2 W_1^{(0)} + \lambda^2 p_{t,\xi}^{(2)} - \lambda^2 u_{1,\zeta\zeta}^{(2)} - \lambda^2{\Omega_0}^2 u_1^{(2)} =\\
&\qquad\qquad = \left(\lambda^2(\lambda^2-1) + p\lambda^2 + 1\right) u_{1,\xi\xi}^{(0)} + \lambda^2 p u_{2,\xi\zeta}^{(0)} + 2\,\lambda\left(\lambda^2-1\right) u_{1,\xi\zeta}^{(1)},\\
&u_{1,\xi}^{(2)} + u_{2,\zeta}^{(2)} = 0,
\end{aligned} \tag{6.84}$$

subject to the zero displacement boundary conditions (6.23). From equation $(6.84)_1$ we obtain

$$p_{t,\zeta}^{(2)} = {\Omega_0}^2 u_2^{(0)} + \left(p + 2(\lambda^2-1)\right) u_{1,\xi\zeta}^{(0)} + (p+1)\, u_{2,\zeta\zeta}^{(0)} + \lambda\,{\Omega_0}^2 u_1^{(1)} - \lambda p_{t,\xi}^{(1)} + \lambda\, u_{1,\zeta\zeta}^{(1)}. \tag{6.85}$$

Taking into account the solutions of the leading order problem (6.62) and the second order problem (6.81) we deduce that

$$p_t^{(2)} = -\frac{P_{t,\xi\xi}^{(0,0)}\zeta^2}{2} - \frac{P_{t,\xi\xi}^{(0,0)}\zeta}{2} + P_t^{(0,2)}. \tag{6.86}$$

To facilitate our analysis we establish that the general solution of the equations (6.84) subject to the boundary conditions (6.23) can be expressed as

$$\begin{aligned}
u_1^{(2)} &= U_{1c}^{(0,2)}\cos(\Omega\zeta) + U_{1s}^{(0,2)}\sin(\Omega\zeta) + U_{1c}^{(1,2)}\zeta\cos(\Omega\zeta) + U_{1s}^{(1,2)}\zeta\sin(\Omega\zeta)\\
&+ U_{1c}^{(2,2)}\zeta^2\cos(\Omega\zeta) + U_{1s}^{(2,2)}\zeta^2\sin(\Omega\zeta) + v_1^{(0,2)} + v_1^{(1,2)}\zeta + v_1^{(2,2)}\zeta^2,\\
u_2^{(2)} &= U_{2c}^{(0,2)}\cos(\Omega\zeta) + U_{2s}^{(0,2)}\sin(\Omega\zeta) + U_{2c}^{(1,2)}\zeta\cos(\Omega\zeta) + U_{2s}^{(1,2)}\zeta\sin(\Omega\zeta)\\
&+ U_{2c}^{(2,2)}\zeta^2\cos(\Omega\zeta) + U_{2s}^{(2,2)}\zeta^2\sin(\Omega\zeta) + v_2^{(0,2)} + v_2^{(1,2)}\zeta + v_2^{(2,2)}\zeta^2 + v_2^{(3,2)}\zeta^3,
\end{aligned} \tag{6.87}$$

where cut-off frequencies $\Omega = \Omega_0$ are given by transcendental equation (3.13). Employing representations (6.87) and (6.86), each of equations (6.84) takes the following similar general form

$$(A^{(1)}\zeta^2 + B^{(1)}\zeta + C^{(1)})\sin(\Omega\zeta) + (D^{(1)}\zeta^2 + E^{(1)}\zeta + F^{(1)})\cos(\Omega\zeta)$$
$$+ K^{(1)}\zeta^2 + L^{(1)}\zeta + M^{(1)} = 0, \tag{6.88}$$

yielding for each equation

$$A^{(1)} = 0, \quad B^{(1)} = 0, \quad C^{(1)} = 0, \quad D^{(1)} = 0, \quad E^{(1)}, \quad F^{(1)} = 0$$
$$K^{(1)} = 0, \quad L^{(1)} = 0, \quad M^{(1)} = 0. \tag{6.89}$$

We note that in the system (6.89) some of the equations are identically zero. From the equations of motion $(6.84)_1$ and $(6.84)_2$ we obtain the following relations

$$2\,\lambda^2{\Omega_0}^2U_{1c}^{(2,2)} + (\lambda^2-1)^2P_{t,\xi\xi\xi}^{(0,0)} = 0, \tag{6.90}$$
$$(\lambda^4 - 3\lambda^2+1))P_{t,\xi\xi\xi}^{(0,0)} - 4\lambda(\lambda^2+1)P_{t,\xi\xi}^{(0,1)} - 4\lambda^2\Omega_0(U_{1s}^{(2,2)} - \Omega_0U_{1c}^{(1,2)}) + \lambda^2W_1^0 = 0, \tag{6.91}$$
$$4\,\lambda^2\Omega_0U_{1s}^{(2,2)} - (\lambda^2-1)^2P_{t,\xi\xi\xi}^{(0,0)} = 0, \tag{6.92}$$
$$2\,\lambda^2W_1^0 - 2(\lambda^2(\lambda^2-1)+1)P_{t,\xi\xi\xi}^{(0,0)} + 4\,\lambda^2{\Omega_0}^3U_{1s}^{(1,2)} + 4\,\lambda^2{\Omega_0}^2U_{1c}^{(2,2)}$$
$$+ 2\,{\Omega_0}^2\lambda(\lambda^2-1)P_{t,\xi\xi}^{(0,1)} - {\Omega_0}^2(\lambda^2-1)^2P_{t,\xi\xi\xi}^{(0,0)} = 0, \tag{6.93}$$
$$2v_1^{(2,2)}{\Omega_0}^2 + P_{t,\xi\xi\xi}^{(0,0)} = 0, \tag{6.94}$$
$$2v_1^{(1,2)}{\Omega_0}^2 + P_{t,\xi\xi\xi}^{(0,0)} = 0, \tag{6.95}$$
$$(\lambda^2(\lambda^2-1)+1)P_{t,\xi\xi\xi}^{(0,0)} + \lambda^2{\Omega_0}^4v_1^{(0,2)} - \lambda^2W_1^0 + 2\lambda^2{\Omega_0}^2v_1^{(2,2)} - \lambda^2P_{t,\xi}^{(0,2)}{\Omega_0}^2 = 0. \tag{6.96}$$

The incompressibility condition $(6.84)_3$ also provides

$$\Omega_0U_{2c}^{(2,2)} - U_{1s,\xi}^{(2,2)} = 0, \tag{6.97}$$
$$2U_{2s}^{(2,2)} - \Omega_0U_{2c}^{(1,2)} + U_{1s,\xi}^{(1,2)} = 0, \tag{6.98}$$
$$U_{1s,\xi}^{(0,2)} + U_{2s}^{(1,2)} - \Omega_0U_{2c}^{(0,2)} = 0, \tag{6.99}$$
$$U_{1c,\xi}^{(2,2)} + \Omega_0U_{2s}^{(2,2)} = 0, \tag{6.100}$$
$$\Omega_0U_{2s}^{(1,2)} + 2U_{2c}^{(2,2)} + U_{1c,\xi}^{(1,2)} = 0, \tag{6.101}$$
$$U_{1c,\xi}^{(0,2)} + \Omega_0U_{2s}^{(0,2)} + U_{2c}^{(1,2)} = 0, \tag{6.102}$$
$$v_{1,\xi}^{(2,2)} + 3v_2^{(3,2)} = 0, \tag{6.103}$$
$$2v_2^{(2,2)} + v_{1,\xi}^{(1,2)} = 0, \tag{6.104}$$
$$v_2^{(1,2)} + v_{1,\xi}^{(0,2)} = 0. \tag{6.105}$$

Finally, the zero displacement boundary conditions (6.23) enable us to establish the following relations

$$U_{1c}^{(0,2)} + v_1^{(0,2)} = 0, \tag{6.106}$$

$$(4-{\Omega_0}^2)U_{1c}^{(0,2)} - 4U_{1s}^{(0,2)}\Omega_0 - (4-{\Omega_0}^2)U_{1c}^{(1,2)} + 4U_{1s}^{(1,2)}\Omega_0 + (4-{\Omega_0}^2)U_{1c}^{(2,2)}$$
$$+ (4+{\Omega_0}^2)v_1^{(0,2)} - 4U_{1s}^{(2,2)}\Omega_0 - (4+{\Omega_0}^2)v_1^{(1,2)} + (4+{\Omega_0}^2)v_1^{(2,2)} = 0, \tag{6.107}$$

$$U_{2c}^{(0,2)} + v_2^{(0,2)} = 0, \tag{6.108}$$

$$(4-{\Omega_0}^2)U_{2c}^{(0,2)} - 4U_{2s}^{(0,2)}\Omega_0 - (4-{\Omega_0}^2)U_{2c}^{(1,2)} + 4U_{2s}^{(1,2)}\Omega_0$$
$$+ (4-{\Omega_0}^2)U_{2c}^{(2,2)} - 4\Omega_0 U_{2s}^{(2,2)} + (4+{\Omega_0}^2)v_2^{(0,2)} - (4+{\Omega_0}^2)v_2^{(1,2)}$$
$$+ (4+{\Omega_0}^2)v_2^{(2,2)} - (4+{\Omega_0}^2)v_2^{(3,2)} = 0. \tag{6.109}$$

