

PRISMS AND PRISMATIC COHOMOLOGY

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In the memory of Jean-Marc Fontaine

ABSTRACT. We introduce the notion of a prism, which may be regarded as a “deperfection” of the notion of a perfectoid ring. Using prisms, we attach a ringed site — the prismatic site — to a p -adic formal scheme. The resulting cohomology theory specializes to (and often refines) most known integral p -adic cohomology theories.

As applications, we prove an improved version of the almost purity theorem allowing ramification along arbitrary closed subsets (without using adic spaces), give a co-ordinate free description of q -de Rham cohomology as conjectured by the second author in [Sch17], and settle a vanishing conjecture for the p -adic Tate twists $\mathbf{Z}_p(n)$ introduced in [BMS19].

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1. INTRODUCTION

Fix a prime p . In this article, we give a new and unified construction of various p -adic cohomology theories, including étale, de Rham and crystalline cohomology, as well as the more recent constructions in [BMS18], [BMS19] and the so far conjectural q -de Rham cohomology from [Sch17]. The key innovation is the following definition, which is a “deperfection” of perfectoid rings.

Definition 1.1. A *prism* is a pair (A, I) where A is a δ -ring (see Remark 1.2) and $I \subset A$ is an ideal defining a Cartier divisor in $\mathrm{Spec}(A)$, satisfying the following two conditions.

- (1) The ring A is (p, I) -adically complete.¹
- (2) The ideal $I + \phi_A(I)A$ contains p ; here ϕ_A refers to the lift of Frobenius on A induced by its δ -structure (Remark 1.2).

A map $(A, I) \rightarrow (B, J)$ of prisms is given by a map of δ -rings carrying I into J .

Before giving examples, we briefly comment on the notion of δ -rings used above.

Remark 1.2. The notion of δ -rings was introduced by Joyal [Joy85] and studied extensively by Buium [Bui97] under the name of “ p -derivations”; see also [Bor16]. This notion provides a convenient language for discussing rings with a lift of the Frobenius, and is reviewed in depth in §2. Roughly, a δ -ring structure on a commutative ring A is a map $\delta_A : A \rightarrow A$ satisfying certain identities ensuring that the associated map $\phi_A : A \rightarrow A$ given by $\phi_A(x) = x^p + p\delta_A(x)$ is a ring homomorphism (which then necessarily lifts the Frobenius on A/p). In particular, if A is a p -torsionfree commutative ring, then a δ -ring structure on A is equivalent to the datum of a ring map $\phi_A : A \rightarrow A$ lifting the Frobenius on A/p . The name “ p -derivation” is explained by observing that δ lowers the p -adic order of vanishing by 1 in a suitable sense (see Remark 2.6 for an example).

Let us give some examples of prisms. In the following examples (and, in fact, all known examples), the ideal $I = (d)$ is actually principal. Under Definition 1.1 (1), condition (2) is then equivalent to the condition that $\delta(d) \in A$ is a unit; we call such an element $d \in A$ a *distinguished element* of A .

Example 1.3. The choice of names below is largely dictated by the comparison theorems for the associated prismatic cohomology theory.

- (1) (Crystalline Prisms) Let A be any p -torsionfree and p -complete ring with a Frobenius lift $\phi : A \rightarrow A$; this induces a unique δ -ring structure on A . Then $(A, (p))$ is a prism.
- (2) (Perfect Prisms) A prism (A, I) is called perfect if A is perfect, i.e. $\phi : A \rightarrow A$ is an isomorphism. The category of perfect prisms is equivalent to the category of perfectoid rings (as defined in say [BMS19, Definition 4.18]) via the functors $(A, I) \mapsto A/I$ and $R \mapsto (A_{\mathrm{inf}}(R), \ker(\theta))$ (Theorem 3.10). Any prism admits a “perfection”, so a general prism can be regarded as a “not necessarily perfect” analog of a perfectoid ring.
- (3) (Breuil-Kisin-type Prisms) Let K/\mathbf{Q}_p be a complete discretely valued extension with perfect residue field k and uniformizer $\pi \in K$. Let $A = \mathfrak{S} = W(k)[[u]]$, regarded as a δ -ring via the

¹We actually allow A to be merely derived (p, I) -adically complete below, although we will usually restrict to situations where the subtle difference between these notions does not come up.

Frobenius lift that extends the Frobenius on $W(k)$ and sends u to u^p . Let $I \subset A$ be the kernel of the surjective map $A = W(k)[[u]] \rightarrow \mathcal{O}_K$ sending u to π . Then (A, I) is a prism.

- (4) (*q*-crystalline Prism) Let $A = \mathbf{Z}_p[[q-1]]$, the $(p, q-1)$ -adic completion of $\mathbf{Z}[q]$, regarded as a δ -ring via the Frobenius lift that sends q to q^p . Let $I = ([p]_q)$, where $[p]_q = \frac{q^p-1}{q-1} = 1 + q + \dots + q^{p-1}$ is the q -analogue of p . Then (A, I) is a prism.

In particular, combining the first example with any of the other examples shows that a given δ -ring A may support many different prism structures. At the opposite extreme, one can show that the local ring $A = W(\mathbf{F}_p[x]/(x^2))$ is naturally a p -adically complete δ -ring (see Remark 2.7 for the δ -structure on the ring of Witt vectors) that does not underlie any prism (A, I) : one can show that any nonzerodivisor $d \in A$ must be a unit².

Our constructions with prisms will often require us to contemplate very large algebras and modules. For technical reasons pertaining to the behaviour of completions in highly nonnoetherian situations (and, relatedly, to avoid introducing derived analogs of prisms), we will restrict attention to the following class of prisms, which includes all the ones in Example 1.3.

Definition 1.4. A prism (A, I) is *bounded* if A/I has bounded p^∞ -torsion, i.e. there is some integer n such that $A/I[p^\infty] = A/I[p^n]$.

Given a bounded prism (A, I) and a smooth p -adic formal scheme X over A/I , our next goal is to describe a ringed site $((X/A)_\Delta, \mathcal{O}_\Delta)$ that we call the *prismatic site* of X . Roughly, this category is defined by “probing” X using prisms over (A, I) . Before giving the formal definition, it is very useful to note the following rigidity property of prisms over (A, I) .

Proposition 1.5. *If $(A, I) \rightarrow (B, J)$ is a map of prisms, then $J = IB$.*

In other words, when working over a fixed base prism, the ideal I is not varying anymore. We can now give the promised definition.

Definition 1.6. Fix a bounded prism (A, I) and a smooth p -adic formal scheme X over A/I .

- (1) A map $(A, I) \rightarrow (B, J)$ of prisms is called (*faithfully*) *flat* if $A \rightarrow B$ is (p, I) -completely (faithfully) flat.
- (2) The *prismatic site* $(X/A)_\Delta$ is the opposite of the category of prisms (B, J) with a map $(A, I) \rightarrow (B, J)$ and a map $\mathrm{Spf}(B/J) \rightarrow X$ over $\mathrm{Spf}(A/I)$, endowed with the faithfully flat covers as defined in (1).
- (3) The *structure sheaf* \mathcal{O}_Δ on $(X/A)_\Delta$ is the sheaf taking a pair (B, J) (with maps $(A, I) \rightarrow (B, J)$ and $\mathrm{Spf}(B/J) \rightarrow X$) to B . This is a sheaf of I -torsionfree δ - A -algebras.
- (4) There is also another sheaf $\overline{\mathcal{O}}_\Delta$ of rings on $(X/A)_\Delta$ taking a pair (B, J) to B/J . This is naturally a sheaf of $\mathcal{O}(X)$ -algebras, and we have $\mathcal{O}_\Delta \otimes_A A/I \cong \overline{\mathcal{O}}_\Delta$ by Proposition 1.5.

Remark 1.7. There are many variants of Definition 1.6. For example, one might use the étale or quasisyntomic topologies instead of the flat topology. More interestingly, one can define an

²The maximal ideal of $W(A)$ is generated by $[x]$ and $VW(A)$ and is annihilated by multiplication by the nonzero element $[x]$: we have $[x] \cdot [x] = [x^2] = 0$ and $[x]V(-) = V(F([x]) \cdot -) = 0$ as $F([x]) = [x^p] = 0$.

“absolute” prismatic site $(X)_{\Delta}$ for any p -adic formal scheme X by simply discarding the base (A, I) in Definition 1.6, i.e., as the category of prisms (B, J) together with a map $\mathrm{Spf}(B/J) \rightarrow X$. While important in arithmetic considerations, this notion does not play a significant role in this paper, and is only used in our study of algebras over a perfectoid ring in an auxiliary way to make certain functorialities transparent. Another variant is the perfect prismatic site $(X/A)_{\Delta}^{\mathrm{perf}}$, which is obtained from $(X/A)_{\Delta}$ by restricting attention to perfect prisms (B, J) ; this variant plays an important role in our results on perfectoid rings, such as Theorem 1.16.

A major goal of this paper is to explain why prismatic cohomology recovers (and often refines) most known integral p -adic cohomology theories:

Theorem 1.8. *Let (A, I) be a bounded prism, and let X be a smooth p -adic formal scheme over A/I . Let*

$$R\Gamma_{\Delta}(X/A) := R\Gamma((X/A)_{\Delta}, \mathcal{O}_{\Delta}) ,$$

which is a commutative algebra in the derived category $D(A)$ of A -modules, and comes equipped with a ϕ_A -linear map ϕ .

- (1) (Crystalline Comparison) *If $I = (p)$, then there is a canonical ϕ -equivariant isomorphism*

$$R\Gamma_{\mathrm{crys}}(X/A) \cong \phi_A^* R\Gamma_{\Delta}(X/A) .$$

of commutative algebras in $D(A)$.

- (2) (Hodge-Tate Comparison) *If X is affine, say $X = \mathrm{Spf} R$, there is a canonical R -module isomorphism*

$$\Omega_{R/(A/I)}^i \{i\} \cong H^i(R\Gamma_{\Delta}(X/A) \otimes_A^L A/I) .$$

Here we write $M\{i\} = M \otimes_{A/I} (I/I^2)^{\otimes i}$ for any A/I -module M .

- (3) (de Rham Comparison) *There is a canonical isomorphism*

$$R\Gamma_{\mathrm{dR}}(X/(A/I)) \cong R\Gamma_{\Delta}(X/A) \widehat{\otimes}_{A, \phi_A}^L A/I .$$

of commutative algebras in $D(A)$. Moreover, it can be upgraded naturally to an isomorphism of commutative differential graded algebras.

- (4) (Étale Comparison) *Assume A is perfect. Let X_{η} be the generic fibre of X over \mathbf{Q}_p , as a (pre-)adic space. For any $n \geq 0$, there is a canonical isomorphism*

$$R\Gamma_{\mathrm{ét}}(X_{\eta}, \mathbf{Z}/p^n \mathbf{Z}) \cong (R\Gamma_{\Delta}(X/A)/p^n[\frac{1}{I}])^{\phi=1} .$$

of commutative algebras in $D(\mathbf{Z}/p^n)$.

- (5) (Base Change) *Let $(A, I) \rightarrow (B, J)$ be a map of bounded prisms, and let $Y = X \times_{\mathrm{Spf}(A/I)} \mathrm{Spf}(B/J)$. Then the natural map induces an isomorphism*

$$R\Gamma_{\Delta}(X/A) \widehat{\otimes}_A^L B \cong R\Gamma_{\Delta}(Y/B) ,$$

where the completion on the left is the derived (p, J) -adic completion.

(6) (Image of ϕ) *The linearization*

$$\phi_A^* R\Gamma_{\Delta}(X/A) \rightarrow R\Gamma_{\Delta}(X/A)$$

of ϕ becomes an isomorphism after inverting I . More precisely, if $I = (d)$ is principal, there is a map $V_i : H_{\Delta}^i(X/A) \rightarrow H^i(\phi_A^* R\Gamma_{\Delta}(X/A))$ such that $V_i \phi = \phi V_i = d^i$.

In particular, it follows from the Hodge-Tate comparison that $R\Gamma_{\Delta}(X/A)$ is a perfect complex of A -modules if X is proper.

Let us discuss the consequences of Theorem 1.8 over the prisms discussed in Example 1.3.

Example 1.9. (1) (Crystalline) Assume that $I = (p)$. In this case, prismatic cohomology gives a canonical Frobenius descent of crystalline cohomology by Theorem 1.8 (1). Note that this is no extra information when A is perfect, while it is interesting extra information in the common case that A is a p -completely smooth lift of a smooth k -algebra equipped with a Frobenius lift, for some perfect field k . In particular, it implies restrictions on the possible structure of the torsion in crystalline cohomology (e.g., its length, when finite, has to be a multiple of p).

(2) (BMS1) Let C/\mathbf{Q}_p be an algebraically closed complete extension with ring of integers \mathcal{O}_C , and let $A := A_{\text{inf}} = A_{\text{inf}}(\mathcal{O}_C)$ be Fontaine's ring with $I = \ker(\theta : A_{\text{inf}} \rightarrow \mathcal{O}_C)$. The pair (A, I) is a perfect prism corresponding to the perfectoid ring \mathcal{O}_C . In [BMS18], we defined a complex of A_{inf} -modules $R\Gamma_{A_{\text{inf}}}(X)$ equipped with a “Frobenius” operator ϕ . We shall prove in §17 that there exists a canonical ϕ -equivariant isomorphism

$$R\Gamma_{A_{\text{inf}}}(X) \cong \phi_A^* R\Gamma_{\Delta}(X/A_{\text{inf}}) .$$

Note that in this case ϕ_A is an isomorphism, so the pullback appearing above merely twists the A -module structure. The comparison results from [BMS18] are then immediate consequences of the comparison results above (except for the results on de Rham-Witt theory in [BMS18] that we have not taken up here); note that the A_{crys} -comparison theorem from [BMS18] follows from Theorem 1.8 (1) thanks to Theorem 1.8 (5) applied to the map $(A_{\text{inf}}, \ker(\theta)) \rightarrow (A_{\text{crys}}, (p))$ of bounded prisms.

Remark 1.10. Our search for the prismatic site was substantially motivated by a desire to obtain a site-theoretic construction of the A_{inf} -cohomology. The “Frobenius” operator on this cohomology had slightly mysterious origins: it came, rather indirectly, via the tilting equivalence for perfectoids. Thanks to the relation to prismatic cohomology, the origins of this operator are now clear: it comes from the Frobenius lift on \mathcal{O}_{Δ} .

(3) (Breuil-Kisin, BMS2) Let K/\mathbf{Q}_p be a complete discretely valued field with perfect residue field k and uniformizer π , and let $A = \mathfrak{S} = W(k)[[u]]$ with $I \subset A$ the kernel of the map $A \rightarrow \mathcal{O}_K$ sending u to π . Then in [BMS19] we defined a complex of A -modules $R\Gamma_{\mathfrak{S}}(X)$ equipped with a Frobenius, using topological Hochschild homology. In §15, we construct a canonical ϕ -equivariant isomorphism

$$R\Gamma_{\mathfrak{S}}(X) \cong R\Gamma_{\Delta}(X/\mathfrak{S}) .$$

The above comparison results recover the results of [BMS19] on Breuil-Kisin cohomology.

- (4) (q -crystalline) Let $A = \mathbf{Z}_p[[q-1]]$ with $I = ([p]_q)$ as above, so that $A/I = \mathbf{Z}_p[\zeta_p]$. If R is a p -completely smooth \mathbf{Z}_p -algebra, $R^{(1)} := R \otimes_{\mathbf{Z}_p} \mathbf{Z}_p[\zeta_p]$ and $X = \mathrm{Spf} R^{(1)}$, then for any choice of a p -completely étale map

$$\square : \mathbf{Z}_p\langle T_1, \dots, T_d \rangle \rightarrow R,$$

we construct in §16 a canonical isomorphism

$$q\Omega_R^\square \cong R\Gamma_\Delta(X/A)$$

with the q -de Rham complexes from [Sch17], proving [Sch17, Conjecture 1.1] on the independence (up to canonical quasi-isomorphism) of $q\Omega_R^\square$ from the choice of \square .