We solve the homogeneous system of equations (6.90)–(6.109) with the help of Maple (1996) to determine the unknown functions in the representations (6.87) and (6.86). As a result, we derive the following solution of the third order problem

$$u_1^{(2)} = U_{1c}^{(0,2)}\cos(\Omega_0\zeta) + U_{1s}^{(0,2)}\sin(\Omega_0\zeta) + U_{1c}^{(1,2)}\zeta\cos(\Omega_0\zeta) + U_{1s}^{(1,2)}\zeta\sin(\Omega_0\zeta)$$
$$+ U_{1c}^{(2,2)}\zeta^2\cos(\Omega_0\zeta) + U_{1s}^{(2,2)}\zeta^2\sin(\Omega_0\zeta) + v_1^{(0,2)} + v_1^{(1,2)}\zeta + v_1^{(2,2)}\zeta^2,$$
$$u_2^{(2)} = U_{2c}^{(0,2)}\cos(\Omega_0\zeta) + U_{2s}^{(0,2)}\sin(\Omega_0\zeta) + U_{2c}^{(1,2)}\zeta\cos(\Omega_0\zeta) + U_{2s}^{(1,2)}\zeta\sin(\Omega_0\zeta)$$
$$+ U_{2c}^{(2,2)}\zeta^2\cos(\Omega_0\zeta) + U_{2s}^{(2,2)}\zeta^2\sin(\Omega_0\zeta) + v_2^{(0,2)} + v_2^{(1,2)}\zeta + v_2^{(2,2)}\zeta^2 + v_2^{(3,2)}\zeta^3,$$
$$p_t^{(2)} = P_t^{(0,2)} + P_t^{(1,2)}\zeta + P_t^{(2,2)}\zeta^2, \tag{6.110}$$

where non-zero coefficients are represented by

$$U_{1s}^{(0,2)} = \frac{12v_1^{(0,2)}{\Omega_0}^4\lambda + P_{t,\xi\xi\xi}^{(0,0)}\lambda\left(4+{\Omega_0}^2\right) + 6P_{t,\xi\xi}^{(0,1)}(1-\lambda^2)\left(4+\Omega_0^2\right)}{24\lambda\,{\Omega_0}^3},$$
$$U_{1c}^{(1,2)} = \frac{P_{t,\xi\xi\xi}^{(0,0)}\left(7\,\lambda^2 - 3(\lambda^4+1)\right) + 6P_{t,\xi\xi}^{(0,1)}\lambda\,\left(\lambda^2-1\right)}{6{\Omega_0}^2\lambda^2},$$
$$U_{1s}^{(1,2)} = \frac{P_{t,\xi\xi\xi}^{(0,0)}\left(4\,\lambda^2 + 3\,{\Omega_0}^2(\lambda^2-1)^2\right) + 6P_{t,\xi\xi}^{(0,1)}{\Omega_0}^2\lambda\,\left(1-\lambda^2\right)}{12{\Omega_0}^3\lambda^2},$$
$$U_{1c}^{(2,2)} = -\frac{P_{t,\xi\xi\xi}^{(0,0)}(\lambda^2-1)^2}{2{\Omega_0}^2\lambda^2}, \quad U_{1s}^{(2,2)} = \frac{P_{t,\xi\xi\xi}^{(0,0)}(\lambda^2-1)^2}{4\lambda^2\Omega_0},$$
$$v_1^{(1,2)} = -\frac{P_{t,\xi\xi\xi}^{(0,0)}}{2{\Omega_0}^2}, \quad v_1^{(2,2)} = -\frac{P_{t,\xi\xi\xi}^{(0,0)}}{2{\Omega_0}^2}, \quad U_{1c}^{(0,2)} = -v_1^{(0,2)}, \tag{6.111}$$

$$
\begin{aligned}
U_{2c}^{(0,2)} &= \frac{P_{t,\xi\xi\xi\xi}^{(0,0)}\left({\Omega_0}^2\lambda^2+4(3(\lambda^4+1)-5\lambda^2)\right)-P_{t,\xi\xi\xi}^{(0,1)}6\lambda\left(\lambda^2-1\right)\left(\Omega_0^2+8\right)+12\,\lambda^2P_{t,\xi\xi}^{(0,2)}{\Omega_0}^2}{24\lambda^2{\Omega_0}^4},\\
U_{2s}^{(0,2)} &= \frac{4\,\lambda^2P_{t,\xi\xi}^{(0,2)}+2\,\lambda\left(\lambda^2-1\right)P_{t,\xi\xi\xi}^{(0,1)}-\left(\lambda^2-1\right)^2P_{t,\xi\xi\xi\xi}^{(0,0)}}{4{\Omega_0}^3\lambda^2},\\
U_{2c}^{(1,2)} &= \frac{P_{t,\xi\xi\xi\xi}^{(0,0)}\left(4(3(\lambda^4+1)-5\lambda^2)+3\Omega_0^2(\lambda^2-1)^2\right)+6\,\lambda\,{\Omega_0}^2\left(1-\lambda^2\right)P_{t,\xi\xi\xi}^{(0,1)}}{12\lambda^2{\Omega_0}^4},\\
U_{2s}^{(1,2)} &= -\frac{P_{t,\xi\xi\xi\xi}^{(0,0)}\lambda+6P_{t,\xi\xi\xi}^{(0,1)}\left(1+\lambda^2\right)}{6\lambda\,{\Omega_0}^3},\\
U_{2c}^{(2,2)} &= \frac{P_{t,\xi\xi\xi\xi}^{(0,0)}(\lambda^2-1)^2}{4{\Omega_0}^2\lambda^2},\quad U_{2s}^{(2,2)}=\frac{P_{t,\xi\xi\xi\xi}^{(0,0)}\left(\lambda^2-1\right)^2}{2\lambda^2{\Omega_0}^3},\\
v_2^{(0,2)} &= \frac{P_{t,\xi\xi\xi\xi}^{(0,0)}\left(4(5\lambda^2-3(\lambda^4+1))-{\Omega_0}^2\lambda^2\right)+6\lambda\left(\lambda^2-1\right)(\Omega_0^2+8)P_{t,\xi\xi\xi}^{(0,1)}-12\,\lambda^2P_{t,\xi\xi}^{(0,2)}{\Omega_0}^2}{24\lambda^2{\Omega_0}^4},\\
v_2^{(1,2)} &= \frac{P_{t,\xi\xi\xi\xi}^{(0,0)}\left(5\,\lambda^2-3(\lambda^4+1)\right)-3\,\lambda^2P_{t,\xi\xi}^{(0,2)}{\Omega_0}^2}{3\lambda^2{\Omega_0}^4},\quad v_2^{(2,2)}=\frac{P_{t,\xi\xi\xi\xi}^{(0,0)}}{4{\Omega_0}^2},\quad v_2^{(3,2)}=\frac{P_{t,\xi\xi\xi\xi}^{(0,0)}}{6{\Omega_0}^2},\\
P_t^{(2,2)} &= P_t^{(1,2)}=-\frac{P_{t,\xi\xi}^{(0,0)}}{2}.
\end{aligned}
\tag{6.112}
$$

At third order, we derive the following representation for the derivative $v_{1,\xi}^{(0,2)}$

$$
v_{1,\xi}^{(0,2)}=\frac{P_{t,\xi\xi\xi\xi}^{(0,0)}\left(3(\lambda^4+1)-5\,\lambda^2\right)+3\,\lambda^2P_{t,\xi\xi}^{(0,2)}{\Omega_0}^2}{3\lambda^2{\Omega_0}^4},
\tag{6.113}
$$

enabling the function $v_1^{(0,2)}$ to be expressed as

$$
v_1^{(0,2)}=\frac{P_{t,\xi\xi\xi}^{(0,0)}\left(3(\lambda^4+1)-5\,\lambda^2\right)+3\,\lambda^2P_{t,\xi}^{(0,2)}{\Omega_0}^2}{3\lambda^2{\Omega_0}^4}+v_{01}^{(0,2)}(\tau).
\tag{6.114}
$$

We remark that at third order the functions $P_t^{(0,1)}$, $P_t^{(0,2)}$ and $v_{01}^{(0,2)}(\tau)$ remain undefined.

We are now in a position to derive the governing equation for the pressure increment $P_t^{(0,0)}$. For this purpose we employ the following representation for the derivative $w_{1,\xi}^{(0)}$

$$
w_{1,\xi}^{(0)}=\frac{P_{t,\xi\xi\xi\xi}^{(0,0)}\left(6(\lambda^4+1)-11\,\lambda^2\right)}{3\lambda^2},
\tag{6.115}
$$

which yields the following relation

$$
\begin{aligned}
W_1^{(0)} &= w_1^{(0)}\left(\frac{\sin(\Omega_0\zeta)}{2\Omega_0}-\frac{\cos(\Omega_0\zeta)}{\Omega_0^2}+\frac{1}{\Omega_0^2}\right)=\\
&= \eta^{-2}\left(P_{t,\tau\tau\xi}^{(0,0)}+\Omega_0^2P_{t,\xi}^{(0,0)}\right)\left(\frac{\sin(\Omega_0\zeta)}{2\Omega_0}-\frac{\cos(\Omega_0\zeta)}{\Omega_0^2}+\frac{1}{\Omega_0^2}\right).
\end{aligned}
\tag{6.116}
$$

The representations (6.116) and (6.115) enable us to establish the governing equation for the pressure increment $P_t^{(0,0)}$ in the following form

$$
\eta^{-2}\left(P_{t,\tau\tau\xi\xi}^{(0,0)}+\Omega_0^2P_{t,\xi\xi}^{(0,0)}\right)-\frac{\left(6(\lambda^4+1)-11\,\lambda^2\right)}{3\lambda^2}P_{t,\xi\xi\xi\xi}^{(0,0)}=0.
\tag{6.117}
$$

In contrast to all the previous high frequency models described in this thesis the governing equation (6.117) is of the fourth order. The fourth order equation (6.117) has two additional spurious solutions arising through the increase in order of ξ derivative. The first solution giving $P_t^{(0,0)}$ as a constant is clearly not a vibration type. The second solution, representing $P_t^{(0,0)}$ as the linear function of ζ, corresponds to the case in which u_1 is independent of ξ and $u_2 = 0$. This type of motion is clearly not associated with long wave high frequency motion. Therefore it is possible to remove two of the derivatives with respect to ξ and express (6.117) as

$$\eta^{-2}\left(P_{t,\tau\tau}^{(0,0)} + \Omega_0^2 P_t^{(0,0)}\right) - \frac{\left(6(\lambda^4+1) - 11\,\lambda^2\right)}{3\lambda^2} P_{t,\xi\xi}^{(0,0)} = 0. \tag{6.118}$$

Employing the dimensionless frequency (5.69), the governing equation (6.118) can be represented in original non-scaled variables t and x_1 as

$$\frac{\rho h^2}{\mu}\frac{\partial^2 P_t^{(0,0)}}{\partial^2 t^2} + \Omega_0^2 P_t^{(0,0)} - h^2 \frac{\left(6(\lambda^4+1) - 11\,\lambda^2\right)}{3\lambda^2}\frac{\partial^2 P_t^{(0,0)}}{\partial^2 x_1^2} = 0. \tag{6.119}$$

We remark that the governing equation (6.119) is of hyperbolic type due to the following quantity being positive for all λ

$$\mathcal{B}_e^{(2)} = \frac{\left(6(\lambda^4+1) - 11\,\lambda^2\right)}{3\lambda^2} > 0. \tag{6.120}$$

Also when $\epsilon = 0$ the function $\mathcal{B}_e^{(2)}$ is consistent with the corresponding quantity in the model by Nolde and Rogerson (2002).

If we specify the function $P_t^{(0,0)}$ in the form (1.72) and substitute it into the governing equation (6.119) we obtain the following expansion for squared non-dimensional frequency

$$\Omega^2 = \Omega_0^2 + \mathcal{B}_e^{(2)}\eta^2 + O(\eta^4). \tag{6.121}$$

The approximation (3.29) can be expressed in a form

$$\Omega^2 = \Omega_0^2 + 2\Omega_0\Omega_{2fh}^{(2)}\eta^2 + O(\eta^4), \tag{6.122}$$

where $\Omega_{2fh}^{(2)}$ is given by (3.28). If we use relation $\epsilon = (\lambda^2 - 1)/\lambda$ it is easy to see that expansions (6.121) and (6.122) coincide. This fact demonstrates the consistency of the derived dynamic one-dimensional model with asymptotic analysis of dispersion relation (3.5) when the cut-off frequencies are given by (3.25).

6.3 Asymptotically approximate equations for the layer with one free and one fixed faces

6.3.1 Asymptotic scaling and dimensionless equations

In this section we derive a simplified model for long wave high frequency motion in a layer with one free and fixed faces, i.e. the so-called mixed boundary conditions (3.6). The corresponding cut-off frequencies are expressed in relation (3.39). The associated relative asymptotic orders of displacements and incremental pressure are given by (3.60) and hence equivalent to the corresponding relation (3.51). Therefore we employ the non-dimensional equations of motion (6.56) with associated non-dimensional incompressibility condition (5.6). The equations (6.56) and (5.6) are subjected to the fixed face boundary condition at the upper surface of the layer given by

$$u_1^* = 0, \quad u_2^* = 0, \quad \text{at} \quad \zeta = 0, \tag{6.123}$$

and the traction free boundary condition at lower surface of the layer expressed as

$$\begin{aligned}
&t_1 = t_{13}\eta^3 + t_{12}\eta^2 + t_{11}\eta + t_{10} = 0, \quad t_2 = t_{23}\eta^3 + t_{22}\eta^2 + t_{21}\eta + t_{20} = 0, \quad \text{at} \quad \zeta = -1,\\
&t_{10} = \lambda p_t^*, \quad t_{20} = \lambda^2 p_t^*, \quad t_{11} = -\lambda^2 u_{1,\zeta}^*, \quad t_{21} = \lambda u_{1,\zeta}^*,\\
&t_{12} = \lambda(1-\lambda^2)u_{1,\xi}^* - \lambda(p+1)u_{2,\zeta}^*, \quad t_{22} = (\lambda^2 - 1)u_{1,\xi}^* - \lambda^2(p+1)u_{2,\zeta}^*,\\
&t_{13} = \left(1 - (1+p)\lambda^2\right)u_{2,\xi}^*, \quad t_{23} = \lambda\left((p+1) - \lambda^2\right)u_{2,\xi}^*.
\end{aligned} \tag{6.124}$$

The solutions of equations (6.56) and (5.6) subject to boundary conditions (6.123), (6.124) are sought in the form of series (4.11).

6.3.2 Leading order problem

At leading order, the non-dimensional equations of motion and associated incompressibility condition are given by (6.57) and subject to the following boundary conditions

$$u_1^{(0)} = 0, \quad u_2^{(0)} = 0, \quad \text{at} \quad \zeta = 0; \quad p_t^{(0)} = 0, \quad \text{at} \quad \zeta = -1. \tag{6.125}$$

From the equations of motion $(6.56)_1$, $(6.56)_2$ we deduce that

$$p_t^{(0)} = 0, \quad u_1^{(0)} = U_{1c}^{(0,0)} \cos\left(\frac{(2n+1)\pi\zeta}{2}\right) + U_{1s}^{(0,0)} \sin\left(\frac{(2n+1)\pi\zeta}{2}\right). \tag{6.126}$$

The incompressibility condition $(6.56)_3$ also provides the following relation

$$u_2^{(0)} = -\frac{2U_{1c,\xi}^{(0,0)}}{(2n+1)\pi} \sin\left(\frac{(2k+1)\pi\zeta}{2}\right) + \frac{2U_{1s,\xi}^{(0,0)}}{(2n+1)\pi} \cos\left(\frac{(2n+1)\pi\zeta}{2}\right) + v_2^{(0,0)}. \tag{6.127}$$

Taking into account the boundary conditions (6.125), together with the relations

$$\sin((2n+1)\pi/2) = (-1)^n, \quad \cos((2n+1)\pi/2) = 0, \tag{6.128}$$

we deduce that

$$U_{1c}^{(0,0)} = 0, \quad U_{2c}^{(0,0)} = \frac{2U_{1s,\xi}^{(0,0)}}{(2n+1)\pi}, \quad v_2 = -\frac{2U_{1s,\xi}^{(0,0)}}{(2n+1)\pi}. \tag{6.129}$$

As a result, the solution of the leading order problem is given by

$$u_1^{(0)} = U_{1s}^{(0,0)} \sin\left(\frac{(2n+1)\pi\zeta}{2}\right), \quad u_2^{(0)} = \frac{2U_{1s,\xi}^{(0,0)}}{(2k+1)\pi} \cos\left(\frac{(2n+1)\pi\zeta}{2}\right) - \frac{2U_{1s,\xi}^{(0,0)}}{(2n+1)\pi},$$
$$p_t^{(0)} = 0. \tag{6.130}$$

6.3.3 Second order problem

The second order problem consists of the non-dimensional equations of motion and associated incompressibility condition for which the equations are given by (6.63) and subject to boundary conditions

$$u_1^{(1)} = 0, \quad u_2^{(1)} = 0, \quad \text{at} \quad \zeta = 0; \quad p_t^{(1)} = \lambda u_{1,\zeta}^{(0)}, \quad \lambda p_t^{(1)} = -u_{1,\zeta}^{(0)} \quad \text{at} \quad \zeta = -1. \tag{6.131}$$

Employing the leading order solution (6.130) it is possible to deduce that at second order the incremental pressure $p_t^{(1)}$ is zero.