In [FM87], Fontaine-Messing gave a description of crystalline cohomology in terms of syntomic cohomology of a sheaf of divided power envelopes. The corresponding idea in mixed characteristic was taken up in [BMS19], where it was rephrased in terms of quasisyntomic descent from a class of semiperfectoid rings. To adapt these results to the prismatic context, we prove the following result about the prismatic site of semiperfectoid rings; we note that part (2) below says that there is a notion of “perfection” even in mixed characteristic (see also Theorem 1.16 (2) for a more general statement).

Theorem 1.11. *Let S be a semiperfectoid ring, i.e., S is a (derived) p -adically complete quotient of a perfectoid ring.*

- (1) (Existence of universal prisms) *The category $(S)_\Delta$ of prisms (A, I) with a map $S \rightarrow A/I$ admits an initial object $(\Delta_S^{\mathrm{init}}, I)$, and $I = (d)$ is principal.*
- (2) (The perfectoidization of S) *Let $(\Delta_S^{\mathrm{init}})_{\mathrm{perfd}}$ be the (p, I) -completed perfection of Δ_S^{init} , and $S_{\mathrm{perfd}} = (\Delta_S^{\mathrm{init}})_{\mathrm{perfd}}/I$. Then S_{perfd} is a perfectoid ring and the map $S \rightarrow S_{\mathrm{perfd}}$ is the universal map to a perfectoid ring from S . Moreover, the map $S \rightarrow S_{\mathrm{perfd}}$ is surjective.*

The surjectivity of $S \rightarrow S_{\mathrm{perfd}}$ in Theorem 1.11 (2) implies that the notions of “Zariski closed” and “strongly Zariski closed” subsets of affinoid perfectoid spaces agree, contrary to a claim made by the second author in [Sch15, Section II.2]. The proof of this result uses an important flatness lemma of André [And18a] for perfectoid rings (for which we offer a direct prismatic proof in Theorem 7.12).

Remark 1.12 (Almost mathematics with respect to any closed set). Theorem 1.11 (2) implies that there is a good notion of “almost mathematics” with respect to any ideal of a perfectoid ring, and not merely $\sqrt{p}R$ as in the classical setup. More precisely, if $J \subset R$ is any p -complete ideal in a perfectoid ring, Theorem 1.11 (2) implies that $R \rightarrow (R/J)_{\mathrm{perfd}}$ is a surjective map of perfectoid rings. General properties of perfectoid rings then show that the kernel $J_{\mathrm{perfd}} := \ker(R \rightarrow (R/J)_{\mathrm{perfd}})$ satisfies $J_{\mathrm{perfd}} \widehat{\otimes}_R^L J_{\mathrm{perfd}} \simeq J_{\mathrm{perfd}}$, which leads to a notion of “ J -almost mathematics” for p -complete R -modules and p -complete objects of the derived category $D(R)$ (see §10.1).

The prism Δ_A^{init} from Theorem 1.11 is poorly behaved in general, and we do not know any explicit description. However, if S has a well-behaved cotangent complex, then one can do better, as this prism can be described as the derived prismatic cohomology of S itself.

Theorem 1.13. *Let S be a semiperfectoid ring. Assume now that S is quasiregular, i.e. S has bounded p^∞ -torsion and the cotangent complex $L_{S/\mathbf{Z}_p}[-1]$ is p -completely flat. Following notation as in Theorem 1.11, write $\Delta_S = \Delta_S^{\text{init}}$ with the understanding that under these hypotheses, Δ_S is better behaved.*

- (1) (The conjugate filtration) *The S -algebra Δ_S/I is p -completely flat and for any choice of a perfectoid R mapping to S , it admits an increasing (“conjugate”) filtration $\text{Fil}_i^{\text{conj}} \Delta_S/I$ with*

$$\text{gr}_i^{\text{conj}} \Delta_S/I \cong (\wedge^i L_{S/R}[-i])^\wedge \otimes_{S/I} (I/I^2)^{\otimes i} .$$

- (2) (The Nygaard filtration) *The ring Δ_S admits a natural decreasing (“Nygaard”) filtration*

$$\text{Fil}_N^j \Delta_S = \{x \in \Delta_S \mid \phi(x) \in I^j \Delta_S\} .$$

For a generator $d \in I$ coming from a generator of $\ker \theta : A_{\text{inf}}(R) \rightarrow R$ for R as in (1), the composite map

$$\text{Fil}_N^j \Delta_S \xrightarrow{\phi/d^j} \Delta_S \rightarrow \Delta_S/I$$

has image $\text{Fil}_j^{\text{conj}} \Delta_S/I$, inducing an isomorphism

$$\text{gr}_N^j \Delta_S \cong \text{Fil}_j^{\text{conj}} \Delta_S/I .$$

- (3) (Comparison with the topological theory) *The ring $\widehat{\Delta}_S = \pi_0 \text{TP}(S; \mathbf{Z}_p)$ defined in [BMS19] is ϕ -equivariantly isomorphic to the completion of Δ_S with respect to its Nygaard filtration, and in particular admits a functorial δ -ring structure.*

Remark 1.14. Let S be a quasiregular semiperfectoid ring (see Theorem 1.11). The formalism of topological Hochschild homology endows the commutative ring $\pi_0 \text{TP}(S; \mathbf{Z}_p)$ with an endomorphism ϕ_S often called the “Frobenius” (see [NS18]). Despite the name, it is not clear from the definitions that ϕ_S lifts the Frobenius on $\pi_0 \text{TP}(S; \mathbf{Z}_p)/p$. Thanks to Theorem 1.11 (5), an even better statement is now available: ϕ_S is the Frobenius lift attached to a δ -structure. (Note that this assertion need not be true for more general S .)

We give three applications of the above results on semiperfectoid rings. Our first application concerns the Nygaard filtration on prismatic cohomology in the smooth case; the relevance of semiperfectoid rings to this question is that the quasiregular semiperfectoid rings form a basis for the quasisyntomic site. Note that part (3) below implies the result on the image of ϕ in Theorem 1.8.

Theorem 1.15. *Let (A, I) be a bounded prism and let $X = \text{Spf}(R)$ be an affine smooth p -adic formal scheme over A/I .*

- (1) (Existence of the Nygaard filtration) *On the quasisyntomic site X_{qsyn} , one can define a sheaf of (p, I) -completely flat δ - A -algebras $\Delta_{-/A}$ equipped with a Nygaard filtration on its Frobenius twist*

$$\Delta_{-/A}^{(1)} := \Delta_{-/A} \widehat{\otimes}_{A, \phi}^L A$$

given by

$$\text{Fil}_N^i \Delta_{-/A}^{(1)} = \{x \in \Delta_{-/A}^{(1)} \mid \phi(x) \in I^i \Delta_{-/A}\} .$$

(2) (Quasisyntomic descent) *There is a canonical isomorphism*

$$R\Gamma_{\Delta}(X/A) \cong R\Gamma(X_{\text{qsyn}}, \Delta_{-/A})$$

and we endow prismatic cohomology with the Nygaard filtration

$$\text{Fil}_N^i R\Gamma_{\Delta}(X/A)^{(1)} = R\Gamma(X_{\text{qsyn}}, \text{Fil}_N^i \Delta_{-/A}^{(1)}) .$$

(3) (Graded pieces of the Nygaard filtration) *There are natural isomorphisms*

$$\text{gr}_N^i R\Gamma_{\Delta}(X/A)^{(1)} \cong \tau^{\leq i} \overline{\Delta}_{R/A} \{i\}$$

for all $i \geq 0$.

(4) (Frobenius is an isogeny) *The Frobenius ϕ on $R\Gamma_{\Delta}(X/A)$ factors as*

$$\phi_A^* R\Gamma_{\Delta}(X/A) = R\Gamma_{\Delta}(X/A)^{(1)} \xrightarrow{\tilde{\phi}} L\eta_I R\Gamma_{\Delta}(X/A) \rightarrow R\Gamma_{\Delta}(X/A) ,$$

using the décalage functor $L\eta_I$ as e.g. in [BMS18]. The map

$$\tilde{\phi} : \phi_A^* R\Gamma_{\Delta}(X/A) \rightarrow L\eta_I R\Gamma_{\Delta}(X/A)$$

is an isomorphism.

Our second application is to the study of perfectoid rings. Generalizing Theorem 1.11 (2), we prove that there is a “perfectoidization” for *any* finite algebra over a perfectoid ring, and this operation behaves like the perfection in characteristic p in many ways.

Theorem 1.16. *Let R be a perfectoid ring and let S be an integral R -algebra.*

- (1) (Perfectoidizations of integral algebras) *There is a perfectoid ring S_{perfd} with a map $S \rightarrow S_{\text{perfd}}$ such that any map from S to a perfectoid ring factors uniquely over S_{perfd} .*
- (2) (Almost purity) *Assume $R \rightarrow S$ is finite étale away from $V(J) \subset \text{Spec}(R)$, where $J \subset R$ is a finitely generated ideal. Then $R \rightarrow S_{\text{perfd}}$ is J -almost finite étale (see Remark 1.12 for the terminology, and §10 for a precise formulation).*

Theorem 1.16 (2) reduces to the usual almost purity theorem for perfectoid algebras when $J = (p)$, and improves on the perfectoid Abhyankar lemma [And18b] when $J = (g)$ for some $g \in R$; moreover, our proof avoids adic spaces (and thus perfectoid spaces), so we get a new proof of both results. As a corollary, we deduce that for a perfectoid ring R , the \mathbf{F}_p -étale cohomological dimension of $\text{Spec}(R[1/p])$ is ≤ 1 (Theorem 11.1).

The third application of our results on semiperfectoid rings is a resolution of the vanishing conjecture on the complexes $\mathbf{Z}_p(n)$ made in [BMS19, Conjecture 7.18]. By the main results of [BMS19] and [CMM18], these are related to p -adic algebraic K -theory.

Theorem 1.17. *The quasisyntomic sheaves of complexes $\mathbf{Z}_p(n)$ of [BMS19] are concentrated in degree 0 and p -torsionfree. In particular, locally in the quasisyntomic topology of any quasisyntomic \mathbf{Z}_p -algebra, the mod p algebraic K -theory $K(-)/p$ functor is concentrated in even degrees.*

Theorem 1.17 (or, rather, its proof) yields vanishing results in concrete situations. For instance, if C/\mathbf{Q}_p is a complete and algebraically closed field, then $\pi_* K(\mathcal{O}_C/p^n; \mathbf{Z}_p)$ vanishes in odd degrees for any $n \geq 0$ (Corollary 14.3).

Leitfaden of the paper. We begin with the theory of δ -rings in §2. Having established enough language, we introduce prisms in §3 and the prismatic site in §4. The crystalline and Hodge-Tate comparison results in characteristic p are proven next in §5. From this, we deduce the Hodge-Tate comparison in general in §6, which implies the compatibility with base change, and the de Rham comparison (under a small technical hypothesis).

We then proceed in §7 to study semiperfectoid rings S via derived prismatic cohomology (i.e., using simplicial resolution by smooth algebras) to prove Theorem 1.11. These results are then applied in §8 — §11 to studying the “generic fibre”. In particular, we prove the étale comparison theorem for prismatic cohomology in §9, and Theorem 1.16 in §10.

In §12, we analyze the Nygaard filtration explicitly for quasiregular semiperfectoid rings. This analysis is used to prove the comparison with [BMS19] in §13 and Theorem 1.17 in §14. Descending back to the smooth case, we deduce Theorem 1.15 in §15, which also implies the results on the image of ϕ and the general version of the de Rham comparison in Theorem 1.8.

Finally, in §16, we introduce q -crystalline cohomology; this theory is computed by q -de Rham complexes in the presence of co-ordinates, which resolves some conjectures from [Sch17]. This theory is also identified with prismatic cohomology in certain situations, which leads to concrete representatives computing prismatic cohomology. These concrete representatives are used in §17 to relate prismatic cohomology to the $A\Omega$ -complexes from [BMS18]. We end in §18 by proving a strong uniqueness result for comparison isomorphisms (so any diagram chase involving such comparison isomorphisms necessarily commutes).

Notation. Given an abelian group M equipped with a set of commuting endomorphisms $\{f_i : M \rightarrow M\}_{i \in I}$, we may regard M as a module over the polynomial algebra $\mathbf{Z}[f_i | i \in I]$ by letting the variable f_i acts as the endomorphism f_i . If we are further given a total ordering of I , we define the (homological) Koszul complex

$$\mathrm{Kos}(M; (f_i)_{i \in I}) = \dots \rightarrow \bigoplus_{i < j} M \rightarrow \bigoplus_{i \in I} M \xrightarrow{(f_i)_{i \in I}} M \rightarrow 0 \quad (1)$$

which represents $M \otimes_{\mathbf{Z}[f_i | i \in I]}^L \mathbf{Z}$ in the derived category, and dually the cohomological Koszul complex

$$\mathrm{Kos}_c(M; (f_i)_{i \in I}) = 0 \rightarrow M \xrightarrow{(f_i)_{i \in I}} \bigoplus_{i \in I} M \rightarrow \bigoplus_{i < j} M \rightarrow \dots \quad (2)$$

representing $R\mathrm{Hom}_{\mathbf{Z}[f_i | i \in I]}(\mathbf{Z}, M)$ in the derived category. Note that if I is finite, then these two complexes are isomorphic up to shift.

We will often take various completions. Usually, these are taken in the derived sense. We refer to [Sta, Tag 091N] for the following assertions about derived completion. Recall that if A is a ring with a finitely generated ideal $I \subset A$, then a complex M of A -modules is derived I -adically complete (often abbreviated to derived I -complete) if the natural map

$$M \rightarrow \widehat{M} := R\lim_n \mathrm{Kos}(M; f_1^n, \dots, f_r^n)$$

is an isomorphism, where $f_1, \dots, f_r \in I$ are generators of I ; this condition is independent of the choice of generators.³ We shall write $D_{I\text{-comp}}(A)$ for the full subcategory of $D(A)$ spanned by I -complete complexes; if the ideal I is clear from context, we shall simply write $D_{\text{comp}}(A)$ instead. A complex M of A -modules is derived I -complete if and only if each $H^i(M)$ is derived I -complete. In particular, the category of derived I -complete A -modules is an abelian category stable under passage to kernels, cokernels, images and extensions in the category of all A -modules. Any classically I -adically complete A -module M is derived I -complete, and conversely if M is derived I -complete and I -adically separated, then M is classically I -adically complete. In general, for an A -module M it can happen that its derived I -completion \widehat{M} is not concentrated in degree zero. However, if I is principal and M has bounded I^∞ -torsion, i.e. there is some integer n such that $M[I^\infty] = M[I^n]$, then \widehat{M} is concentrated in degree zero.

A complex M of A -modules is I -completely flat if for any I -torsion A -module N , the derived tensor product $M \otimes_A^L N$ is concentrated in degree 0. This implies in particular that $M \otimes_A^L A/I$ is concentrated in degree 0 and a flat A -module. Moreover, if M is I -completely flat, then it is J -completely flat for any ideal J that contains I^n for some n . Note that if M is a flat A -module, then the derived I -completion \widehat{M} is still I -completely flat.

A complex M of A -modules is I -completely faithfully flat if it is I -completely flat and $M \otimes_A^L A/I$ (which is automatically a flat A/I -module concentrated in degree zero) is a faithfully flat A/I -module.

A derived I -complete A -algebra R is I -completely étale (resp. I -completely smooth, I -completely ind-smooth) if $R \otimes_A^L A/I$ is étale (resp. smoth, ind-smooth). We note that by Elkik's algebraization results, R is I -completely étale (resp. smooth) if and only if it is the derived I -completion of an étale (resp. smooth) A -algebra, which is the condition that we have used in [BMS18], [BMS19].

We will sometimes use simplicial rings, e.g., to resolve arbitrary algebras by ind-smooth algebras as in the definition of the cotangent complex. (All our rings, including the simplicial ones, are assumed to be commutative.) Following topological terminology, we say that a simplicial ring A is *discrete* if it is concentrated in degree 0, i.e. $\pi_i A = 0$ for $i > 0$; in other words, a discrete simplicial ring is equivalent to the usual ring $\pi_0 A$. Occasionally, we use the same terminology more generally for any object of the derived category: A complex M is *discrete* if $H^i(M) = 0$ for $i \neq 0$, in which case M is isomorphic to the module $H^0(M)$; likewise, a complex M is *connective* (resp. *coconnective*) if $H^i(M) = 0$ for $i > 0$ (resp. $i < 0$).