To facilitate our analysis we establish that the solution of equations (6.63) subject to boundary conditions (6.131) can be expressed by (6.64) with cut-off frequencies given by (3.39). Using representation (6.64), the equation of motion, incompressibility and boundary conditions take the form (6.65). The equations (6.63) and (6.131) yields a homogeneous system (6.66) to determine unknown functions in (6.64).

The equations of motion $(6.63)_1$ and $(6.63)_2$ provide the following relations

$$U_{1c}^{(1,1)} = 0, \tag{6.132}$$

$$(\lambda^2 - 1)U_{1s,\xi}^{(0,0)} + \lambda U_{1s}^{(1,1)} = 0, \tag{6.133}$$

$$v_1^{(1,1)} = 0, \tag{6.134}$$

$$v_1^{(0,1)} = 0. \tag{6.135}$$

The incompressibility condition $(6.63)_3$ may now be used to yield the following equations

$$(2n+1)\pi U_{2c}^{(1,1)} - 2U_{1s,\xi}^{(1,1)} = 0, \tag{6.136}$$
$$2U_{1s,\xi}^{(0,1)} - (2n+1)\pi U_{2c}^{(0,1)} + 2U_{2s}^{(1,1)} = 0, \tag{6.137}$$
$$(2n+1)\pi U_{2s}^{(1,1)} + 2U_{1c,\xi}^{(1,1)} = 0, \tag{6.138}$$
$$(2n+1)\pi U_{2s}^{(0,1)} + 2U_{2c}^{(1,1)} + 2U_{1c,\xi}^{(0,1)} = 0, \tag{6.139}$$
$$v_{1,\xi}^{(1,1)} = 0, \tag{6.140}$$
$$v_2^{(1,1)} + v_{1,\xi}^{(0,1)} = 0, \tag{6.141}$$

with the following additional equations coming from boundary conditions (6.131)

$$U_{1c}^{(0,1)} + v_1^{(0,1)} = 0, \tag{6.142}$$
$$U_{2c}^{(0,1)} + v_2^{(0,1)} = 0. \tag{6.143}$$

We employ Maple (1996) to solve the homogeneous system of equations (6.132)–(6.143). To conclude, at second order the displacement components and incremental pressure can be expressed as

$$\begin{aligned} u_1^{(1)} &= U_{1s}^{(0,1)} \sin\left(\frac{(2n+1)\pi\zeta}{2}\right) + U_{1s}^{(1,1)} \zeta \sin\left(\frac{(2n+1)\pi\zeta}{2}\right), \\ u_2^{(1)} &= U_{2c}^{(0,1)} \cos\left(\frac{(2n+1)\pi\zeta}{2}\right) + U_{2s}^{(0,1)} \sin\left(\frac{(2n+1)\pi\zeta}{2}\right) + \\ &\quad + U_{2c}^{(1,1)} \zeta \cos\left(\frac{(2n+1)\pi\zeta}{2}\right) + v_2^{(0,1)}, \quad p_t^{(1)} = 0, \end{aligned} \tag{6.144}$$

where the non-zero coefficients are given by

$$\begin{aligned} U_{1s}^{(1,1)} &= \frac{U_{1s,\xi}^{(0,0)}(1-\lambda^2)}{\lambda}, \quad U_{2c}^{(0,1)} = \frac{2U_{1s,\xi}^{(0,1)}}{(2n+1)\pi}, \\ U_{2s}^{(0,1)} &= \frac{4U_{1s,\xi\xi}^{(0,0)}\left(\lambda^2-1\right)}{\lambda\,(2n+1)^2\pi^2}, \quad U_{2c}^{(1,1)} = \frac{2U_{1s,\xi\xi}^{(0,0)}\left(1-\lambda^2\right)}{\lambda(2n+1)\pi}, \quad v_2^{(0,1)} = -\frac{2U_{1s,\xi}^{(0,1)}}{(2n+1)\pi}. \end{aligned} \tag{6.145}$$

6.3.4 Third order problem

At third order, the non-dimensional equations of motion and associated incompressibility condition are given by (6.84) and subject to the following boundary conditions

$$\begin{aligned} &u_1^{(2)} = 0, \quad u_2^{(2)} = 0, \quad \text{at} \quad \zeta = 0, \\ &p_t^{(2)} = \left(1-\lambda^2\right) u_{1,\xi}^{(0)} - (p+1)\, u_{2,\zeta}^{(0)} - \lambda u_{1,\zeta}^{(1)}, \quad \text{at} \quad \zeta = -1, \\ &\lambda^2 p_t^{(2)} = \left(1-\lambda^2\right) u_{1,\xi}^{(0)} + \lambda^2\,(p+1)\, u_{2,\zeta}^{(0)} - \lambda\, u_{1,\zeta}^{(1)}, \quad \text{at} \quad \zeta = -1. \end{aligned} \tag{6.146}$$

We remark that to determine third order solution, and derive the governing equation for long wave amplitude $U_{1s}^{(0,0)}$, it is necessary to consider at fourth order one equation of motion and boundary conditions at the lower surface of the layer. These fourth order boundary conditions consist of the in-plane fourth order traction component being equal to zero at the lower surface of the layer $\zeta = -1$, thus

$$\lambda^2 u_{1,\zeta}^{(2)} - \lambda p_t^{(3)} = \left(1 - \lambda^2(p+1)\right) u_{2,\xi}^{(0)} + \lambda \left(1 - \lambda^2\right) u_{1,\xi}^{(1)} - \lambda \left(p+1\right) u_{2,\zeta}^{(1)}, \tag{6.147}$$

with the normal fourth order traction component vanishing at $\zeta = -1$, i.e.

$$\lambda\, u_{1,\zeta}^{(2)} + \lambda^2 p_t^{(3)} = \lambda \left(\lambda^2 - p - 1\right) u_{2,\xi}^{(0)} - \left(\lambda^2 - 1\right) u_{1,\xi}^{(1)} + \lambda^2 \left(p+1\right) u_{2,\zeta}^{(1)}. \tag{6.148}$$

The fourth order equation of motion can be expressed as

$$\begin{aligned} p_{t,\zeta}^{(3)} = & \frac{\left(\lambda^2(\lambda^2-1) + p\lambda^2 + 1\right)}{\lambda} u_{1,\xi\xi}^{(0)} + \frac{\left(p\lambda^2 - 2(1-\lambda^2)\right)}{\lambda} u_{2,\xi\zeta}^{(0)} + \\ & + \left(2(\lambda^2-1) + p\right) u_{1,\xi\zeta}^{(1)} + (1+p)\, u_{2,\zeta\zeta}^{(1)} + \frac{(2n+1)^2\pi^2}{4} u_2^{(1)} - \\ & - \lambda W_1^{(0)} - \lambda p_{t,\xi}^{(2)} + \lambda u_{1,\zeta\zeta}^{(2)} + \frac{(2n+1)^2\pi^2\lambda}{4} u_1^{(2)}. \end{aligned} \tag{6.149}$$

The above equation (6.149) with the help of leading order solution (6.130) and second order solution (6.144) yields the following representation for the fourth order incremental pressure

$$p_t^{(3)} = -\frac{(2n+1)\pi}{2} U_{1s,\xi}^{(0,1)} \zeta + P_t^{(0,3)}. \tag{6.150}$$

Employing the equation of motion $(6.84)_1$ and boundary conditions $(6.146)_2$, $(6.146)_3$ we derive the following relation

$$\begin{aligned} p_{t,\zeta}^{(2)} = & \frac{(2n+1)^2\pi^2}{4} u_2^{(0)} + \left(p + 2(\lambda^2-1)\right) u_{1,\xi\zeta}^{(0)} + (p+1)\, u_{2,\zeta\zeta}^{(0)} + \\ & + \frac{\lambda(2n+1)^2\pi^2}{4} u_1^{(1)} - \lambda p_{t,\xi}^{(1)} + \lambda\, u_{1,\zeta\zeta}^{(1)}. \end{aligned} \tag{6.151}$$

Using the relation (6.151) together with the leading order solution (6.130) and second order solution (6.144), we derive the following representation for the incremental pressure $p_t^{(2)}$

$$p_t^{(2)} = -\frac{(2n+1)\pi}{2} U_{1s,\xi}^{(0,0)} \zeta + U_{1s,\xi}^{(0,0)} \left((-1)^n \left(1+p\right) - \frac{(2n+1)\pi}{2} \right). \tag{6.152}$$

To simplify our analysis we note that the solution of equations (6.84), subject to boundary conditions (6.146), is given by (6.87) with the cut-off frequencies Ω given by (3.39). Employing the expressions (6.87) and (6.86) the equations (6.84) can be expressed in the form (6.88). The equations of motion $(6.84)_1$ and $(6.84)_2$ yield the following relations

$$U_{1c}^{(2,2)} = 0, \tag{6.153}$$

$$(\lambda^4 - 3\lambda^2 + 1)U_{1s,\xi\xi}^{(0,0)} - 2\,\lambda^2 U_{1s}^{(2,2)} + \lambda^2(2n+1)\pi U_{1c}^{(1,2)} + \lambda^2 W_1^{(0)} = 0, \tag{6.154}$$

$$2\,\lambda^2 U_{1s}^{(2,2)} - (\lambda^2 - 1)^2 U_{1s,\xi\xi}^{(0,0)} = 0, \tag{6.155}$$

$$((2n+1)\pi\lambda^2 - 2)U_{1s,\xi}^{(0,1)} + \lambda(2n+1)\pi U_{1s}^{(1,2)} + 2\lambda U_{1c}^{(2,2)} = 0, \tag{6.156}$$

$$v_1^{(2,2)} = 0, \tag{6.157}$$

$$(2n+1)\pi v_1^{(1,2)} + 2U_{1s,\xi\xi}^{(0,0)} = 0, \tag{6.158}$$

$$8v_1^{(2,2)} + (2n+1)^2\pi^2 v_1^{(0,2)} + 2((2n+1)\pi - 2(p+1)\,(-1)^n)U_{1s,\xi\xi}^{(0,0)} = 0, \tag{6.159}$$

with the incompressibility condition $(6.84)_3$ providing

$$(2n+1)\pi U_{2c}^{(2,2)} - 2U_{1s,\xi}^{(2,2)} = 0, \tag{6.160}$$

$$4U_{2s}^{(2,2)} - (2n+1)\pi U_{2c}^{(1,2)} + 2U_{1s,\xi}^{(1,2)} = 0, \tag{6.161}$$

$$2U_{1s,\xi}^{(0,2)} + 2U_{2s}^{(1,2)} - (2n+1)\pi U_{2c}^{(0,2)} = 0, \tag{6.162}$$

$$2U_{1c,\xi}^{(2,2)} + (2n+1)\pi U_{2s}^{(2,2)} = 0, \tag{6.163}$$

$$(2n+1)\pi U_{2s}^{(1,2)} + 4U_{2c}^{(2,2)} + 2U_{1c,\xi}^{(1,2)} = 0, \tag{6.164}$$

$$2U_{1c,\xi}^{(0,2)} + (2n+1)\pi U_{2s}^{(0,2)} + 2U_{2c}^{(1,2)} = 0, \tag{6.165}$$

$$v_{1,\xi}^{(2,2)} + 3v_2^{(3,2)} = 0, \tag{6.166}$$

$$2v_2^{(2,2)} + v_{1,\xi}^{(1,2)} = 0, \tag{6.167}$$

$$v_2^{(1,2)} + v_{1,\xi}^{(0,2)} = 0, \tag{6.168}$$

and the boundary conditions (6.146) yielding

$$U_{1c}^{(0,2)} + v_1^{(0,2)} = 0, \tag{6.169}$$

$$U_{2c}^{(0,2)} + v_2^{(0,2)} = 0. \tag{6.170}$$

The boundary conditions (6.147) and (6.147) yields the following equations

$$
\begin{aligned}
&4((2n+1)\pi(-1)^n(\lambda^2+1+p)+2(p(\lambda^2+1)-1))U_{1s,\xi\xi}^{(0,0)}\\
&+\lambda^2(2n+1)\pi((2n+1)^2\pi^2-4)v_1^{(1,2)}+4(2n+1)\pi\lambda P_t^{(0,3)}\\
&+2(2n+1)\pi(2(-1)^n(\lambda(\lambda^2-2)-p)+(2n+1)\pi)U_{1s,\xi}^{(0,1)}-\lambda^2(2n+1)^3\pi^3v_1^{(2,2)}\\
&+4\lambda^2(2n+1)\pi\,(-1)^n\,U_{1s}^{(1,2)}-\lambda^2(2n+1)^3\pi^3v_1^{(0,2)}-2\lambda^2(2n+1)^2\pi^2\,(-1)^n\,U_{1c}^{(1,2)}\\
&-4(2n+1)\pi\lambda^2\,(-1)^n\,W_1^{(0)}+6\,(-1)^k\,\lambda^2(2n+1)^2\pi^2U_{1c}^{(2,2)}\\
&-2\lambda^2(2n+1)^2\pi^2\,(-1)^n\,U_{1c}^{(0,2)}=0,
\end{aligned}
\tag{6.171}
$$

$$
\begin{aligned}
&4((2n+1)\pi(-1)^n(\lambda^2(p+2)-1-\lambda^4(\lambda^2-2))+2\lambda^2(\lambda^2-(p+1)))U_{1s,\xi\xi}^{(0,0)}\\
&+2\lambda^2(2n+1)^2\pi^2(-1)^n(4\lambda^2+1)U_{1c}^{(2,2)}+4(-1)^n(2n+1)\pi\lambda^2(2(\lambda^2+1)-1)U_{1s}^{(1,2)}\\
&+2(2n+1)\pi\lambda(\lambda^2\pi(2n+1)-2(-1)^n(\lambda^2-1))U_{1s,\xi}^{(0,1)}\\
&+\lambda^2(2n+1)\pi(8(1-\lambda^2)-\lambda^2(2n+1)^2\pi^2)v_1^{(2,2)}\\
&-4\lambda^4(2n+1)\pi W_1^{(0)}\,(-1)^n+4\lambda^2(2n+1)\pi v_1^{(1,2)}\\
&+4\lambda^3(2n+1)\pi P_t^{(0,3)}-4\,(-1)^n\,\lambda^4(2n+1)^2\pi^2U_{1c}^{(1,2)}-8\,(-1)^n\,(2n+1)\pi\lambda^3U_{1s,\xi}^{(0,1)}\\
&+\lambda^4(2n+1)^3\pi^3v_1^{(1,2)}-\lambda^4(2n+1)^3\pi^3v_1^{(0,2)}=0.
\end{aligned}
\tag{6.172}
$$

Solving equations (6.153)–(6.170), with the help of Maple (1996), we determine the coefficients in the representation (6.87). As a result, the solution of third order problem can be represented as

$$
\begin{aligned}
u_1^{(2)}&=U_{1c}^{(0,2)}\cos\left(\frac{(2n+1)\pi\zeta}{2}\right)+U_{1s}^{(0,2)}\sin\left(\frac{(2n+1)\pi\zeta}{2}\zeta\right)+U_{1c}^{(1,2)}\zeta\cos\left(\frac{(2n+1)\pi\zeta}{2}\right)\\
&+U_{1s}^{(1,2)}\zeta\sin\left(\frac{(2n+1)\pi\zeta}{2}\right)+U_{1s}^{(2,2)}\zeta^2\sin\left(\frac{(2n+1)\pi\zeta}{2}\right)+v_1^{(0,2)}+v_1^{(1,2)}\zeta,\\
u_2^{(2)}&=U_{2c}^{(0,2)}\cos\left(\frac{(2n+1)\pi\zeta}{2}\right)+U_{2s}^{(0,2)}\sin\left(\frac{(2n+1)\pi\zeta}{2}\right)+U_{2c}^{(1,2)}\zeta\cos\left(\frac{(2n+1)\pi\zeta}{2}\right)\\
&+U_{2s}^{(1,2)}\zeta\sin\left(\frac{(2n+1)\pi\zeta}{2}\right)+U_{2c}^{(2,2)}\zeta^2\cos\left(\frac{(2n+1)\pi\zeta}{2}\right)+v_2^{(0,2)}+v_2^{(1,2)}\zeta+v_2^{(2,2)}\zeta^2,\\
p_t^{(2)}&=P_t^{(0,2)}+P_t^{(1,2)}\zeta,
\end{aligned}
\tag{6.173}
$$