Finally, we occasionally use ∞ -categorical techniques (especially when constructing and using derived prismatic cohomology). We shall follow the following standard convention in these situations: given a commutative (or E_∞ -) ring A , we write $\mathcal{D}(A)$ for its derived ∞ -category. Similarly, given a finitely generated ideal $I \subset \pi_0(A)$, we write $\mathcal{D}_{I\text{-comp}}(A)$ (or simply $\mathcal{D}_{\text{comp}}(A)$ if I is clear) for the full ∞ -subcategory of $\mathcal{D}(A)$ spanned by I -complete objects.

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³Beware that in general the completion of M is not given by $R\lim_n M \otimes_A^L A/I^n$; however, this happens if I is generated by a regular sequence, or if A is noetherian (by Artin-Rees).

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2. δ -RINGS

Fix a prime p . In this section, we discuss the theory of δ -rings. In §2.1 and §2.2, which are essentially review, we discuss the definitions and basic properties of δ -rings [Joy85]; a good reference for this material is [Bor16]. In §2.3, we introduce distinguished elements, which are essential to defining prisms. Basic properties of perfect δ -rings are then the subject of §2.4. In §2.5, we establish a relationship between divided power envelopes and δ -structures; this relationship plays an important role in many subsequent computations in this paper, thanks in large part to the nice commutative algebra properties of divided power envelopes of regular sequences, which are reviewed in §2.6.

All our rings are $\mathbf{Z}_{(p)}$ -algebras. We write $\text{rad}(A)$ for the Jacobson radical of a ring A . In the following, note that the expression

$$\frac{x^p + y^p - (x + y)^p}{p} \in \mathbf{Z}[x, y]$$

can be evaluated in any ring.

2.1. Definition and basic properties.

Definition 2.1. A δ -ring is a pair (R, δ) where R is a commutative ring and $\delta : R \rightarrow R$ is a map of sets with $\delta(0) = \delta(1) = 0$, satisfying the following two identities

$$\delta(xy) = x^p \delta(y) + y^p \delta(x) + p \delta(x) \delta(y) \quad \text{and} \quad \delta(x + y) = \delta(x) + \delta(y) + \frac{x^p + y^p - (x + y)^p}{p}.$$

There is an evident category of δ -rings. (In the literature, a δ -structure is often called a p -derivation.)

Remark 2.2 (δ -structures give Frobenius lifts). Given a δ -ring (R, δ) , we write $\phi : R \rightarrow R$ for the map defined by $\phi(x) = x^p + p\delta(x)$; the identities on δ ensure that this is a ring homomorphism that lifts Frobenius on R/p . In fact, the identities on δ are reverse engineered from this requirement: if R is a p -torsionfree ring, then any lift $\phi : R \rightarrow R$ of the Frobenius on R/p comes from a unique δ -structure on R , given by the formula $\delta(x) = \frac{\phi(x) - x^p}{p}$. Note that as the condition of being a lift of Frobenius is vacuous if p is invertible, a δ -ring over \mathbf{Q} is the same thing as a \mathbf{Q} -algebra with an endomorphism. In general, given a δ -ring R , we shall write $\phi : R \rightarrow R$ for its Frobenius endomorphism; if there is potential for confusion, we shall denote this map by ϕ_R instead.

Remark 2.3 (δ -structures and λ -structures). By a theorem of Wilkerson [Wil82, Proposition 1.2], giving a λ -ring structure on a flat \mathbf{Z} -algebra R is equivalent to specifying commuting Frobenius lifts ϕ_p on R for all primes p . Borger has extended this in [Bor11] in multiple ways (including allowing all \mathbf{Z} -algebras). In particular, it follows from his work that a δ -structure on a ring R is the same as a p -typical λ -structure. This motivates the following terminology: an element x in a δ -ring B has *rank* 1 if $\delta(x) = 0$; such elements satisfy $\phi(x) = x^p$ (and the converse holds true if B is p -torsionfree).

Remark 2.4 (δ -rings via $W_2(-)$, following Rezk [Rez14]). For any ring R , the ring $W_2(R)$ of p -typical length 2 Witt vectors is defined as follows: we have $W_2(R) = R \times R$ as sets, and addition and multiplication are defined via

$$(x, y) + (x', y') := (x + x', y + y' + \frac{x^p + (x')^p - (x + x')^p}{p}) \quad \text{and} \quad (x, y) \cdot (x', y') = (xx', x^p y' + x'^p y + p y y').$$

Ignoring the second component gives a ring homomorphism $\epsilon : W_2(R) \rightarrow R$. It is immediate from the definitions that specifying a δ -structure on R is the same as specifying a ring map $w : R \rightarrow W_2(R)$ such that $\epsilon \circ w = \text{id}$: the correspondence attaches the map $w(x) = (x, \delta(x))$ to a δ -structure $\delta : R \rightarrow R$ on R .

Remark 2.5 (Derived Frobenius lifts give δ -structures). Let R be any $\mathbf{Z}_{(p)}$ -algebra. Then specifying a δ -structure on R is the same as specifying a map $\phi : R \rightarrow R$ and a path in the space $\text{End}_{SCR_{\mathbf{F}_p}}(R \otimes_{\mathbf{Z}}^L \mathbf{F}_p)$, between the points defined by ϕ and the Frobenius; here $SCR_{\mathbf{F}_p}$ denotes the ∞ -category of simplicial commutative \mathbf{F}_p -algebras. In other words, giving a δ -structure on R is the same as specifying a lift of Frobenius in the derived sense. To prove this, first note that for a p -torsionfree ring R , the square

$$\begin{array}{ccc} W_2(R) & \xrightarrow{F} & R \\ \downarrow & & \downarrow \\ R & \xrightarrow{\phi} & R/p \end{array}$$

is a pullback square of rings, where the lower map is the composite of the projection $R \rightarrow R/p$ with the Frobenius of R/p . Passing to simplicial resolutions, this implies that for any simplicial ring, there is a functorial pullback square

$$\begin{array}{ccc} W_2(R) & \xrightarrow{F} & R \\ \downarrow & & \downarrow \\ R & \xrightarrow{\phi} & R \otimes_{\mathbf{Z}}^L \mathbf{F}_p \end{array}$$

of simplicial rings. Using this pullback square, Remark 2.4 translates into the desired description of δ -rings.

Example 2.6 (The initial δ -ring). The identity map on the p -torsionfree ring $\mathbf{Z}_{(p)}$ is its unique endomorphism and lifts the Frobenius modulo p , so $\mathbf{Z}_{(p)}$ carries a unique δ -structure given by $\delta(x) = \frac{x-x^p}{p}$. In fact, this is the initial object in the category of δ -rings (as we work with $\mathbf{Z}_{(p)}$ -algebras). One checks that δ on $\mathbf{Z}_{(p)}$ lowers the p -adic valuation by 1 for non-units. In particular, $\delta^n(p^n)$ is a unit for all n . As $\mathbf{Z}_{(p)}$ is the initial δ -ring, it follows that there is no nonzero δ -ring where $p^n = 0$ for some $n \geq 0$.

Remark 2.7 (Limits and colimits of δ -rings, and Witt vectors). One can show that the category of δ -rings admits all limits and colimits, and these are computed at the level of underlying rings: this is easy for limits, and for colimits it follows via the characterization in Remark 2.4 as there is a natural map $\text{colim}_i W_2(-) \rightarrow W_2(\text{colim}_i -)$ of functors. By general nonsense, the forgetful functor from δ -rings to rings has both left and right adjoints. The left adjoint provides one with a notion of “free objects” and is studied in Lemma 2.11 below. The right adjoint is given by the (always p -typical) Witt vector functor $W(-)$ by a result of Joyal [Joy85]. Thus, for each δ -ring R , we have a natural map $R \xrightarrow{w_R} W(R)$ of δ -rings by adjunction; the first two components of this map (in standard Witt co-ordinates) are given by $x \mapsto (x, \delta(x))$, as in Remark 2.4.

Notation 2.8. We shall use the symbols $\{\}$ and $()_\delta$ to denote the adjoining and killing of elements in the theory of δ -rings. Thus, $\mathbf{Z}_{(p)}\{x\}$ is the free δ -ring on one generator x , $\mathbf{Z}_{(p)}\{x, y\}/(f)_\delta$ is defined by a pushout square

$$\begin{array}{ccc} \mathbf{Z}_{(p)}\{t\} & \xrightarrow{t \mapsto f} & \mathbf{Z}_{(p)}\{x, y\} \\ \downarrow t \mapsto 0 & & \downarrow \\ \mathbf{Z}_{(p)} & \longrightarrow & \mathbf{Z}_{(p)}\{x, y\}/(f)_\delta, \end{array}$$

etcetera.

Regarding the quotients that one takes here, we note the following lemma.

Lemma 2.9 (Quotients). *Let A be a δ -ring. Let $I \subset A$ be an ideal. Then I is stable under δ if and only if there exists a (necessarily unique) δ -structure on A/I compatible with the one on A .*

Proof. The “if” direction is clear. For the “only if” direction, we must show that if $a \in A$ and $f \in I$, then $\delta(a) \equiv \delta(a + f) \pmod{I}$; but this follows immediately from the additivity formula. \square

Example 2.10 (Quotients in δ -rings). Let A be a δ -ring, and let $I \subset A$ be an ideal. Then the universal δ - A -algebra B with $IB = 0$ is given by $B = A/J$, where J is the δ -stabilization of I , i.e., the ideal generated by $\cup_n \delta^n(I)$.

Lemma 2.11 (Free δ -rings). *The ring $\mathbf{Z}_{(p)}\{x\}$ is a polynomial ring on the set $\{x, \delta(x), \delta^2(x), \dots\}$ and its Frobenius endomorphism is faithfully flat. The ring $\mathbf{Q}\{x\} = \mathbf{Z}_{(p)}\{x\}[\frac{1}{p}]$ is also a polynomial ring on the set $\{x, \phi(x), \phi^2(x), \dots\}$.*

Proof. Consider the polynomial ring $A = \mathbf{Z}_{(p)}[x_0, x_1, x_2, \dots]$ on countably many generators. The assignment $x_i \mapsto x_i^p + px_{i+1}$ gives an endomorphism ϕ of A that lifts the Frobenius on A/p . As A is p -torsionfree, there is a unique δ -structure on A described by $\delta(x_i) = x_{i+1}$. Given any δ -ring R with an element $f \in R$, thanks to the universal property of the polynomial ring, there is a unique ring homomorphism $\eta_f : A \rightarrow R$ defined by $\eta_f(x_i) = \delta^i(f)$. By construction, we have $\eta_f(\delta(x_i)) = \delta(\eta_f(x_i))$, so η_f is a map of δ -rings, as the relations imposed on δ in Definition 2.1 determine the behaviour on the $\mathbf{Z}_{(p)}$ -algebra generated by the x_i from the behaviour on the x_i . It is then also clear that η_f is uniquely determined as a map of δ -rings by the requirement $\eta_f(x_0) = f$. It follows that setting $\mathbf{Z}_{(p)}\{x\} = A$ with $x = x_0$ gives the free δ -ring on a generator x , so this ring is indeed a polynomial ring on $\{x, \delta(x), \delta^2(x), \dots\}$ as asserted. Inverting p easily gives the desired assertion for $\mathbf{Q}\{x\}$ as well.

We explain the faithful flatness assertion. The map $\phi : A \rightarrow A$ can be written as the filtered colimit of the maps

$$\phi_i : \mathbf{Z}_{(p)}[x, \delta(x), \dots, \delta^i(x)] \rightarrow \mathbf{Z}_{(p)}[x, \delta(x), \dots, \delta^{i+1}(x)],$$

so it is enough to prove that each ϕ_i is faithfully flat. By the fibrewise criterion for flatness, it suffices to see that $\phi_i[1/p]$ and ϕ_i/p are faithfully flat. But $\phi_i[1/p]$ agrees with the map

$$\mathbf{Q}[x, \phi(x), \dots, \phi^i(x)] \rightarrow \mathbf{Q}[x, \phi(x), \dots, \phi^{i+1}(x)]$$

shifting generators, which is evidently faithfully flat. On the other hand, ϕ_i/p agrees with the composite of the Frobenius on $\mathbf{F}_p[x, \delta(x), \dots, \delta^i(x)]$ with the inclusion

$$\mathbf{F}_p[x, \delta(x), \dots, \delta^i(x)] \hookrightarrow \mathbf{F}_p[x, \delta(x), \dots, \delta^{i+1}(x)] ,$$

both of which are faithfully flat. \square

Corollary 2.12 (Frobenius is fpqc locally surjective). *Fix a δ -ring A and an element $x \in A$. Then there exists a faithfully flat map $A \rightarrow B$ of δ -rings such that the image of x in B has the form $\phi(y)$ for some $y \in B$.*

Proof. Set B to be the pushout of the diagram $\mathbf{Z}_{(p)}\{s\} \leftarrow \mathbf{Z}_{(p)}\{t\} \rightarrow A$ of δ -rings, where the first map sends t to $\phi(s)$ and the second map sends t to x . The resulting map $A \rightarrow B$ is faithfully flat by Lemma 2.11 (recalling that pushouts of δ -rings can be computed on the level of underlying rings by Remark 2.7), and the image of x equals that of $\phi(s)$ in B . \square

Remark 2.13 (Joyal's δ_n -operations). As δ -rings R admit a natural map $w : R \rightarrow W(R)$ to their ring of p -typical Witt vectors, there are natural functorial operations $\delta_n : R \rightarrow R$ for $n \geq 0$ on any δ -ring R such that

$$w(x) = (\delta_0(x), \delta_1(x), \delta_2(x), \dots) \in W(R)$$

in Witt vector coordinates; in particular $\delta_0(x) = x$ and $\delta_1(x) = \delta(x)$. These operations can be characterised by the following universal identity: for each element x in each δ -ring R , we have

$$\phi^n(x) = \delta_0(x)^{p^n} + p\delta_1(x)^{p^{n-1}} + \dots + p^n\delta_n(x).$$

In general, one can express $\delta_n(-)$ as a monic polynomial of degree n in δ .

Remark 2.14 (The free δ -ring via Joyal's operations). As the operation δ_n from Remark 2.13 is a monic degree n polynomial in δ , one can reformulate the first part of Lemma 2.11 as the following assertion: the ring $\mathbf{Z}_{(p)}\{x\}$ is a polynomial ring on the set $\{\delta_n(x)\}_{n \geq 0}$.

2.2. Extending δ -structures.

Lemma 2.15 (Localizations). *Let A be a δ -ring. Let $S \subset A$ be a multiplicative subset such that $\phi(S) \subset S$. Then the localization $S^{-1}A$ admits a unique δ -structure compatible with the map $A \rightarrow S^{-1}A$. Moreover, the map $A \rightarrow S^{-1}A$ is initial amongst all δ - A -algebras B such that each element of S is invertible in B .*

Proof. We first explain the argument when A is p -torsionfree. In this case, the localization $S^{-1}A$ is also p -torsionfree. The map $\phi_A : A \rightarrow A$ carries S to itself, and hence induces a map $\phi_{S^{-1}A} : S^{-1}A \rightarrow S^{-1}A$. As $A \rightarrow S^{-1}A$ is a localization, it is easy to see that $\phi_{S^{-1}A}$ is a lift of Frobenius as ϕ_A is so; this gives the first part of the lemma. The second part is clear.

In general, given a pair (A, S) as in the lemma, choose a surjection $\alpha : F \twoheadrightarrow A$ with F being a free δ -ring on some set. Then F is p -torsionfree (by Lemma 2.11). The preimage $T := \alpha^{-1}(S) \subset F$ is a multiplicative closed subset of F (as α is multiplicative) that is ϕ -stable (as α commutes with ϕ). The localization $T^{-1}F$ carries a unique δ -structure compatible with the one on F by the preceding paragraph. The formula $S^{-1}A = T^{-1}F \otimes_F A$, and the fact that colimits of δ -rings coincide with those of the underlying rings, then shows that $S^{-1}A$ also carries a unique δ -structure compatible with the one on A . The last part is clear. \square

Remark 2.16 (Localizations in the p -local world). In a δ -ring A with $p \in \text{rad}(A)$, the formula $\phi(f) = f^p + p\delta(f)$ shows that if f is a unit, so is $\phi(f)$. Thus, for any δ -ring A and any multiplicative subset $S \subset A$, the p -localization $(S^{-1}A)_{(p)}$ of the localization $S^{-1}A$ of A coincides with the p -localization of $T^{-1}A$ where $T = \{S, \phi(S), \phi^2(S), \dots\}$. By Lemma 2.15, it follows that $(S^{-1}A)_{(p)}$ carries a unique δ -structure compatible with the one on A , and can be characterized as the initial object in the category of all δ - A -algebras B with $p \in \text{rad}(B)$ where S becomes invertible.