where the non-zero coefficients are represented by

$$
\begin{aligned}
U_{1c}^{(0,2)} &= \frac{2U_{1s,\xi\xi}^{(0,0)}\left((2n+1)\pi - 2\left(-1\right)^n (p+1)\right)}{(2n+1)^2\pi^2},\\
U_{1c}^{(1,2)} &= \frac{2U_{1s,\xi\xi}^{(0,0)}\left((2n+1)\pi - 4\left(-1\right)^n (p+1)\right)}{(2n+1)^2\pi^2},\\
U_{1s}^{(1,2)} &= \frac{U_{1s,\xi}^{(0,1)}\left(1-\lambda^2\right)}{\lambda}, \quad U_{1s}^{(2,2)} = \frac{U_{1s,\xi\xi}^{(0,0)}\left(\lambda^2-1\right)^2}{2\lambda^2},\\
v_1^{(0,2)} &= \frac{2U_{1s,\xi\xi}^{(0,0)}\left(2\left(-1\right)^n (p+1) - (2n+1)\pi\right)}{(2n+1)^2\pi^2}, \quad v_1^{(1,2)} = -\frac{2U_{1s,\xi\xi}^{(0,0)}}{(2n+1)\pi},
\end{aligned}
\tag{6.174}
$$

$$
\begin{aligned}
U_{2c}^{(0,2)} &= 2\frac{U_{1s,\xi}^{(0,2)}((2n+1)^3\pi^3\lambda^2 + 4U_{1s,\xi\xi\xi}^{(0,0)}\left((2n+1)\pi\left(\lambda^2(1-\lambda^2)-1\right) + 4\left(-1\right)^n \lambda^2 (p+1)\right)}{(2n+1)^4\pi^4\lambda^2},\\
U_{2s}^{(0,2)} &= 4\frac{U_{1s,\xi\xi\xi}^{(0,0)}\lambda\left(2\left(-1\right)^n (p+1) - (2n+1)\pi\right) + U_{1s,\xi\xi}^{(0,1)}(2n+1)\pi\left(\lambda^2-1\right)}{(2n+1)^3\pi^3\lambda},\\
U_{2s}^{(1,2)} &= 4\frac{U_{1s,\xi\xi\xi}^{(0,0)}\left(4\left(-1\right)^n \lambda^2 (p+1) - (2n+1)\pi\left(\lambda^2(\lambda^2-1)+1\right)\right)}{(2n+1)^3\pi^3\lambda^2},\\
U_{2c}^{(1,2)} &= \frac{2U_{1s,\xi\xi}^{(0,1)}\left(1-\lambda^2\right)}{\lambda(2n+1)\pi}, \quad U_{2c}^{(2,2)} = \frac{U_{1s,\xi\xi\xi}^{(0,0)}(\lambda^2-1)^2}{\lambda^2(2n+1)\pi},\\
v_2^{(0,2)} &= 2\frac{U_{1s,\xi}^{(0,2)}(2n+1)^3\pi^3\lambda^2 + 4U_{1s,\xi\xi\xi}^{(0,0)}\left((2n+1)\pi\left(\lambda^2(\lambda^2-1)+1\right) - 4\left(-1\right)^n \lambda^2 (p+1)\right)}{(2n+1)^4\pi^4\lambda^2},\\
v_2^{(1,2)} &= \frac{2U_{1s,\xi\xi\xi}^{(0,0)}\left((2n+1)\pi - 2\left(-1\right)^n (p+1)\right)}{(2n+1)^2\pi^2}, \quad v_2^{(2,2)} = \frac{2U_{1s,\xi\xi\xi}^{(0,0)}}{(2n+1)\pi},
\end{aligned}
\tag{6.175}
$$

$$
P_t^{(1,2)} = -\frac{U_{1s,\xi}^{(0,0)}(2n+1)\pi}{2}, \quad P_t^{(0,2)} = \frac{1}{2}U_{1s,\xi}^{(0,0)}\left(2\left(-1\right)^n (1+p) - (2n+1)\pi\right). \tag{6.176}
$$

We remark that the functions $U_{1s}^{(0,1)}$ and $U_{1s}^{(0,2)}$ remain undefined at third order. However, at third order we are able to obtain the following representations for the function $P_t^{(0,3)}$

$$
\begin{aligned}
P_t^{(0,3)} = &\frac{U_{1s,\xi\xi}^{(0,0)}\left(2(1-\lambda^2) + \left(-1\right)^n (2n+1)\pi(\lambda^2+1-(p+1))\right)}{\lambda(2n+1)\pi} +\\
&+ \frac{1}{2}U_{1s,\xi}^{(0,1)}\left(2(-1)^n(p+1) - (2n+1)\pi\right).
\end{aligned}
\tag{6.177}
$$

Hence, the fourth order pressure increment (6.150) can be expressed as

$$
\begin{aligned}
p_t^{(3)} = &-\frac{(2n+1)\pi}{2}U_{1s,\xi}^{(0,1)}\zeta + \frac{U_{1s,\xi\xi}^{(0,0)}(2(1-\lambda^2) + \left(-1\right)^n (2n+1)\pi(\lambda^2+1-(p+1)))}{\lambda(2n+1)\pi} +\\
&+ \frac{1}{2}U_{1s,\xi}^{(0,1)}\left(2(-1)^n(p+1) - (2n+1)\pi\right).
\end{aligned}
\tag{6.178}
$$

Now we are ready to derive the governing equation for the long wave amplitude $U_{1s}^{(0,0)}$. The function $W_1^{(0)}$ can be represented in the following form

$$W_1^{(0)} = \eta^{-2}(u_{1,\tau\tau}^{(0)} + \frac{(2n+1)^2\pi^2}{4} u_1^{(0)}) =$$
$$= \eta^{-2}(U_{1s,\tau\tau}^{(0,0)} + \frac{(2n+1)^2\pi^2}{4} U_{1s}^{(0,0)}) = U_{1s,\xi\xi}^{(0,0)} \left(\frac{8\,(-1)^n\,(p+1) - (2n+1)\pi}{(2n+1)\pi} \right). \tag{6.179}$$

The above equation yields the following non-dimensional governing equation for long wave amplitude $U_{1s}^{(0,0)}$

$$\eta^{-2} \left(U_{1s,\tau\tau}^{(0,0)} + \frac{(2n+1)^2\pi^2}{4} U_{1s}^{(0,0)} \right) - \left(\frac{8\ (-1)^n\,(p+1) - (2n+1)\pi}{(2n+1)\pi} \right) u_{1s,\xi\xi}^{(0,0)} = 0. \tag{6.180}$$

We remark here that the nature of the governing equation (6.180) is dependent on the following quantity

$$\mathcal{C}_e = \frac{8\ (-1)^n\,(p+1) - (2n+1)\pi}{(2n+1)\pi}, \tag{6.181}$$

being hyperbolic or elliptic according to $\mathcal{C}_e > 0$ or $\mathcal{C}_e < 0$. The governing equation (6.180) can be represented in original non-scaled variables as

$$\frac{\rho h^2}{\mu} \frac{\partial^2 U_{1s}^{(0,0)}}{\partial^2 t^2} + \frac{(2n+1)^2\pi^2}{4} U_{1s}^{(0,0)} - h^2 \left(\frac{8\ (-1)^n\,(p+1) - (2n+1)\pi}{(2n+1)\pi} \right) \frac{\partial^2 U_{1s}^{(0,0)}}{\partial^2 x_1^2} = 0. \tag{6.182}$$

If we specify the function $U_{1s}^{(0,0)}$ in the form (1.72) and substitute it into the governing equation (6.182) we obtain the following expansion for squared non-dimensional frequency

$$\Omega^2 = \frac{(2n+1)^2\pi^2}{4} + \mathcal{C}_e\eta^2 + O(\eta^4). \tag{6.183}$$

The approximation (3.42) can be represented in a form

$$\Omega^2 = \frac{(2n+1)^2\pi^2}{4} + (2n+1)\pi\Omega_{mh}^{(2)}\eta^2 + O(\eta^4), \tag{6.184}$$

where second order corrections $\Omega_{mh}^{(2)}$ are given by (3.41). The fact that expansions (6.183) and (6.184) coincide proves the consistency of the derived dynamic one-dimensional model with asymptotic analysis of dispersion relation (3.9).