In the paper, we will use the preceding remark mostly with p -localization replaced with p -completion. Regarding completions, we have the following general result.

Lemma 2.17 (Completions). *Let A be a δ -ring, and let $I \subset A$ be a finitely generated ideal containing p . Then the map $\delta : A \rightarrow A$ is I -adically continuous; more precisely, for each n there is some m such that for all $x \in A$, one has $\delta(x + I^m) \subset \delta(x) + I^n$.*

Moreover, the classical I -adic completion of A acquires a unique δ -structures compatible with the one on A .

Proof. Once we have proved that δ is I -adically continuous, it follows that δ extends to a continuous map on the I -adic completion \widehat{A} of A , which will by continuity still be a δ -structure. This extension is also unique as the δ -structure on \widehat{A} must be $I\widehat{A}$ -adically continuous by the same result applied to \widehat{A} .

For continuity, we note that the additivity formula implies that

$$\delta(x + I^m) - \delta(x) \subset \delta(I^m) + I^m,$$

so it suffices to see that for any n there is some $m \geq n$ such that $\delta(I^m) \subset I^n$. Note that the product (and addition) formula for δ imply that for any two ideals J_1, J_2 ,

$$\delta(J_1 J_2) \subset J_1 + J_2 + p\delta(J_1)\delta(J_2)A.$$

In particular, taking $J_1 = J_2 = I$, we see that $\delta(I^2) \subset I$ as $p \in I$ by assumption. Taking $J_1 = J_2 = I^{2^n}$ then shows inductively that $\delta(I^{2^{n+1}}) \subset I^{2^n}$, as desired. \square

Lemma 2.18 (Étale maps). *Let A be a δ -ring equipped with a finitely generated ideal $I \subset A$ containing p . Assume that B is an A -algebra such that B is derived I -complete, and $A \rightarrow B$ is I -completely étale. Then B admits a unique δ -structure compatible with the one on A .*

This lemma implies, in particular, that the δ -structure on A passes uniquely up to its derived I -completion for any $I \subset A$ containing a power of p .

Proof. We use the characterization of δ -rings in terms of W_2 as in Remark 2.4. By Elkik's algebraization theorem, we can write B as the derived I -completion of some étale A -algebra B' . Then $W_2(A) \rightarrow W_2(B')$ is étale by van der Kallen's theorem, cf. [BMS18, Theorem 10.4]. We claim that $W_2(B)$ is the derived I -completion of $W_2(B')$, when regarded as A -algebras via $A \xrightarrow{w_A} W_2(A) \rightarrow W_2(B')$. For this, note that there is a short exact sequence

$$0 \rightarrow \phi_* B' \rightarrow W_2(B') \rightarrow B' \rightarrow 0$$

of A -modules, and derived I -completion agrees with derived $\phi(I)$ -completion as $p \in I$. In particular $W_2(B)$ is derived I -complete and I -completely étale over $W_2(A)$.

Considering the diagram

$$\begin{array}{ccc} A & \longrightarrow & W_2(B) \\ \downarrow & & \downarrow \\ B' & \longrightarrow & B \end{array}$$

and using that $A \rightarrow B'$ is étale while $W_2(B) \rightarrow B$ is a pro-infinitesimal thickening, we see that there is a unique lift $B' \rightarrow W_2(B)$ making the diagram commute, which then extends to a unique map $w_B : B \rightarrow W_2(B)$ as $W_2(B)$ is derived I -complete. This gives the desired unique δ -structure on B compatible with the one on A . \square

2.3. Distinguished elements. The following notion plays a central role in this paper:

Definition 2.19. An element d of a δ -ring A is *distinguished* if $\delta(d)$ is a unit.

Any morphism of δ -rings preserves distinguished elements.

Example 2.20. The following examples of distinguished elements are crucial for cohomological purposes.

- (1) *Crystalline cohomology.* Take $A = \mathbf{Z}_p$ with $d = p$. Indeed, $\delta(p) = 1 - p^{p-1} \in \mathbf{Z}_p^*$.
- (2) *q -de Rham cohomology.* Take $A = \mathbf{Z}_p[q]$, $d = [p]_q := \frac{q^p - 1}{q - 1} \in A$, with δ -structure determined by $\phi(q) = q^p$. The distinguishedness of d can be seen directly, or by simply observing that $\delta([p]_q) \equiv \delta(p) \pmod{(q - 1)}$, so the claim follows from (1) and $(q - 1)$ -adic completeness.
- (3) *A_{inf} -cohomology.* For a perfectoid field C/\mathbf{Q}_p , take $A = A_{\text{inf}}(\mathcal{O}_C)$ with $d = \xi$ being any generator of Fontaine's map $A_{\text{inf}} \rightarrow \mathcal{O}_C$. The ring A carries a unique δ -structure given by the usual lift of Frobenius. The distinguishedness of d can be seen like (2) via specialization along the δ -map $A_{\text{inf}} \rightarrow W(k)$, where k is the residue field of C .
- (4) *Breuil-Kisin cohomology.* Fix a discretely valued extension K/\mathbf{Q}_p with uniformizer π . Let $W \subset \mathcal{O}_K$ be the maximal unramified subring. Take $A = W[[u]]$ with δ -structure determined by the canonical one on W and satisfying $\phi(u) = u^p$. There is a W -equivariant surjection $A \rightarrow \mathcal{O}_K$ determined by $u \mapsto \pi$. Any generator $d \in A$ of the kernel of this map is distinguished; this can be seen like (2) via specialization along the δ -map $W[[u]] \xrightarrow{u \mapsto 0} W$.

Example 2.21. Consider the universal δ -ring equipped with a distinguished element, i.e., the δ -ring $\mathbf{Z}_{(p)}\{d, \delta(d)^{-1}\}$ equipped with the element d . By Lemma 2.15 and Lemma 2.11, we can also describe this ring explicitly as the localization $S^{-1}\mathbf{Z}_{(p)}\{d\}$, where $S = \{\delta(d), \phi(\delta(d)), \phi^2(\delta(d)), \dots\}$. In particular, the universal distinguished element d is a nonzerodivisor modulo p .

Remark 2.22 (Viewing a distinguished element as a deformation of p). There is a relatively easy way to map any distinguished element to the distinguished element p (up to units). Let A be a derived p -complete δ -ring equipped with a distinguished element d . Consider the composite

$$s : A \xrightarrow{\phi} A \xrightarrow{w} W(A) \xrightarrow{\text{can}} W(A/d),$$

where w comes from Remark 2.7. As s is a map of δ -rings, the Witt vector $s(d) \in W(A/d)$ has the form $F(0, \delta(d), \dots) \in W(A/d)$, where F is the Witt vector Frobenius. Thus, we can write

$s(d) = F(V(\delta(d), \dots)) = p \cdot (\delta(d), \dots)$. As $\delta(d)$ is a unit and $W(A/d)$ is complete along the kernel of $W(A/d) \rightarrow A/d$ (by the derived p -completeness of A/d and the analogous assertion for \mathbf{Z}/p^n -algebras), the Witt vector $(\delta(d), \dots) \in W(A/d)$ is a unit as well, so $s(d) = pu$ for a unit $u \in W(A/d)$. Thus, s gives the promised map. However, the passage from A to $W(A/d)$ is fairly drastic, and one often loses control on the algebraic properties of this map. A refinement of s which is better behaved homologically is presented in Construction 6.1 (under some extra hypotheses on A).

Lemma 2.23. *Let A be a δ -ring. Fix a distinguished element $d \in A$ and a unit $u \in A^*$. If $d, p \in \text{rad}(A)$, then ud is distinguished.*

Proof. We want to show that $\delta(ud)$ is a unit. Expanding, we have

$$\delta(ud) = u^p \delta(d) + d^p \delta(u) + p \delta(u) \delta(d).$$

The first term on the right side is a unit and the other two terms lie in $\text{rad}(A)$, so the whole expression is also a unit. \square

Lemma 2.24. *Let A be a δ -ring with a distinguished element $d \in A$. Assume that we can write $d = fh$ for some $f, h \in A$ such that $f, p \in \text{rad}(A)$. Then f is distinguished and h is a unit.*

Proof. Applying δ to $d = fh$ gives

$$\delta(d) = f^p \delta(h) + h^p \delta(f) + p \delta(f) \delta(h).$$

The left side is a unit, while the first and last terms of the right side lie in $\text{rad}(A)$, so $h^p \delta(f)$ is a unit, which proves both claims. \square

Lemma 2.25. *Fix a δ -ring A and an element $d \in A$ such that $d, p \in \text{rad}(A)$. Then d is distinguished if and only if $p \in (d, \phi(d))$. In particular, the property “ d is distinguished” only depends on the ideal (d) .*

Proof. Assume first that d is distinguished, so $\delta(d)$ is a unit. Then the formula $\phi(d) = d^p + p\delta(d)$ immediately shows that $p \in (d, \phi(d))$.

Conversely, assume we can write $p = ad + b\phi(d)$ for some $a, b \in A$. We want to show $\delta(d)$ is invertible. As $d, p \in \text{rad}(A)$, it suffices to show that $\delta(d)$ is invertible modulo (d, p) ; equivalently, it suffices to show that $A/(p, d, \delta(d)) = 0$. To prove this, we may replace A with its $(p, d, \delta(d))$ -adic completion (or just a suitable ind-Zariski localization) to assume that $p, d, \delta(d) \in \text{rad}(A)$. Simplifying the equation $p = ad + b\phi(d)$ using the definition of ϕ then yields an equation of the form $p(1 - b\delta(d)) = cd$ for suitable $c \in A$. As p is distinguished and $\delta(d) \in \text{rad}(A)$, the left hand side is distinguished by Lemma 2.23. But then Lemma 2.24 implies that d is distinguished, as desired. \square

2.4. Perfect δ -rings.

Definition 2.26. A δ -ring A is *perfect* if ϕ is an isomorphism.

Remark 2.27 (Perfection). The inclusion of perfect δ -rings into all δ -rings has left and right adjoints given by the perfection functors $A \mapsto A_{\text{perf}} = \text{colim}_{\phi} A$ and $A \mapsto A^{\text{perf}} = \text{lim}_{\phi} A$ respectively.

A pleasant feature of this theory is that Frobenius kills the p -torsion.

Lemma 2.28 (p -torsion in δ -rings). *Fix a δ -ring A . Then A is p -torsionfree if one of the following holds true:*

- (1) *The Frobenius ϕ is injective (e.g., A is perfect).*
- (2) *A is reduced.*

Proof. For (1), we shall prove a stronger statement: given $x \in A$ with $px = 0$, we have $\phi(x) = 0$. To see this, apply δ to $px = 0$ to get

$$0 = p^p \delta(x) + x^p \delta(p) + p \delta(x) \delta(p) = p^p \delta(x) + \phi(x) \delta(p).$$

As $\delta(p)$ is a unit, it is enough to show that $p^p \delta(x) = 0$. But we have

$$p^p \delta(x) = p^{p-1}(\phi(x) - x^p) = \phi(p^{p-1}x) - p^{p-1}x^p,$$

and this vanishes as $px = 0$ and $p \geq 2$.

For (2), say A is a reduced δ -ring and $x \in A$ with $px = 0$. We have seen above that $\phi(x) = 0$ and that $p^p \delta(x) = 0$. The latter implies that $p \delta(x) = 0$ as A is reduced. On the other hand, since $\phi(x) = 0$, we have $x^p = -p \delta(x)$, so $x^p = 0$, whence $x = 0$ by reducedness of A . \square

Remark 2.29. A ring R of characteristic p is reduced if and only if its Frobenius endomorphism is injective. Thus, one might wonder if assumptions (1) and (2) in Lemma 2.28 are equivalent. In fact, they are mutually non-comparable, i.e., neither implies the other. For example, the reduced ring $A = \mathbf{Z}_p[x]/(x^p - 1)$ supports a unique δ -structure $\phi(x) = x^p = 1$; the Frobenius lift ϕ on A is not injective since $\phi(x - 1) = 0$. Conversely, the non-reduced ring $B = \mathbf{Z}_p[x]/(x^2)$ supports a unique δ -structure with $\phi(x) = px$; since $\phi(a + bx) = a + pbx$ for $a, b \in \mathbf{Z}_p$, the map ϕ is injective.

Remark 2.30. Lemma 2.28 (1) can also be conceptually understood using the fact that ϕ is a derived Frobenius lift, as in Remark 2.5. To see this, given a δ -ring (A, δ) , write B for simplicial commutative \mathbf{F}_p -algebra $A \otimes_{\mathbf{Z}}^L \mathbf{F}_p$ obtained by base change from A . Then the map $\phi_B : B \rightarrow B$ induced by base change from $\phi : A \rightarrow A$ coincides with the Frobenius endomorphism of B . Now it is known that the Frobenius is always zero on $\pi_i(B)$ for $i > 0$, cf. e.g. [BS17, Proposition 11.6]. In particular, it follows that $\pi_1(\phi_B) : \pi_1(B) \rightarrow \pi_1(B)$ is zero. But the standard resolution of \mathbf{F}_p over \mathbf{Z} identifies $\pi_1(B)$ with $A[p]$ in a ϕ -compatible manner, so it follows that $\phi(A[p]) = 0$, as wanted.

Corollary 2.31 (Perfect δ -rings). *The following categories are equivalent:*

- (1) *The category \mathcal{C}_1 of perfect p -complete δ -rings.*
- (2) *The category \mathcal{C}_2 of p -adically complete and p -torsionfree rings A with A/p being perfect.*
- (3) *The category \mathcal{C}_3 of perfect \mathbf{F}_p -algebras.*

The functor relating (1) and (2) is the forgetful functor; in particular, any ring homomorphism between two perfect p -complete δ -rings is automatically a δ -map. The functors relating (2) and (3) are $A \mapsto A/p$ and $R \mapsto W(R)$; in particular, there is only one δ -structure on $W(R)$ for R perfect of characteristic p .

Proof. Lemma 2.28 ensures that forgetting the δ -structure gives a faithful functor $F_{12} : \mathcal{C}_1 \rightarrow \mathcal{C}_2$. The functor $F_{23} : \mathcal{C}_2 \rightarrow \mathcal{C}_3$ is given by reduction modulo p , i.e. by $A \mapsto A/p$. Finally, the functor $F_{31} : \mathcal{C}_3 \rightarrow \mathcal{C}_1$ is given by $R \mapsto W(R)$. It is well-known (e.g., by deformation theory) that the

resulting functor $F_{32} := F_{12} \circ F_{31}$ is an inverse to F_{23} . As F_{12} is faithful and the equivalence F_{32} factors over F_{12} , it formally follows that F_{12} is also an equivalence. \square

Lemma 2.32 (Perfect elements have rank 1). *Fix a δ -ring A and some $x \in A$. Then $\delta(x^{p^n}) \in p^n A$ for all n . In particular, if A is p -adically separated and $y \in A$ admits a p^n -th root for all $n \geq 0$, then $\delta(y) = 0$, i.e., y has rank 1.*

The conclusion of last part is false if we do not impose some form of p -locality on A : any endomorphism of the ring $\mathbf{Q}[x^{\frac{1}{p^\infty}}]$ determines a δ -structure, but very few of them make x have rank 1. An explicit example is given by taking the Frobenius to be the identity.

Proof. The last assertion is automatic from the first one. For the first one, by reduction to the universal case from Lemma 2.11, we may assume A is p -torsionfree. Thus, $\delta(x^{p^n}) \in p^n A$ if and only if $p\delta(x^{p^n}) \in p^{n+1}A$. But showing the latter is equivalent to checking that $\phi(x^{p^n}) \equiv x^{p^{n+1}} \pmod{p^{n+1}A}$. For $n = 0$, this is true by definition. In general, given elements a, b of a ring C such that $a \equiv b \pmod{p^k C}$ for some $k \geq 0$, we have $a^p \equiv b^p \pmod{p^{k+1}C}$ using the binomial theorem. Applying this inductively to $\phi(x) \equiv x^p \pmod{pA}$ then gives the desired claim. \square

Lemma 2.33 (Distinguished elements in perfect δ -rings). *Let A be a perfect p -complete δ -ring, and fix $d \in A$. Then d is distinguished if and only if the coefficient of p in the Teichmüller expansion of d (defined via Corollary 2.31) is a unit.*

Proof. Say $d = \sum_{i=0}^{\infty} [a_i]p^i$ is the Teichmüller expansion of some $d \in A = W(R)$ for a perfect \mathbf{F}_p -algebra R . One then has

$$\delta(d) = \frac{1}{p} \cdot \left(\sum_{i=0}^{\infty} [a_i^p]p^i - \left(\sum_{i=0}^{\infty} [a_i]p^i \right)^p \right).$$

Reducing modulo p , this gives

$$\delta(d) \equiv a_1^p \pmod{pA},$$

so, by p -completeness of A , d is distinguished exactly when $a_1 \in R$ is a unit. \square

Lemma 2.34. *Let A be a p -torsionfree and p -adically separated δ -ring with A/p reduced. (For example, A could be perfect and p -complete.) Fix $d \in A$ that is distinguished. Then*

- (1) *The element $d \in A$ is a nonzerodivisor.*
- (2) *The ring $R = A/d$ has bounded p^∞ -torsion; in fact, we have $R[p] = R[p^\infty]$.*

Proof. For (1), assume $fd = 0$ for some $f \in A$. We must show $f = 0$. If not, then since A is p -torsionfree and p -adically separated, we may assume $p \nmid f$ (by dividing f by a suitable power of p). Applying δ to $fd = 0$ gives

$$f^p \delta(d) + \delta(f) \phi(d) = 0.$$

Multiplying by $\phi(f)$ and using $\phi(fd) = 0$ gives $f^p \phi(f) \delta(d) = 0$, and hence $f^p \phi(f) = 0$ as $\delta(d)$ is a unit. Reducing modulo p yields $f^{2p} = 0 \pmod{pA}$, whence $f = 0 \pmod{pA}$ as A/pA has an injective Frobenius. But we assumed $p \nmid f$, so we get a contradiction.

For (2), it is enough to show that $R[p] = R[p^2]$. Lifting to A , we must show the following: given $f, g \in A$ with $p^2 f = gd$, we must have $p \mid g$ and hence $pf \in dA$ since A is p -torsionfree. Applying

δ to the containment $gd \in p^2A$ gives $\delta(d)g^p + \delta(g)\phi(d) \in pA$. Multiplying by $\phi(g)$, and using that $\phi(dg) \in pA$, gives $\delta(d)g^p\phi(g) \in pA$. As $\delta(d)$ is a unit, this gives $g^p\phi(g) \in pA$ and hence $g^{2p} \in pA$. Finally, as Frobenius is injective modulo p , we conclude that $g \in pA$, as wanted. \square

2.5. Relation to divided power algebras. In a p -torsionfree $\mathbf{Z}_{(p)}$ -algebra A , we let

$$\gamma_n(x) = \frac{x^n}{n!} \in A[\frac{1}{p}]$$

be the usual divided powers.

Lemma 2.35. *Let A be a p -torsionfree δ -ring. Fix $z \in A$ with $\gamma_p(z) \in A$. Then $\gamma_n(z) \in A$ for all $n \geq 0$.*

Proof. We first explain why $\gamma_{p^2}(z) \in A$. As $\text{val}_p(p^2!) = p+1$, we must check that $\frac{z^{p^2}}{p^{p+1}} \in A$. As A is a δ -ring, we have $\delta(\frac{z^p}{p}) \in A$. Using the definition of δ , we get

$$\delta\left(\frac{z^p}{p}\right) = \frac{1}{p} \left(\frac{\phi(z)^p}{p} - \frac{z^{p^2}}{p^p} \right) = \frac{(z^p + p\delta(z))^p}{p^2} - \frac{z^{p^2}}{p^{p+1}} \in A.$$

Now $\frac{z^p + p\delta(z)}{p} \in A$ by assumption, which gives $\frac{(z^p + p\delta(z))^p}{p^2} = p^{p-2} \cdot \left(\frac{z^p + p\delta(z)}{p}\right)^p \in A$ as well (as $p \geq 2$), so the above formula for $\delta(\frac{z^p}{p})$ shows that $\frac{z^{p^2}}{p^{p+1}} \in A$.

We now inductively prove that $\gamma_n(z) \in A$ for all n and for all pairs (A, z) as in the statement; let us call such a pair *admissible* for the purposes of this proof. It is clearly enough to check that $\gamma_n(z) \in A$ for n divisible by p , so assume $n = kp$, and thus $k < n$. But one checks (by induction on k) that $\gamma_{kp}(z) = u \cdot \gamma_k(\gamma_p(z))$ for a unit $u \in \mathbf{Z}_{(p)}$. Using the same formula for $k = p$, the previous paragraph shows that the pair $(A, \gamma_p(z))$ is admissible. By induction, we know that $\gamma_k(\gamma_p(z)) \in A$ as $k < n$, so the claim follows. \square

Lemma 2.36. *The ring $C := \mathbf{Z}_{(p)}\{x, \frac{\phi(x)}{p}\}$ identifies with the pd -envelope $D := D_{\mathbf{Z}_{(p)}\{x\}}(x)$.*

Proof. By definition, we have a pushout square

$$\begin{array}{ccc} \mathbf{Z}_{(p)}\{y\} & \xrightarrow{y \mapsto pz} & \mathbf{Z}_{(p)}\{z\} \\ y \mapsto \phi(x) \downarrow & & \downarrow \\ \mathbf{Z}_{(p)}\{x\} & \longrightarrow & \mathbf{Z}_{(p)}\{x, z\}/(\phi(x) - pz)_\delta =: C. \end{array}$$

The left vertical map identifies abstractly with the map ϕ on $\mathbf{Z}_{(p)}\{x\}$, and is thus faithfully flat by Lemma 2.11. The right vertical map is then also faithfully flat, so C has no p -torsion. Moreover, as the top horizontal map is an isomorphism after inverting p , the same is true for the bottom horizontal map. We may thus view C as the smallest δ -subring of $\mathbf{Z}_{(p)}\{x\}[\frac{1}{p}]$ that contains $\mathbf{Z}_{(p)}\{x\}$ and $\frac{\phi(x)}{p}$. Equivalently, as $\frac{\phi(x)}{p} = \frac{x^p}{p} + \delta(x)$, we can also view C as the smallest δ -subring of $\mathbf{Z}_{(p)}\{x\}[\frac{1}{p}]$ containing $\mathbf{Z}_{(p)}\{x\}$ and $\frac{x^p}{p}$.

Now consider the pd-envelope D of $\mathbf{Z}_{(p)}\{x\}$ along (x) . As x is a free variable, it is standard that D is p -torsionfree, and may thus be viewed as the smallest subring of $\mathbf{Z}_{(p)}\{x\}[\frac{1}{p}]$ that contains $\mathbf{Z}_{(p)}\{x\}$ and $\frac{x^n}{n!}$ for all $n \geq 1$. We shall check $D = C$ as subrings of $\mathbf{Z}_{(p)}\{x\}[\frac{1}{p}]$.

The containment $D \subset C$ is immediate from Lemma 2.35. To show $C \subset D$, since $\frac{x^p}{p} \in D$, it is enough to show that ϕ preserves D and that the resulting endomorphism of D gives a δ -structure (or, equivalently, that ϕ restricts to a Frobenius lift on D). Observe that

$$\phi\left(\frac{x^n}{n!}\right) = \frac{(x^p + p\delta(x))^n}{n!} = \frac{\sum_{i=0}^n \binom{n}{i} x^{pi} p^{n-i} \delta(x)^{n-i}}{n!} = \frac{\sum_{i=0}^n \binom{n}{i} \cdot (pi)! \cdot p^{n-i} \cdot \frac{x^{pi}}{(pi)!} \cdot \delta(x)^{n-i}}{n!}.$$

To prove ϕ preserves D , it suffices to prove that the coefficient

$$\frac{\binom{n}{i} \cdot (pi)! \cdot p^{n-i}}{n!}$$

is p -integral. But we have

$$\frac{\binom{n}{i} \cdot (pi)! \cdot p^{n-i}}{n!} = \frac{(pi)!}{i!} \cdot \frac{p^{n-i}}{(n-i)!}.$$

Both terms lie in $\mathbf{Z}_{(p)}$, and the first factor is divisible by p for $i > 0$, while the second factor is divisible by p if $i < n$ (as it is a divided power of p). This proves that ϕ preserves D , and that

$$\phi\left(\frac{x^n}{n!}\right) \equiv 0 \pmod{pD}$$

for $n > 0$. We also have

$$\left(\frac{x^n}{n!}\right)^p = \gamma_p\left(\frac{x^n}{n!}\right) \cdot p! = 0 \pmod{pD}$$

for $n > 0$. In particular, the endomorphism ϕ of D reduces modulo p to Frobenius on all generators, and hence all elements, of D , as wanted. \square

Remark 2.37. The proof of Lemma 2.36 shows also that the simplicial commutative ring obtained by freely adjoining $\frac{\phi(x)}{p}$ to $\mathbf{Z}_{(p)}\{x\}$ is discrete, and thus coincides with the ring $\mathbf{Z}_{(p)}\{x, \frac{\phi(x)}{p}\} \simeq D_{\mathbf{Z}_{(p)}\{x\}}(x)$ above.

Corollary 2.38. *Let A be a p -torsionfree δ -ring. Fix $f_1, \dots, f_r \in A$ that define a regular sequence in A/p . Then $A\{\frac{\phi(f_1)}{p}, \dots, \frac{\phi(f_r)}{p}\}$ identifies with the pd-envelope $D_A(I)$ of $I = (f_1, \dots, f_r) \subset A$ (as subrings of $A[\frac{1}{p}]$). In particular, $D_A(I)$ is a δ -ring, and is finitely presented (over A) as such.*

Proof. By induction, we may assume that $r = 1$; set $f = f_1$. Then

$$A\left\{\frac{\phi(f)}{p}\right\} = A \otimes_{\mathbf{Z}_{(p)}\{x\}} \mathbf{Z}_{(p)}\left\{x, \frac{\phi(x)}{p}\right\}.$$

By Lemma 2.36, this can be rewritten as

$$A \otimes_{\mathbf{Z}_{(p)}\{x\}} D_{\mathbf{Z}_{(p)}\{x\}}(x).$$

But the formation of divided power algebras for elements that are regular modulo p commutes with base change, so this tensor product agrees with $D_A(f)$, as desired. \square

Remark 2.39. Corollary 2.38 shows that the pd-envelope $D_A(I)$ is the ϕ -pullback of $A\{\frac{f_1}{p}, \dots, \frac{f_r}{p}\}$. This last ring only depends on the ideal $I = (f_1, \dots, f_r)$, and not on the specific sequence chosen: it can be characterized as the universal p -torsionfree δ - A -algebra B where $p \mid I$. In particular, the pd-envelopes that arise in this fashion have a canonical ϕ -descent.

Warning 2.40. Corollary 2.38 shows that the subring $A\{\frac{\phi(f_1)}{p}, \dots, \frac{\phi(f_r)}{p}\} \subset A[\frac{1}{p}]$, which also equals $A\{\frac{f_1^p}{p}, \dots, \frac{f_r^p}{p}\}$, is independent of the δ -structure on A . The Frobenius appearing in the preceding sentence is crucial, and the claim does not hold without the ϕ -twist. For example, consider the ring $A = \mathbf{Z}_{(p)}[x]$ with $f = x$. If we use the Frobenius lift $x \mapsto x^p$ to define a δ -structure δ_1 , then $R_1 := A\{\frac{x}{p}\} \subset A[\frac{1}{p}]$ contains $\delta_1(\frac{x}{p}) = \frac{x^p}{p^{p+1}} \cdot (p^{p-1} - 1)$. On the other hand, if we use the Frobenius lift $x \mapsto x^p + p$ to define a δ -structure δ_2 , then $R_2 := A\{\frac{x}{p}\} \subset A[\frac{1}{p}]$ contains $\delta_2(\frac{x}{p}) = \frac{x^p}{p^{p+1}} \cdot (p^{p-1} - 1) + \frac{1}{p}$. Any subring of $A[\frac{1}{p}]$ that contains both $\delta_1(\frac{x}{p})$ and $\delta_2(\frac{x}{p})$ must then also contain $\frac{1}{p}$. On the other hand, it is easy to see using Corollary 2.38 that neither R_1 nor R_2 contain $\frac{1}{p}$. In particular, we have $R_1 \not\subset R_2$ and $R_2 \not\subset R_1$.

2.6. Flatness for pd-envelopes. For future reference, we record some well-known facts about divided power envelopes. We formulate these in the generality of simplicial commutative rings as it will be convenient for applications.

Recall the following notion: a map $A \rightarrow B$ of simplicial commutative rings is *flat* if $\pi_0(B)$ is flat over $\pi_0(A)$ and $\pi_i(A) \otimes_{\pi_0(A)} \pi_0(B) \rightarrow \pi_i(B)$ is an isomorphism for all i ; this condition can be detected after derived base change along $A \rightarrow \pi_0(A)$.

Fix a simplicial commutative ring A and a finitely generated ideal $I = (f_1, \dots, f_n) \subset \pi_0(A)$. Let B be a derived I -complete simplicial commutative A -algebra. Then it follows from the definitions that B is *I -completely flat over A* if and only if the map $\text{Kos}(A; f_1, \dots, f_n) \rightarrow \text{Kos}(B; f_1, \dots, f_n)$ is a flat map of simplicial commutative rings. Here the Koszul complexes are defined by

$$\text{Kos}(A; f_1, \dots, f_n) = A \otimes_{\mathbf{Z}[f_1, \dots, f_n]}^L \mathbf{Z}[f_1, \dots, f_n] / (f_1, \dots, f_n) .$$

We shall need a variant.

Definition 2.41. In the preceding situation, a sequence $x_1, \dots, x_r \in \pi_0(B)$ is *I -completely regular relative to A* if the map $\text{Kos}(A; f_1, \dots, f_n) \rightarrow \text{Kos}(B; f_1, \dots, f_n, x_1, \dots, x_r)$ is a flat map of simplicial commutative rings.

It is immediate that these notations are stable under I -complete base changes. A key source of examples is:

Lemma 2.42. *Let A be a simplicial commutative ring. Let B be a p -completely flat A -algebra equipped with a sequence $x_1, \dots, x_r \in \pi_0(B)$ that is p -completely regular relative to A . Let D be the simplicial commutative ring obtained by freely adjoining the divided powers of the x_i 's to B in the ∞ -category of p -complete simplicial commutative A -algebras. Then $A \rightarrow D$ is p -completely flat.*

Proof. To show that $A \rightarrow D$ is p -completely flat, it suffices to show that $\pi_0(A/p) \rightarrow \pi_0(A/p) \otimes_A^L D$ is flat. As the formation of D from B commutes with base change, and because our hypotheses are stable under base change, we may base change along $A \rightarrow \pi_0(A/p)$ to reduce to the case where A

is a discrete \mathbf{F}_p -algebra. In this case, B is a discrete flat A -algebra, and $x_1, \dots, x_r \in B$ is a sequence such that $\text{Kos}(B; x_1, \dots, x_r)$ is A -flat. In this case, it is well-known that D is a free $B/(x_1^p, \dots, x_r^p)$ -module, so we are reduced to showing that $B/(x_1^p, \dots, x_r^p)$ is a flat A -module. But it is easy to see by induction on $\sum_i n_i$ that $\text{Kos}(B; x_1^{n_1}, \dots, x_r^{n_r})$ is A -flat for any r -tuple (n_1, \dots, n_r) of positive integers. \square

Corollary 2.43. *Let A be a p -complete simplicial commutative δ -ring. Let B be a p -completely flat simplicial commutative δ - A -algebra. Fix a sequence $x_1, \dots, x_r \in \pi_0(B)$ that is p -completely regular relative to A . Then the p -complete simplicial commutative δ - A -algebra $C := B\{\frac{x_i}{p}\}^\wedge$ obtained by freely adjoining $\frac{x_i}{p}$ to B is p -completely flat over A .*

Proof. Consider the commutative diagram

$$\begin{array}{ccccccc}
 A & \longrightarrow & A\{x_1, \dots, x_r\}^\wedge & \longrightarrow & B & \longrightarrow & B\{\frac{x_1}{p}, \dots, \frac{x_r}{p}\}^\wedge =: C \\
 \parallel & & \downarrow \psi & & \downarrow \psi_B & & \downarrow \psi_C \\
 A & \longrightarrow & A\{x_1, \dots, x_r\}^\wedge & \longrightarrow & B' & \longrightarrow & B'\{\frac{\phi(x_1)}{p}, \dots, \frac{\phi(x_r)}{p}\}^\wedge =: C'
 \end{array}$$

Here the second vertical map ψ is relative Frobenius, i.e., the unique δ - A -algebra map defined by $x_i \mapsto \phi(x_i)$, and B' and C' are simplicial commutative δ -rings defined by requiring the middle and right squares above to be derived pushout squares in p -complete simplicial commutative rings. The map ψ is p -completely faithfully flat, and hence the same holds for ψ_B and ψ_C . It is thus enough to show that C' is p -completely flat over A . But this follows from Lemma 2.42 and Corollary 2.38. \square

3. PRISMS

In this section, we define prisms (Definition 3.2) and prove some basic properties; notably, we establish the equivalence between perfectoid rings and perfect prisms (Lemma 3.9), and prove a flatness result for certain “prismatic envelopes” (Corollary 3.14) that plays an important role later.