Conclusion

The aim of the conclusion is to summarize the novel aspects of this study. In this thesis we consider wave propagation in an incompressible elastic layer subject to primary simple shear deformation. An interesting feature of a simple shear deformation is that no principal axes is normal to faces of a layer or a half space. The main results of this thesis are an asymptotic analysis of dispersion relations and a derivation of simplified dynamic models for the two dimensional long wave motion. All these aspects are novel in comparison to the study by Connor and Ogden (1996) which is restricted to stability analysis and numerical investigation of dispersion for the neo-Hookean material. We generalize results by Connor and Ogden (1996) for any material model and investigate numerically and analytically effects of different boundary conditions. In particular, we perform numerical and analytical analysis of fixed faces boundary value problem which was not considered in Connor and Ogden (1996) .

Most importantly, in contract with the study by Connor and Ogden (1996) we demonstrate the alternative approach to investigate the propagation of travelling waves. The main advantages of our approach are the derivation of all the governing equations in a layer coordinates and the explicit representation of hydrostatic pressure. With the help of our approach we construct simplified dynamic models which are consistent with the analysis of corresponding dispersion relation and previously published results. The lower dimensional dynamic models with minimal number of essential parameters provide significant simplification to describe two dimensional motion in terms of one dimensional governing equation. Also analytical solutions of incremental pressure, in-plane and normal displacements components are derived considering three asymptotic orders of the problem. Finally we remark that investigation of dispersive waves performed in this thesis is significantly different from the previous works by Connor and Ogden (1995) and Destrade and Ogden (2005) devoted to analysis of surface waves in the equivalent configuration.

In this thesis we consider long wave low and high frequency motion a layer with free, fixed and one fixed one free face boundary conditions. In comparison to an isotropic and a pure homogeneous strain cases motion cannot be decomposed into symmetric and antisymmetric components. We perform the asymptotic analysis of dispersion relation and derive the relative asymptotic orders of displacements and incremental pressure. For the long wave high frequency motion the results are broadly similar to those previously published in regards to pure homogeneous strain, see Kaplunov et al. (2002). However, the results for long wave low frequency motion, which exist only in a layer with free faces, show a significant departure from the previous investigations. In this case it is not possible to establish analogue of classical extension and bending. The reason for that is that whenever the amount of shear is finite, within the long wave low frequency regime both in-plane and normal displacement components have the same asymptotic order. This contrasts with the classical case of extension and bending, see Kaplunov et al. (1998) together with their homogeneously strained counterparts, for which the in-plane and normal displacement component is asymptotically leading, see Kaplunov et al. (2000).

Based on knowledge of relative asymptotic orders of displacements lower dimensional dynamic models with minimal essential parameters were constructed to elucidate long wave low and high frequency motion. In all constructed dynamic models the governing equations were derived only at the third order problem seemingly due to the lack of symmetry in considered system. This is the novel aspect of all derived models in comparison to previously published results by Kaplunov et al (2000, 2002) and Pichugin and Rogerson (2001, 2002). In addition the influence of mixed boundary conditions on the derivation of the long wave high frequency dynamic model was investigated. In respect of the other long wave high frequency models the governing equations are of the same form and consistent with previously published results by Kaplunov et al. (2002) and Pichugun and Rogerson (2001).

In the long wave low frequency dynamic model derived for neo-Nookean material and then generalized the governing equation is of novel type. The one dimensional vector governing equations involves the second derivatives in respect to in-plane coordinate and time of two essential parameters long wave amplitudes of in-plane and normal displacement components. Such equation arises because both the in-plane and normal displacement components are of the same asymptotic order. Hence this type of governing equation is the consequence of the simple shear deformation. The facts that the governing equation is of vector type and there are two essential parameters in the dynamic model are significant novel aspects in comparison to previously derived long wave low frequency dynamic models in respect of a pure homogeneous strain by Kaplunov et al. (2000) and Pichugun and Rogerson (2002).

Bibliography

Amirova, S. R. & Rogerson, G. A. (2008) The influence of simple shear deformation on long-wave motion in a pre-stressed incompressible elastic layer. *Journal of Mechanics of Materials and Structures*, **3**, 831-851.

Beatty, M. F. (1987) Topics in finite elasticity: hyperelasticity of rubber, elastomers and biological tissues-with examples. *Appl. Mech. Reviews*, **40**, 1699-1734.

Beklemishev, D. V. (1971) *The course of analytical geometry and linear algebra.* Nauka, Moscow.

Biot, M. A. (1965) *Mechanics of incremental deformations: theory of elasticity and viscoelasticity of initially stressed solids and fluids, including thermodynamic foundations and applications to finite strain.* New York, London, Sidney, John Wiley and Sons Inc.

Boulanger, P. H. & Hayes, M. (2002) A note on maximum shear. *J. Elasticity*, **69**, 215-222.

Boulanger, P. H. & Hayes, M. (2007) Consequences of a result on pure shear. *Int. J. Nonlinear Mech.*, **42**, 376-380.

Boulanger, P. H. & Hayes, M. (2000) On finite shear. *1 Arch. Ration. Mech. An.*, **151**, 125-185.

Boulanger, P. H. & Hayes, M. (2004) On pure shear. *J. Elasticity*, **77**, 83-89.

Boulanger, P. H. & Hayes, M. (1997) Wave propagation in sheared rubber. *Acta Mech.*, **122**, 75-87.

Chadwick, P. (1999) *Continuum mechanics: Concise Theory and Problems.* Dover Publications, New York.

Connor, P. & Ogden, R. W. (1995) The effect of shear on the propagation of elastic surface waves. *Int. J. Engng Sci.*, **33**, 973-982.

Connor, P. & Ogden, R. W. (1996) The influence of shear strain and hydrostatic stress on stability and elastic waves in a layer. *Int. J. Engng Sci.*, **34**, 375-397.

Destrade, M. & Ogden, R. W. (2005) Surface waves in a stretched and sheared incompressible material. *Int. J.Non-Linear Mech.*, **40**, 241-253.

Dowaikh, M. A. & Ogden, R. W. (1990) On the surface waves and deformations in a pre-stressed incompressible elastic solid. *IMA J. of Appl. Math*, **44**, 261-284.

Fang, N., Xi, D. J. & Xu, J. Y. (2006) Ultrasonic metamaterials with negative modulus. *Nature Materials*, **5**, 452-456.

Fu, Y. B. (2005) An explicit expression for the surface-impedance matrix of a generally anisotropic incompressible elastic material in a state of plain strain. *Int. J.Non-Linear Mech*, **40**, 229-239.

Fu, Y. B. (2007) Linear and nonlinear wave propagation in coated or uncoated elastic half-spaces, in M. Destrade and G. Saccomandi (Eds.) *Waves in nonlinear pre-stressed materials, CISM Courses and Lectures*, **495**, 103-127.

Fu, Y. B. & Ogden, R. W. (1999) Nonlinear analysis of pre-stressed elastic bodies. *Continuum Mech Therm*, **11**, 141-172.

Furness, D. N., Doborah, E., Zetes, C., Hackney, M. & Steele, C. R. (1997) Kinematic analysis of shear displacement as a means for operating mechanotransduction channels in the contact region between adjacent stereocilia of mammalian cochlear hair cells. *Proc. R. Soc. Lond.*, **B 264**, 45-51.

Gasser, T. C., Ogden, R. W. & Holzapfel, G. A. (2006) Hyperelastic modelling of arterial layers with distributed collagen fibre orientations. *1 J. R. Soc. Interface*, **3**, 15-35.

Gere, J. M. & Timoshenko, S. P. (1991) *Mechanics of materials.* Chapman and Hall, London.

Gessner, K., Wijns, C. & Moresi, L. (2007) Significance of strain localization in the lower crust for structural evolution and thermal history of metamorphic core complexes. *Tectonics*, **26**, TC2012.

Goldenveizer, A. L., Kaplunov, J. D. & Nolde, E. V. (1993) On Timoshenko-Reissner type theories of plates and shells. *Int. J. Solids Struct.*, **30**, 675-694.

Graff, K. F. (1991) *Wave motion in elastic solids.* University Press, Oxford, Dover Edition.

Green, A. E. & Adkins, J. E. (1960) *Large elastic deformations.* Oxford, Clarendon Press.

Green, A. E., Rivlin, R. S. & Shield, R. T. (1952) General theory of small elastic deformations superimposed on finite elastic deformations. *Proc. R. Soc. Lond.*, **A 211**, 128-154.

Green, A. E. & Zerna, W. (1954) *Theoretical Elasticity.* Oxford, Clarendon Press.

Hard, P. & Toll, S. (2008) Open-open cellular solids: A constitutive equation for hyperelasticity with deformation induced anisotropy. *Int.J. Solids Structures*, **45**, 1978-1992.