Before defining prisms, let us record the following criterion for when an ideal in a δ -ring can be generated by a distinguished element.

Lemma 3.1. *Fix a pair (A, I) where A is a δ -ring and $I \subset A$ is an ideal that is locally principal. Assume that p and I lie in $\text{rad}(A)$. The following are equivalent:*

- (1) *We have $p \in I^p + \phi(I)A$.*
- (2) *We have $p \in I + \phi(I)A$.*
- (3) *There exists a faithfully flat map $A \rightarrow A'$ of δ -rings that is an ind-Zariski localization such that IA' is generated by a distinguished element d and $d, p \in \text{rad}(A')$.*

Proof. We trivially have (1) \Rightarrow (2).

For (2) \Rightarrow (3), choose $A' = \prod_{i=1}^n \widetilde{A[1/g_i]}$ where $g_1, \dots, g_n \in A$ is a sequence of elements generating the unit ideal of A such that $IA[1/g_i]$ is principal, and $\widetilde{A[1/g_i]}$ denotes the Zariski localization of $A[1/g_i]$ along $V(p, I)$, so p and IA' lie in $\text{rad}(A')$. By Remark 2.16, the ring A' admits a unique δ - A -algebra structure. As p and I lie in $\text{rad}(A)$, it trivially follows that $A \rightarrow A'$ is faithfully flat. By construction, $IA' = (d)$ for some $d \in A'$, and this element is distinguished by Lemma 2.25.

For (3) \Rightarrow (1): under the hypothesis of (3), we must check that $p = 0$ in $A/(I^p + \phi(I)A)$. This can be checked after base change to A' , so it is enough to show that $p = 0$ in $A'/(d^p, \phi(d))$ where $d \in IA'$ is a distinguished generator. But this follows immediately as d is distinguished. \square

Definition 3.2 (The category of prisms). Fix a pair (A, I) comprising a δ -ring A and an ideal $I \subset A$; the collection of all such pairs forms a category called the category of δ -pairs.

- (1) The pair (A, I) is a *prism* if $I \subset A$ defines a Cartier divisor on $\text{Spec}(A)$ such that A is derived (p, I) -complete, and $p \in I + \phi(I)A$. The category of prisms is the corresponding full subcategory of all δ -pairs.
- (2) A prism (A, I) is called
 - *perfect* if A is a perfect δ -ring, i.e., $\phi : A \rightarrow A$ is an isomorphism.
 - *bounded* if A/I has bounded p^∞ -torsion.
 - *orientable* if the ideal I is principal, and the choice of a generator of I is called an *orientation*.
 - *crystalline* if $I = (p)$; any such prism is bounded and orientable.
- (3) A map $(A, I) \rightarrow (B, J)$ of prisms is (*faithfully*) *flat* if the map $A \rightarrow B$ is (p, I) -completely (*faithfully*) flat.

Example 3.3. For any p -torsionfree and p -complete δ -ring A , the pair $(A, (p))$ is a crystalline prism, and conversely any crystalline prism is of this form.

Example 3.4. Let $A_0 = \mathbf{Z}_{(p)}\{d, \delta(d)^{-1}\}$ be the displayed localization of the free δ -ring on a variable d . Let A be the (p, d) -completion of A_0 , and let $I = (d) \subset A$. Then the pair (A, I) is a bounded

prism, and the element $d \in I$ is a distinguished generator (and thus defines an orientation). In fact, this pair is the universal oriented prism (but not the universal orientable prism). For future reference, we note the following: in this prism, the sequence p, d is regular and the Frobenius map $\phi : A/p \rightarrow A/p$ is d -completely flat.

An important property of prisms is the rigidity of the ideal I .

Lemma 3.5 (Rigidity of maps). *If $(A, I) \rightarrow (B, J)$ is a map of prisms, then the natural map induces an isomorphism $I \otimes_A B \cong J$. In particular, $IB = J$.*

Conversely, if $A \rightarrow B$ is a map of δ -rings with B being derived (p, I) -complete, then (B, IB) is a prism exactly when $B[I] = 0$.

Proof. We use Lemma 3.1 to choose faithfully flat maps $A \rightarrow A'$ and $B \rightarrow B'$ such that $IA' = (d)$ and $JB' = (e)$ are principal and $d, p \in \text{rad}(A')$, $p, e \in \text{rad}(B')$. Note that necessarily d and e are nonzerodivisors, as I and J define Cartier divisors. We may also assume that the map $A \rightarrow B$ extends to a map $A' \rightarrow B'$ (replacing B' by a localization of $A' \otimes_A B'$ if necessary). To see that $I \otimes_A B \rightarrow J$ is an isomorphism, it suffices to see that the similar claim holds for the map of δ -pairs $(A', (d)) \rightarrow (B', (e))$, by faithfully flat descent. But now $d = ef$ for some $f \in B'$ and $p, e \in \text{rad}(B')$ by assumption, so Lemma 2.24 says that f is a unit, and thus $(d) = (e)$ in B' .

For the second, note that $B[I] = 0$ if and only if $I \otimes_A B \simeq IB$ via the natural map. It then follows from the first part that if (B, IB) is a prism, then $B[I] = 0$. Conversely, if $I \otimes_A B \simeq IB$, then $IB \subset B$ is an invertible B -module, so it defines a Cartier divisor on $\text{Spec}(B)$. Also, the containment $p \in IB + \phi(I)B$ comes from the corresponding containment on A and B is derived (p, I) -complete by hypothesis, so (B, IB) is indeed a prism. \square

One can get close to proving that the ideal I is necessarily principal:

Lemma 3.6. *Let (A, I) be a prism. Then the ideal $\phi(I)A \subset A$ is principal and any generator is a distinguished element. Moreover, the invertible A -modules $\phi^*(I) = I \otimes_{A, \phi} A$ and I^p are trivial, i.e. isomorphic to A .*

Proof. Once we know one generator of $\phi(I)A$ is distinguished, it follows from Lemma 2.24 that all generators are distinguished. So for the first part it suffices to show that $\phi(I)A$ is generated by a distinguished element. By Lemma 3.1, we can write $p = a + b$ where $a \in I^p$ and $b \in \phi(I)A$. We claim that b generates $\phi(I)A$, i.e., the map $A \rightarrow \phi(I)A$ defined by $1 \mapsto b$ is surjective. Choose a faithfully flat map $A \rightarrow B$ as in Proposition 3.1 (3). It is enough to show that the map $B \rightarrow \phi(I)B$ induced by $1 \mapsto b$ is surjective and that b is distinguished. We have $IB = (d)$ for a distinguished element $d \in B$. Since $a \in I^p$ and $b \in \phi(I)$, we can write $a = xd^p$ and $b = y\phi(d)$ for suitable $x, y \in B$. Our task is to show that y is a unit. As $d, p \in \text{rad}(B)$, it suffices to show that y is a unit modulo (d, p) or equivalently that $B/(d, p, y) = 0$. If not, by localizing along $\text{Spec}(B/(d, p, y)) \subset \text{Spec}(B)$, we may assume that $d, p, y \in \text{rad}(B)$. The equation $p = a + b = xd^p + y\phi(d)$ simplifies to show

$$p(1 - y\delta(d)) = d^p(x + y) = d \cdot (d^{p-1}(x + y))$$

Now $1 - y\delta(d)$ is a unit as $y \in \text{rad}(B)$, so the left side is distinguished by Lemma 2.23. Lemma 2.24 then implies that $d^{p-1}(x + y)$ is a unit, whence d is a unit, which contradicts $d \in \text{rad}(B)$.

Note that $\phi^*(I)$ and I^p become isomorphic over A/p as in characteristic p Frobenius pullback of a line bundle is its p -th power. As $p \in \text{rad}(A)$, it follows that $\phi^*(I)$ and I^p are isomorphic, so it suffices to prove that $\phi^*(I)$ is isomorphic to A . There is a natural surjection $\phi^*(I) \rightarrow \phi(I)A \subset A$, and we proved that $\phi(I)A$ is principal. Let $f \in \phi^*(I)$ be any element mapping to a generator of $\phi(I)A$. We claim that the induced map $A \rightarrow \phi^*(I), 1 \mapsto f$ is an isomorphism. As it is a map of invertible modules, it suffices to prove that it induces a surjection on fibres at all closed points of $\text{Spec}(A)$ (all of which lie in $\text{Spec}(A/p)$ as $p \in \text{rad}(A)$). In particular, letting B be the p -adic completion of A_{perf} , it suffices to see that the map $B \rightarrow \phi^*(I) \otimes_A B, 1 \mapsto f$ is surjective. But now the natural map $\phi^*(I) \otimes_A B \rightarrow \phi(I)B$ is an isomorphism as any distinguished element in a perfect δ -algebra is a nonzerodivisor by Lemma 2.34, and evidently $B \rightarrow \phi(I)B, 1 \mapsto f$ is surjective as we chose f to be a generator of $\phi(I)A$. \square

Lemma 3.7 (Properties of bounded prisms). *Let (A, I) be a bounded prism.*

- (1) *The ring A is classically (p, I) -complete.*
- (2) *Fix a (p, I) -completely flat A -complex $M \in D(A)$. Then M is discrete and classically (p, I) -complete. For any $n \geq 0$, we have $M[I^n] = 0$ and $M/I^n M$ has bounded p^∞ -torsion.*
- (3) *The category of (faithfully) flat maps $(A, I) \rightarrow (B, J)$ of prisms identifies with the category of (p, I) -completely (faithfully) flat δ - A -algebras B by the functor sending such an A -algebra B to (B, IB) .*
- (4) *(Bounded prisms are locally orientable) There exists a (p, I) -completely faithfully flat map $A \rightarrow B$ of δ -rings such that $IB = (d)$ for a distinguished element $d \in B$ that is a nonzerodivisor. In fact, we may choose $A \rightarrow B$ to be the derived (p, I) -completion of an ind-Zariski localization of A . In particular, $(A, I) \rightarrow (B, (d))$ is a faithfully flat map of bounded prisms.*

Proof. For (1), we must show that $A \simeq \lim_k A/(I^k, p^k)$. But this follows from the following sequence of isomorphisms

$$\begin{aligned}
 A &\simeq R \lim_n R \lim_m \text{Kos}(A; I^n, p^m) \\
 &\simeq R \lim_n R \lim_m \text{Kos}(A/I^n; p^m) \\
 &\simeq R \lim_n R \lim_m A/(I^n, p^m) \\
 &\simeq \lim_k A/(I^k, p^k),
 \end{aligned}$$

where the first equality comes from derived (p, I) -completeness of A , the second from the fact that $I^n \subset A$ is locally generated by a nonzerodivisor, the third by virtue of A/I^n having bounded p^∞ -torsion (by devissage and the fact that A/I has this property by assumption), and the last by a simple cofinality argument.

For (2), note that $M \otimes_A^L A/I^n$ is a p -completely flat A/I^n -complex for any $n \geq 0$. Since A/I has bounded p^∞ -torsion, the same holds true for A/I^n for any $n \geq 0$. But then [BMS19, Lemma 4.7] implies that $M \otimes_A^L A/I^n \simeq M/I^n M$ is a discrete A/I^n -module with bounded p^∞ -torsion; in particular, $M[I^n] = 0$ for all $n \geq 0$. The discreteness of M now follows from the derived I -completeness of M and the discreteness of $M \otimes_A^L A/I^n$. Finally, to prove that M is classically (p, I) -complete, we may proceed as in (1).

For (3), note that if $(A, I) \rightarrow (B, J)$ is a (faithfully) flat map of prisms, then B is a (p, I) -completely flat A -algebra by hypothesis, and $J = IB$ by Lemma 3.5. Conversely, say B is a (p, I) -completely (faithfully) flat A -algebra. By Lemma 3.5, it suffices to see that $B[I] = 0$, which follows from part (2).

For (4), take B to be the derived (p, I) -completion of the ring A' from Lemma 3.1 (3), so $IB = (d)$ for a distinguished element $d \in B$. By (2), the ring B is discrete, B/IB has bounded p^∞ -torsion, and $B[I] = 0$, so (B, IB) defines a prism by Lemma 3.5. \square

Lemma 3.8 (Properties of perfect prisms). *Let (A, I) be a perfect prism.*

- (1) *The ideal I is principal and any generator is a distinguished element.*
- (2) *The prism (A, I) is bounded. In particular, the ring A is classically (p, I) -complete.*

Proof. (1) follows from Lemma 3.6 as $\phi : A \rightarrow A$ is an isomorphism by hypothesis. For (2), choose a generator $d \in I$. By Lemma 3.7, it is enough to check that A/d has bounded p^∞ -torsion. Now A is p -torsionfree by Lemma 2.28. As A/p is perfect, it follows from Lemma 2.34 that A/d has bounded p^∞ -torsion. \square

Lemma 3.9 (Perfection of a prism). *Let (A, I) be a prism. Let $A_{\text{perf}} = \text{colim}_\phi A$ be the perfection of A . Then $IA_{\text{perf}} = (d)$ is generated by a distinguished element, d and p are nonzerodivisors in A_{perf} , and $A_{\text{perf}}/d[p^\infty] = A_{\text{perf}}/d[p]$. In particular, the derived (p, I) -completion A_∞ of A_{perf} agrees with its classical (p, I) -adic completion, and $(A, I)_{\text{perf}} := (A_\infty, IA_\infty)$ is the universal perfect prism under (A, I) .*

Proof. As $A \rightarrow A_{\text{perf}}$ factors over $\phi : A \rightarrow A$, it follows that from Lemma 3.6 that IA_{perf} is generated by a distinguished element $d \in A_{\text{perf}}$. As A_{perf} is perfect, p is a nonzerodivisor, and by Lemma 2.34, d is a nonzerodivisor and $A/d[p^\infty] = A/d[p]$. Passing to p -adic completions first, the derived and classical completions agree by p -torsionfreeness, and d is still a nonzerodivisor by Lemma 2.34. Note that as A_{perf} is p -adically complete, any quotient A_{perf}/d^n is derived p -complete, and it also has bounded p^∞ -torsion, so that it is in fact classically p -complete. This implies that we may now also pass to the d -adic completion, where again the derived and classical completions agree, and it is classically (p, d) -adically complete. The universal property of (A_∞, IA_∞) is clear. \square

Theorem 3.10 (Perfectoid rings = perfect prisms). *The following two categories are equivalent:*

- *The category of perfectoid rings R (in the sense of [BMS18, §3]).*
- *The category of perfect prisms (A, I) .*

The functors are $R \mapsto (A_{\text{inf}}(R), \ker(\theta))$ and $(A, I) \mapsto A/I$ respectively.