Hirst, A. J. (1969) *Rubber suspensions in transport*, in L.R.Mernagh (Ed.), *Rubber handbook*, 79-86, West Wickham, Morgan-Grampian Ltd.

Holzapfel, G. A. & Ogden, R. W. (2006) Mechanics of Biological Tissue. *Proceedings of the IUTAM Symposium*, Graz, Austria, Springer.

Hussain, W. & Odgen, R. W. (1999) On the reflection of plane waves at the boundary of an elastic half-space subject to simple shear. *Int. J. Engng Sci.*, **37**, 1549-1576.

Ide, S., Beroza, G. C., Shelly, D. R. & Uchide, T. (2007) A scaling law for slow earthquakes. *Nature*, **447**, 76-79.

Kaplunov, J. D. & Markushevich, D. G. (1993) Plane vibrations and radiation of an elastic layer lying on a liquid half-space. *Wave Motion*, **17**, 199-211.

Kaplunov, J. D., Kossovich, L. Y. & Nolde, E. V. (1998) *Dynamics of thin walled elastic bodies.* Academic Press.

Kaplunov, J. D. & Nolde, E. V. (2002) Long wave vibrations of nearly incompressible isotropic plate with fixed faces. *Q. J. Mech. Appl. Math.*, **55**, 345-356.

Kaplunov, J. D., Nolde, E. V. & Rogerson, G. A. (2000) A low frequency model for dynamic motion in pre-stressed incompressible elastic structures. *Proc. R. Soc. Lond.*, **A 456**, 2589-2610.

Kaplunov, J. D., Nolde, E. V. & Rogerson, G. A. (2002) An asymptotically consistent model for long wave high frequency motion in a pre-stressed elastic plate. *Math Mech Solids*, **7**, 581-606.

Korn, G. A. & Korn, T. M. (1968) *Mathematical handbook for scientists and engineers, definitions, theorems and formulas for reference and review*, McGraw-Hill Book Company, New York.

Kossovitch, L. Y., Moukhomodiarov, R. R. & Rogerson, G. A. (2002) Long wave asymptotic integration in incompressible transversely isotropic elastic structures. *Acta Mech.*, **159**, 53-64.

Kossovitch, L. Y., Moukhomodiarov, R. R. & Rogerson, G. A. (2006) Symmetric low-frequency motion in incompressible transversely isotropic elastic plates. *Mechanics Research Communications*, **33**, 460-469.

Lamb, H. (1917) On waves in an elastic plate. *Proc. R. Soc. Lond.*, **A 93**, 114-128.

Liang, C., Lippman, H. & Nazar, H. (1993) Effects of artificially induced vibrations on the prevention of coal mine bumps. *Proceedings of the 3rd International Symposium on Rockbursts and Seismicity in Mines*, 91-94, Balkema.

Love, A. E. H. (1927) *A treatise on the mathematical theory of elasticity*, Cambridge University Press.

Lord Rayleigh (1877) *The theory of sound*. London, Macmillan and Co.

Lord Rayleigh (1885) On waves propagated along the plane surface of an elastic solid. *Proc. Lond. Math. Soc.*, **17**, 4-11.

Lord Rayleigh (1889) On the free vibrations of an infinite plate of homogeneous isotropic elastic matter. *Proc. Lond. Math. Soc.*, **20**, 225-234.

Manoussaki, D., Dimitriadis, E. K. & Chadwick, R. S. (2006) Cochlea's Graded Curvature Effect on Low Frequency Waves. *Phys Rev Lett* , **96**, 088701.

Mindlin, R. D. (1960) *Waves and vibrations in isotropic elastic plates*, in J. N. Goodier and N. Hoff (Eds.), *Structural mechanics*, Pergamont Press, Oxford.

Nolde, E. V. (2007) Qualitative analysis of initial-value problems for a thin elastic strip. *IMA J.Appl.Math.*, **72**, 348-375.

Nolde, E. V., Prikazchikova, L. A. & Rogerson, G. A. (2004) Dispersion of small amplitude waves in a pre-stressed compressible elastic plate. *J. Elasticity*, **75**, 1-29.

Nolde, E. V. & Rogerson, G. A. (2002) Long wave asymptotic integration for the governing equations for a pre-stressed incompressible elastic layer with fixed faces. *Wave motion*, **36**, 287-304.

Ogden, R. W. (1984) *Non-linear Elastic Deformations.* Ellis Horwood, Chichester.

Ogden, R. W. (2002) *Nonlinear elasticity with application to material modelling.* Lecture course, Warsaw.

Ogden, R. W., Gutkowski, W. & Kowalewski, T. A. (2005) Mechanics of rubberlike solids, Mechanics of the 21st Century. *Proceedings of the 21st ICTAM*, Warsaw, 263-274, Springer, Dordrecht.

Odgen, R. W. & Roxburgh, D. G. (1993) The effect of pre-stress on the vibration and stability of elastic plates. *Int. J. Eng. Sci.*, **31**, 1611-1639.

Picu, R. C. (2002) On the functional form of non-local elasticity kernels. *J. Mech. Phys. Solids*, **50**, 1923-1939.

Pichugin, A. V. & Rogerson, G. A. (2002) An asymptotic membrane-like theory for long-wave motion in a pre-stressed elastic plate. *Proc. R. Soc. Lond.*, **A 458**, 1447-1468.

Pichugin, A. V. & Rogerson, G. A. (2001) A two-dimensional model for extensional motion of a pre-stressed incompressible elastic layer near cut-off frequencies. *IMA J. Appl. Math.*, **66**, 357-385.

Prendergast, J. (1995) Seismic isolation in bridges. *ASCE Civil Engineering*, **65**, 58-61.

Prikazchikova, L. A. (2004) Small amplitude waves in a pre-stressed compressible elastic plate. *Ph.D. Thesis*, University of Salford, UK.

Reissner, E. (1944) On the derivation of theory of thin elastic shells. *J. Math. Phys.*, **43**, 263-277.

Rivlin, R. S. & Sanders, D. W. (1951) Large elastic deformation of isotropic materials VII. Experiments on the deformation of rubber. *Phil. Trans. R. Soc. Lond.*, **A 243**, 251-288.

Rogerson, G. A. (1992) Penetration of impact waves in six-ply fibre composite laminate. *J Sound Vib*, **158**, 105-120.

Rogerson, G. A. (1997) Some asymptotic expansions of the dispersion relation for an incompressible elastic plate. *Int. J. Solid Structures*, **34**, 2785-2802.

Rogersion, G. A. & Fu, Y. B. (1995) An asymptotic analysis of the dispersion relation of a pre-stressed incompressible elastic plate. *Acta Mech*, **111**, 59-77.

Rogerson, G. A. (1998) On the existence of surface waves and the propagation of plate waves in pre-stressed fibre-reinforced composites. *J. Mech. Phys. Solids*, **46**, 1581-1612.

Rogerson, G. A. & Sandiford, K. J. (2000) Asymptotic representations of the dispersion relation for a two-ply, pre-stressed incompressible elastic laminate. *Acta Mech.*, **143**, 179-201.

Rogerson, G. A. & Amirova, S. R. (2008) Long wave dispersion in a neo-Hookean layer subject to simple shear. *3rd Canadian Conference on Nonlinear Solid Mechanics, CanCNSM*, 85-92, University of Toronto, Ontario, Canada (2008).

Roxburgh, D. G. & Ogden, R. W. (1994) Stability and vibration of pre-stressed compressible elastic plates. *Int. J. Engng. Sci.* **32**, 427-454.

Sandiford, K. J. & Rogerson, G. A. (2000) Some dynamic properties of a pre-stressed, nearly incompressible (rubber-like) elastic material. *Int. J. Non-linear Mech.*, **35**, 849-868.

Sheridan, P. M., James, F. O. & Miller, T. S. (1992) Design of components, in A.N. Gent. *Engineering with rubber*, 209-235, Munich, Hanser.

Spencer, A. J. M. (1980) *Continuum mechanics.* Dover Publications, New York.

Scott, N. (1986) The slowness surface of incompressible and nearly incompressible elastic materials. *J. Elasticity*, **16**, 239-250.

Timoshenko, S. P. (1938) *Vibration problems in engineering.* New York, D.Van Nostrand Company, Inc.

Treloar, L. G. R. (1975) *The physics of rubber elasticity.* Oxford, Clarendon Press.

Triantafyllidis, N. & Abeyarante, R. (1983) Instabilities of a finitely-deformed elastic material. *ASME Journal of Applied Mechanics*, **50**, 149-156.

Waterloo Maple Software (1996) *Maple V Programming Guide.* Springer-Verlag, New York.

Willson, A. J. (1977) Waves propagation in thin pre-stressed elastic plates. *Int. J. Engng Sci.*, **15**, 245-251.