Proof. For a perfectoid ring R in the sense of [BMS18, §3], the ring $A_{\text{inf}}(R)$ is a perfect δ -ring (clear) and $\ker(\theta)$ is generated by a distinguished element that is a nonzerodivisor by [BMS18, Remark 3.11]. Moreover, $A_{\text{inf}}(R)$ is classically $(p, \ker(\theta))$ -complete by construction, and hence also derived $(p, \ker(\theta))$ -complete. This gives a functor F in the forward direction.

Conversely, fix a perfect prism (A, I) . Lemma 3.8 shows that $I = (d)$ for a distinguished element $d \in A$ that is a nonzerodivisor, and that A is classically (p, d) -complete. To get a functor G in the reverse direction, we shall show that $R = A/d$ is perfectoid. The perfectness of A ensures that the Frobenius on R/p is surjective. As A is perfect, we have $A \cong W(S)$ for a perfect \mathbf{F}_p -algebra

S by Corollary 2.31. Via this isomorphism, we have $d = [a_0] + pu$ for a unit $u \in A$ thanks to Lemma 2.33. If we let $\pi \in R$ be the image of $[a_0^{1/p}]$, then $\pi^p \mid p$ in R . It remains to show that R is p -adically complete. Note that R is derived p -complete as it is the cokernel of the map $A \xrightarrow{d} A$ of derived p -complete modules. As it has bounded torsion by Lemma 2.34, we conclude that R is also p -adically complete in the classical sense. This proves that R is perfectoid.

It is immediate from the construction that $GF \simeq \text{id}$. To prove $FG \simeq \text{id}$, we must check that if (A, I) is a perfect prism, then $(A, I) \simeq (A_{\text{inf}}(A/I), \ker(\theta))$. As $A_{\text{inf}}(A/I) \xrightarrow{\theta} A/I$ is the universal pro-infinitesimal thickening of A/I , there is a unique map $A_{\text{inf}}(A/I) \rightarrow A$ factoring the map down to A/I on either side; in particular, we have $\ker(\theta)A \subset IA$. By Corollary 2.31, it is automatically a map of δ -rings. As $\ker(\theta) \subset A_{\text{inf}}(A/I)$ and $I \subset A$ are both generated by distinguished elements, it follows from Lemma 2.24 that if $\ker(\theta) = dA_{\text{inf}}(A/I)$, then $I = dA$. Moreover, d is a nonzerodivisor on both rings by Lemma 2.34 and both rings are d -adically complete. It remains to simply observe that $A_{\text{inf}}(A/I)/dA_{\text{inf}}(A/I) \rightarrow A/d$ is an isomorphism as both sides identify with A/I . \square

Remark 3.11 (Regular local rings via prisms). Let R be a complete noetherian regular local ring of dimension d with residue field k of characteristic $p > 0$. We explain how the formalism of prisms helps understand the construction of a faithfully flat map $R \rightarrow R_\infty$ with R_∞ perfectoid that appeared in [And18b, Example 3.4.6 (3)] and [Bha18, Proposition 5.2]. We focus on the essential case of mixed characteristic, i.e., we assume R is p -torsionfree.

Choose a Cohen ring W for k with a Frobenius lift ϕ_W . Thanks to the Cohen structure theorem, we may write $R = A/(f)$, where $A := W[[x_1, \dots, x_d]]$ and $f \in \mathfrak{m} - \mathfrak{m}^2$, where $\mathfrak{m} = (p, x_1, \dots, x_d) \subset A$ is the maximal ideal. We first explain how to endow A with the structure of a δ -ring that makes f a distinguished element (and thus $(A, (f))$ becomes a prism). In the $(d+1)$ -dimensional k -vector space $\mathfrak{m}/\mathfrak{m}^2$, we can write $f = a_0p + \sum_{i=1}^d a_i x_i$, with $a_0 \in k$. There are two cases to consider, depending on whether $a_0 = 0$ or not.

If $a_0 \neq 0$, then constant coefficient $f(0)$ has p -adic valuation 1. Consider the Frobenius lift ϕ_A on A extending ϕ_W and satisfying $\phi_A(x_i) = x_i^p$ for all i . We claim that the resulting δ -structure on A makes f distinguished. Indeed, the “evaluating at 0” map $A \xrightarrow{x_i \mapsto 0} W$ is a δ -map with topologically nilpotent kernel that carries f to a distinguished element, so f must be distinguished.

If $a_0 = 0$, then f can be regarded a nonzero W -linear form in the x_i ’s modulo \mathfrak{m}^2 . We can then change variables to assume that $f = x_1 - p$. In this case, the same δ -structure used in the previous paragraph makes f distinguished.

Thus, we have written $R = A/(f)$, where $(A, (f))$ is a prism. As A is a p -torsionfree δ -ring whose mod p reduction $A/p = k[[x_1, \dots, x_d]]$ is regular, the Frobenius $\phi_A : A \rightarrow A$ is p -completely faithfully flat by inspection (or by the easy half of Kunz’s theorem). Write $(A_\infty, (f))$ for the perfection of $(A, (f))$ (Proposition 3.9), and $R_\infty = A_\infty/(f)$ for the corresponding perfectoid ring (Theorem 3.10). As $(A, (f)) \rightarrow (A_\infty, (f))$ is a faithfully flat map of prisms with perfect target, the map $R \rightarrow R_\infty$ is then p -completely faithfully flat with R_∞ perfectoid. As R is noetherian, one checks that $R \rightarrow R_\infty$ is actually faithfully flat [Bha18, Proposition 5.1], thus providing the requisite perfectoid faithfully flat cover of R .

Corollary 3.12 (The site of all prisms). *The category opposite to that of all bounded prisms (A, I) , endowed with the topology where covers are determined by faithfully flat maps of prisms, forms a*

site. The functor that carries (A, I) to A (resp. A/I) forms a sheaf for this topology with vanishing higher cohomology on any (A, I) .

Proof. To check the site axioms, we must show the following: (a) isomorphisms are covers, (b) a composition of covers is a cover, and (c) the pushout of a cover along an arbitrary map is a cover. The first two are automatic. For the last one, using the description from Lemma 3.5, we are reduced to showing the following: given a diagram $(C, IC) \xleftarrow{c} (A, IA) \xrightarrow{b} (B, IB)$ of maps of bounded prisms with b faithfully flat, the map b admits a pushout along c which is also faithfully flat. For this, we simply take D to be the derived (p, I) -completion of $B \otimes_A^L C$. By standard properties, D is (p, I) -completely faithfully flat over C . By Lemma 3.7, it follows that $(C, IC) \rightarrow (D, ID)$ is a faithfully flat map of bounded prisms; one checks that it serves as a pushout of b along c .

Let us check the sheaf axiom now. Say $(A, I) \rightarrow (B, IB)$ is a faithfully flat map of prisms. Let (B^\bullet, IB^\bullet) be the Čech nerve of this map in the category of bounded prisms. By the previous paragraph, this means that B^\bullet is the derived (p, I) -completion of the derived Čech nerve of $A \rightarrow B$. In particular, $\text{Kos}(A; p^n, I^n) \rightarrow \text{Kos}(B^\bullet; p^n, I^n)$ is a limit diagram for all n by faithfully flat descent for maps of simplicial commutative rings. The sheaf axiom and vanishing of higher Čech cohomology for $(A, I) \mapsto A$ now follows by taking a limit over n . The case of $(A, I) \mapsto A/I$ follows (using Lemma 3.5 again). \square

Proposition 3.13 (Existence of prismatic envelopes for regular sequences). *Let (A, I) be a bounded prism, and let B be a (p, I) -completely flat δ - A -algebra. Fix a sequence $x_1, \dots, x_r \in B$ that is (p, I) -completely regular relative to A . Write $J = (I, x_1, \dots, x_r) \subset B$, so there is a map of δ -pairs $(A, I) \rightarrow (B, J)$.*

Then there is a universal map $(B, J) \rightarrow (C, IC)$ of δ -pairs to a prism (C, IC) over (A, I) . Moreover, (C, IC) is flat over (A, I) , and its formation commutes with base changes $(A, I) \rightarrow (A', IA')$. If there is no potential for confusion, we shall write $C := B\{\frac{J}{I}\}^\wedge$.

This construction is compatible with flat localization on B : if $B \rightarrow B'$ is a (p, I) -completely flat map of δ - A -algebras, then $B\{\frac{J}{I}\}^\wedge \hat{\otimes}_B^L B' \cong B'\{\frac{J'}{I}\}^\wedge$, where $J' = JB' \subset B'$.

Proof. Thanks to the base change compatibility and Lemma 3.7, we may work locally on the prism (A, I) , so we may assume $I = (d)$ is principal with a distinguished generator d . Consider the simplicial commutative δ - A -algebra $C := B\{\frac{x_1}{d}, \dots, \frac{x_r}{d}\}^\wedge$ obtained by freely adjoining $\frac{x_i}{d}$ to B in the category of derived (p, I) -complete simplicial δ - A -algebras. We shall check that C is (p, I) -completely flat over A . Granting this, the proposition follows. Indeed, the (p, I) -complete flatness of C forces C to be discrete and d -torsionfree by Lemma 3.7, so $(C, (d))$ is a bounded prism over $(A, (d))$. Moreover, there is a natural map $(B, J) \rightarrow (C, (d))$ over $(A, (d))$ by construction, and its universality is clear: to specify a map from C to any d -torsionfree (p, d) -complete δ -ring D , we must specify a map $B \rightarrow D$ that carries all the x_i , and hence the ideal J , into dD . Finally, the flat localization property is clear from the definition of C (and the fact that the sequence $x_1, \dots, x_r \in B'$ is (p, I) -completely regular relative to A thanks to (p, I) -complete flatness of $B \rightarrow B'$).

We have reduced to checking that C is (p, d) -completely flat over A . Consider the following commutative diagram of derived (p, d) -complete simplicial commutative δ -rings:

$$\begin{array}{ccccccc}
 \mathbf{Z}_p\{z\}^\wedge & \xrightarrow{z \mapsto d} & A & \longrightarrow & B & \longrightarrow & B\{\frac{x_1}{d}, \dots, \frac{x_r}{d}\}^\wedge =: C \\
 \downarrow \psi & & \downarrow & & \downarrow & & \downarrow \\
 \mathbf{Z}_p\{y\}^\wedge & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & B'\{\frac{x_1}{\phi(y)}, \dots, \frac{x_r}{\phi(y)}\}^\wedge =: C' \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & D := A'\{\frac{\phi(y)}{p}\}^\wedge & \longrightarrow & B'' & \longrightarrow & B''\{\frac{x_1}{p}, \dots, \frac{x_r}{p}\}^\wedge =: C''
 \end{array}$$

Here the first row is our input data, the top left vertical map labelled ψ abstractly identifies with the Frobenius on the free (p, z) -complete δ -ring $\mathbf{Z}_p\{z\}^\wedge$, all squares are derived pushout squares of (p, z) -complete simplicial commutative rings over the top left vertex, and description of the bottom right vertex follows by observing that $\phi(y) = p \cdot u \in \pi_0(D)$ for a unit $u \in \pi_0(D)$ by Lemma 2.24.

The map ψ is (p, z) -completely faithfully flat, and hence the same holds true for all the vertical maps relating the first and second rows, so it suffices to show that $A' \rightarrow C'$ is (p, z) -completely flat. The map $D \rightarrow C''$ is (p, z) -completely flat by Corollary 2.43 (as our hypotheses are stable under base change). So it is enough to show that (p, z) -complete flatness, or equivalently (p, y) -complete flatness, over A' can be detected after base change along $A' \rightarrow D$. But Corollary 2.38 ensures that D identifies with the ring obtained by freely adjoining divided powers of y to A' . Since (p, y) -complete flatness of an A' -algebra is defined as flatness after base change to $A' \rightarrow \text{Kos}(A'; p, y)$, it suffices to observe that there is an A' -algebra map $D \rightarrow \text{Kos}(A'; p, y)$ (which easily follows, for instance, by reduction to the universal case $A' = \mathbf{Z}_p\{y\}$). \square

Corollary 3.14 (Existence of prismatic envelopes for smooth algebras). *Fix a bounded prism (A, I) and a p -completely smooth A/I -algebra R . Let B be a (p, I) -completely free δ - A -algebra equipped with an A -algebra surjection $B \rightarrow R$ with kernel J . Then there is a universal map $(B, J) \rightarrow (C, IC)$ of δ -pairs to a prism (C, IC) over (A, I) . Moreover, (C, IC) is flat over (A, I) , and its formation commutes with (p, I) -completely flat base change on B . If there is no potential for confusion, we shall write $C := B\{\frac{J}{I}\}^\wedge$.*

Proof. We shall prove the result for a more general class of pairs δ -pairs (B, J) over (A, I) . For the purposes of this proof, call such a δ -pair *good* if it satisfies the following conditions:

- (1) B is (p, I) -completely flat over A and J is (p, I) -complete.
- (2) There exists a universal map $(B, J) \rightarrow (C, IC)$ of δ -pairs to a prism (C, IC) over (A, I) . Moreover, (C, IC) is flat over (A, I) , and its formation commutes with (p, I) -completely flat base change on B .

Examples of a good pairs are provided by Proposition 3.13. This notion has the following stability properties:

- (a) Good pairs are stable under filtered colimits in the category of all δ -pairs (B, J) over (A, I) with B and J being (p, I) -complete.

- (b) If (B, J) is a δ -pair over (A, I) with B being (p, I) -completely flat over A , and $B \rightarrow B'$ is a (p, I) -completely faithfully flat map with (B', JB'^\wedge) good, then (B, J) is good: this follows by (p, I) -completely faithfully flat descent.

Now take (B, J) as in the corollary. Our goal is to show that (B, J) is good. As B is (p, I) -completely free and R is p -completely smooth, we may use (a) to reduce to the case where B is the (p, I) -completion of a free δ - A -algebra on a finite set. Using the structure of such algebras and suitable truncations, we can then write B as the (p, I) -completed colimit of a filtered diagram $\{B_s\}_{s \in S}$ of (p, I) -completely smooth A -algebras (but not necessarily with δ -structures) with (p, I) -completely faithfully flat transition maps; in fact, we can choose the diagram to be an \mathbf{N} -indexed sequence. By changing indices, we can assume $B_s \rightarrow R$ is surjective for all $s \in S$. Write $J_s \subset B$ for the (p, I) -complete ideal in B generated by the image in B of the kernel of $B_s \rightarrow R$. Then the pair (B, J_s) is good: this follows from the (p, I) -complete faithful flatness of $B_s \rightarrow B$ using Proposition 3.13 as the kernel of $B_s \rightarrow R$ is, locally on $\mathrm{Spf}(B_s)$, generated by a (p, I) -completely regular sequence relative to A (and implicitly using property (b) above to handle the Zariski localization). As the pair (B, J) is the filtered colimit of the pairs (B, J_s) 's, it is also good by property (a). \square

4. THE PRISMATIC SITE AND FORMULATION OF THE HODGE-TATE COMPARISON

Fix a bounded prism (A, I) . Unless otherwise specified, all formal schemes over A are assumed to have the (p, I) -adic topology, and thus formal schemes over A/I have the p -adic topology. Fix a smooth p -adic formal scheme X . In §4.1, we introduce the prismatic site $(X/A)_\Delta$ and discuss its relation to the étale topology of X . A key comparison theorem for prismatic cohomology — the Hodge-Tate comparison — is then formulated in §4.2. We end in §4.3 by discussing the Čech-Alexander approach to computing prismatic cohomology, drawing consequences from the flatness results for prismatic envelopes established earlier.

4.1. The prismatic site.

Definition 4.1 (The prismatic site and its structure sheaf). Let X be a smooth p -adic formal scheme over A/I . Let $(X/A)_\Delta$ be the category of maps $(A, I) \rightarrow (B, IB)$ of bounded prisms together with a map $\mathrm{Spf}(B/IB) \rightarrow X$ over A/I ; the notion of morphism is the obvious one. We shall often denote such an object by

$$(\mathrm{Spf}(B) \leftarrow \mathrm{Spf}(B/IB) \rightarrow X) \in (X/A)_\Delta$$

if no confusion arises. A map $(\mathrm{Spf}(C) \leftarrow \mathrm{Spf}(C/IC) \rightarrow X) \rightarrow (\mathrm{Spf}(B) \leftarrow \mathrm{Spf}(B/IB) \rightarrow X)$ in $(X/A)_\Delta$ is a *flat cover* if $(B, IB) \rightarrow (C, IC)$ is a faithfully flat map of prisms, i.e., C is (p, IB) -completely flat over B . The category $(X/A)_\Delta$ with the topology defined by flat covers is called the *prismatic site* of X/A .

The assignment $(\mathrm{Spf}(B) \leftarrow \mathrm{Spf}(B/IB) \rightarrow X) \mapsto B$ (resp. B/IB) defines a presheaf \mathcal{O}_Δ (resp. $\overline{\mathcal{O}}_\Delta$) of commutative A -algebras (resp. $\mathcal{O}(X)$ -algebras) on $(X/A)_\Delta$. It follows from Corollary 3.12 that $(X/A)_\Delta$ with topology as defined above is indeed a site, and that \mathcal{O}_Δ and $\overline{\mathcal{O}}_\Delta$ are sheaves. We refer to \mathcal{O}_Δ as the *structure sheaf* on $(X/A)_\Delta$.

Thus, given a formal scheme X as above, the data of a object of the prismatic site can be summarized by the following diagram:

$$\begin{array}{ccc} \mathrm{Spf}(B/IB) & \longrightarrow & \mathrm{Spf}(B) \\ \downarrow f & & \downarrow g \\ X & & \\ \downarrow & & \downarrow \\ \mathrm{Spf}(A/I) & \longrightarrow & \mathrm{Spf}(A), \end{array}$$

where f and g are the maps that must be specified (together with the δ - A -algebra B).

Remark 4.2. We only define the prismatic site $(X/A)_\Delta$ above under the assumption that X is smooth. While it would be possible to make the basic definitions without the smoothness assumption, the main theorems (such as the Hodge-Tate comparison) would break down in general. We shall later extend prismatic cohomology to all p -adic formal A/I -schemes via left Kan extension from the smooth case; though we do not do so here, it is possible to recover the same definition using a *derived* prismatic site, i.e. by working with simplicial rings throughout.

Remark 4.3 (The prismatic topos as a slice topos). Let $(*/A)_\Delta$ be the category of all bounded prisms over (A, I) , made into a site as in Corollary 3.12; this also coincides with the site $(\mathrm{Spf}(A/I)/A)_\Delta$ constructed above. The smooth p -adic formal scheme X defines a presheaf sheaf h_X on $(*/A)_\Delta$ by the formula $h_X(B, IB) = \mathrm{Hom}_{A/I}(\mathrm{Spf}(B/IB), X)$. A variant of the argument used to prove Corollary 3.12 shows that h_X is a sheaf on $(*/A)_\Delta$. Unwinding definitions, the prismatic site $(X/A)_\Delta$ introduced above coincides with the category of pairs $((B, IB) \in (*A)_\Delta, \eta \in h_X(B, IB))$, with topology inherited from $(*/A)_\Delta$. In particular, we may view $\mathrm{Shv}((X/A)_\Delta)$ as the slice topos $\mathrm{Shv}((*/A)_\Delta)_{/h_X}$.

Construction 4.4 (Relating prismatic sheaves to étale sheaves). Write $f\mathrm{Sch}/_X$ for the category of p -adic formal schemes U equipped with an adic map $U \rightarrow X$; we endow this category with the étale topology. There is a natural functor $\mu : (X/A)_\Delta \rightarrow f\mathrm{Sch}/_X$ given by sending (B, IB) to $\mathrm{Spf}(B/IB) \rightarrow X$. This functor is cocontinuous by Lemma 2.18: given a bounded prism (B, IB) and a p -completely étale map $B/IB \rightarrow \overline{C}$, there is a unique (p, I) -completely étale map $B \rightarrow C$ of δ -rings lifting the previous map modulo IB . Consequently, this construction defines a morphism

$$\mu_X : \mathrm{Shv}((X/A)_\Delta) \rightarrow \mathrm{Shv}(f\mathrm{Sch}/_X)$$

of topoi by [Sta, Tag 00XO]; we shall simply write μ instead of μ_X if no confusion arises. There is also a natural map $\mathrm{Shv}(f\mathrm{Sch}/_X) \rightarrow \mathrm{Shv}(X_{\mathrm{\acute{e}t}})$ of topoi defined by restriction. Composing with μ_X gives a map

$$\nu_X : \mathrm{Shv}((X/A)_\Delta) \rightarrow \mathrm{Shv}(X_{\mathrm{\acute{e}t}});$$

again, we write ν instead of ν_X if no confusion arises. We shall be interested in the pushforward ν_* along this map, which is described as follows. For any étale map $U \rightarrow X$, the category $(U/A)_\Delta$ is naturally a slice of $(X/A)_\Delta$ in a manner that preserves the notion of coverings. Moreover, restriction along this functor carries sheaves to sheaves. Unwinding definitions then gives that

$$(\nu_* F)(U \rightarrow X) = H^0((U/A)_\Delta, F|_{(U/A)_\Delta}).$$

In other words, the functor ν allows us to localize the prismatic cohomology on $X_{\mathrm{\acute{e}t}}$.

Remark 4.5 (The perfect prismatic site). One can define a version of the prismatic site by restricting to the full subcategory of $(X/A)_\Delta^{\mathrm{perf}} \subset (X/A)_\Delta$ of those objects of the prismatic site $(B, IB; \mathrm{Spf} B/IB \rightarrow X)$ for which B is perfect. Equivalently, this is the category of perfectoid rings R over A/I together with a map $\mathrm{Spf} R \rightarrow X$, and so this admits a description that does not use the notion of prisms. The cohomology of the perfect prismatic site has a different flavour than that of the prismatic site, and we will analyze it in Section 8.

Remark 4.6 (The absolute prismatic site). For any p -adic formal scheme X , we can also consider the absolute prismatic site $(X)_\Delta$; its objects are bounded prisms (B, J) equipped with a map $\mathrm{Spf}(B/J) \rightarrow X$ of p -adic formal schemes. We do not seriously explore this notion in this paper. However, the following observation shall be useful later:

Lemma 4.7. *Let (A, I) be a perfect prism corresponding to a perfectoid ring $R = A/I$. Then for any prism (B, J) , any map $A/I \rightarrow B/J$ of commutative rings lifts uniquely to a map $(A, I) \rightarrow (B, J)$ of prisms.*

Using suggestive notation, this says that $A \in (R)_\Delta$ is initial (or rather final, as we pass to opposite categories in order to get a site), so that $(R)_\Delta = (R/A)_\Delta$.

Proof. Fix a prism (B, J) . We must show that any map $A/I \rightarrow B/J$ of commutative rings lifts uniquely to map $(A, I) \rightarrow (B, J)$ of prisms. Now L_{A/\mathbf{Z}_p} vanishes after derived p -completion as A is ring of Witt vectors of a perfect ring of characteristic p , while both B and B/J are derived p -complete. It follows by deformation theory that $\mathrm{Hom}(A, B) \simeq \mathrm{Hom}(A, B/J) \simeq \mathrm{Hom}(A, B/(J, p))$, so the composite map $A \rightarrow A/I \rightarrow B/J$ lifts uniquely to a map $\alpha : A \rightarrow B$. This argument also proves that α is compatible with the Frobenius lifts on A and B . Also, by construction, we have $\alpha(I) \subset J$. Thus, once we know α is a δ -map (which is not automatic, since B might have p -torsion), we will have found a unique map $(A, I) \rightarrow (B, J)$ of prisms lifting the given map $A/I \rightarrow B/J$, proving the lemma. To check α is a δ -map, note that α factors uniquely over $B^{\mathrm{perf}} := \lim_\phi B \rightarrow B$ by the bijectivity of ϕ on A . As B^{perf} is a perfect δ -ring, it is p -torsionfree. Thus, the ϕ -equivariant map $A \rightarrow B^{\mathrm{perf}}$ is automatically a δ -map. Moreover, the map $B^{\mathrm{perf}} \rightarrow B$ is a δ -map by construction, so the composite $\alpha : A \rightarrow B$ is indeed a δ -map, as wanted. \square

The “dual” to the above assertion is false: given a prism (A, I) and a perfect prism (B, J) , there might be multiple maps $(A, I) \rightarrow (B, J)$ inducing the same map $A/I \rightarrow B/J$.

4.2. The Hodge-Tate comparison theorem: formulation.

Construction 4.8 (The Hodge-Tate comparison map). Using Construction 4.4, we can produce some complexes on $X_{\acute{e}t}$ that will be fundamental to this paper. Write

$$\Delta_{X/A} := R\nu_* \mathcal{O}_\Delta \in D(X_{\acute{e}t}, A) \quad \text{and} \quad \overline{\Delta}_{X/A} := R\nu_* \overline{\mathcal{O}}_\Delta \in D(X_{\acute{e}t}, \mathcal{O}_X).$$

These are commutative algebra objects of the corresponding derived categories, and we have the formula

$$\overline{\Delta}_{X/A} \simeq \Delta_{X/A} \otimes_A^L A/I.$$

In particular, setting $M\{i\} := M \otimes_{A/I} I^i/I^{i+1}$ for an A/I -module M , we have a Bockstein differential

$$\beta_I : H^i(\overline{\Delta}_{X/A})\{i\} \rightarrow H^{i+1}(\overline{\Delta}_{X/A})\{i+1\}$$

from the triangle

$$\overline{\Delta}_{X/A}\{i+1\} \rightarrow \Delta_{X/A} \otimes_A^L I^i/I^{i+2} \rightarrow \overline{\Delta}_{X/A}\{i\}.$$

As i varies, these maps assemble to endow the graded ring $H^*(\overline{\Delta}_{X/A})\{*\}$ with the structure of a differential graded A/I -algebra whose 0-th term is an \mathcal{O}_X -algebra; write $\eta_X^0 : \mathcal{O}_X \rightarrow H^0(\overline{\Delta}_{X/A})$ for the structure map. By the universal property of Kähler differentials, we have an \mathcal{O}_X -module map $\eta_X^1 : \Omega_{X/(A/I)}^1 \rightarrow H^1(\overline{\Delta}_{X/A})\{1\}$ given by sending a local section fdg to $f\beta_I(g)$. To extend this to larger degrees, note that the differential graded algebra $H^*(\overline{\Delta}_{X/A})\{*\}$ is graded commutative by generalities on cohomology of commutative ring objects in a topos. Moreover, we have the following weak form of strict graded commutativity

Lemma 4.9. *For any local section $f \in \mathcal{O}_X(U)$, the differential $\beta_I(f) \in H^1(\overline{\Delta}_{X/A})\{1\}(U)$ of the image of f in $H^0(\overline{\Delta}_{X/A})(U)$ squares to zero.*

This lemma is only non-obvious when $p = 2$. We do not prove it here; it follows from Proposition 6.2, which we shall prove later. Assuming Lemma 4.9, we can use the universal property of the de Rham complex to extend the graded \mathcal{O}_X -module map

$$\eta_X^0 \oplus \eta_X^1 : \mathcal{O}_X \oplus \Omega_{X/(A/I)}^1 \rightarrow H^0(\overline{\Delta}_{X/A}) \oplus H^1(\overline{\Delta}_{X/A})\{1\}$$

(that is compatible with the differential) as defined above to a map

$$\eta_X^* : \Omega_{X/(A/I)}^* \rightarrow H^*(\overline{\Delta}_{X/A})\{*\}$$

of commutative differential graded algebras compatibly with the \mathcal{O}_X -module structure on the terms. When the prism (A, I) is oriented via the choice of a distinguished element $d \in I$, we have $A/I \simeq I^i/I^{i+1}$ via d^i , so we may drop the twists appearing above; we typically do this when $d = p$. For future reference, we remark that the construction of the map η_X^* depends functorially on the commutative algebra object $\Delta_{X/A} \in D(X_{\text{ét}}, A)$ together with the \mathcal{O}_X -algebra structure on $\Delta_{X/A} \otimes_A^L A/I$.

This map forms our main tool for controlling $\Delta_{X/A}$.

Theorem 4.10 (Hodge-Tate Comparison). *The map*

$$\eta_X^* : \Omega_{X/(A/I)}^* \rightarrow H^*(\overline{\Delta}_{X/A})\{*\}$$

constructed above is an isomorphism of differential graded A/I -algebras. In particular, $\overline{\Delta}_{X/A} \in D(X_{\text{ét}}, \mathcal{O}_X)$ is a perfect complex with $H^i(\overline{\Delta}_{X/A}) \cong \Omega_{X/(A/I)}^i\{-i\}$.

Theorem 4.10 will be proven in §5 in characteristic p , and then in §6 in general. In the rest of this subsection, we record some consequences.

Corollary 4.11. *The formation of $\Delta_{X/A} \in D(X_{\text{ét}}, A)$ commutes with base change along a map $(A, I) \rightarrow (B, IB)$ of bounded prisms, i.e., if $g : X_B := X \times_{\text{Spf}(A/I)} \text{Spf}(B/IB) \rightarrow X$ denotes the projection, then $(g^*\Delta_{X/A})^\wedge \cong \Delta_{X_B/B}$, where $(-)^^\wedge$ denotes derived (p, I) -completion.*

Proof. By derived Nakayama, it suffices to prove the base change modulo I , i.e. that

$$(g^*\overline{\Delta}_{X/A})^\wedge \cong \overline{\Delta}_{X_B/B},$$

where the completion is the derived p -completion. This follows from the Hodge-Tate comparison as differentials $\Omega_{X/A}^*$ commute with base change. \square

Remark 4.12 (The Higgs specialization). Assume A is p -torsionfree. If X lifts to a (p, I) -completely flat (p, I) -adic formal scheme \mathfrak{X}/A equipped with a lift of Frobenius, then the affines in $X_{\text{ét}} \simeq \mathfrak{X}_{\text{ét}}$ naturally yield objects of $(X/A)_\Delta$: for each such affine $\mathfrak{U} \rightarrow \mathfrak{X}$, the diagram $(\mathfrak{U} \leftarrow \mathfrak{U} \times_A A/I \rightarrow X)$ is an object of $(X/A)_\Delta$. Passing to global sections, this gives a ϕ -equivariant map $\Delta_{X/A} \rightarrow \mathcal{O}_X$ of E_∞ - A -algebras on $X_{\text{ét}}$. Reducing modulo I , this determines an E_∞ - A/I -algebra map $\overline{\Delta}_{X/A} \rightarrow \mathcal{O}_X$ that splits the structure map $\mathcal{O}_X \rightarrow \overline{\Delta}_{X/A}$; note that this map depends on the choice of \mathfrak{X} .

Remark 4.13 (Obstruction to lifting modulo I^2). One can show that the map $\mathcal{O}_X \rightarrow \tau^{\leq 1}\overline{\Delta}_{X/A}$ splits as an \mathcal{O}_X -module map if and only if X lifts to A/I^2 . To see this, we use the cotangent complex; in this remark, all cotangent complexes are interpreted in the p -complete sense. By

general properties of the cotangent complex of the quotient of a ring by a nonzerodivisor, lifting X to A/I^2 is equivalent to splitting the map $L_{X/A} \rightarrow L_{X/(A/I)}$. The desired claim now follows from:

Proposition 4.14. *There is a natural isomorphism $L_{X/A} \simeq (\tau^{\leq 1} \overline{\Delta}_{X/A})\{1\}[1]$.*

Proof sketch. When $X := \mathrm{Spf}(R)$ is affine, we shall construct a natural map $L_{R/A} \rightarrow \overline{\Delta}_{R/A}\{1\}[1]$ (where we follow Notation 4.15 below). As both objects appearing in the statement of the proposition lie in $D_{\mathrm{perf}}(X)$ (by Theorem 4.10 for the right side), this will yield the desired comparison map on globalizing as $L_{X/A} \in D^{\leq 0}$. For this map, observe that given any object $(R \rightarrow B/IB \leftarrow B) \in (R/A)_{\Delta}$, we have a natural map

$$L_{R/A} \rightarrow L_{(B/IB)/B} \simeq I/I^2 \otimes_{A/I}^L B/IB[1] \simeq B/IB\{1\}[1].$$

Taking derived global sections over $(R/A)_{\Delta}$ then yields the desired comparison map.

It remains to show that the map $\mu : L_{X/A} \rightarrow (\tau^{\leq 1} \overline{\Delta}_{X/A})\{1\}[1]$ constructed above is an isomorphism. Note that the cohomology sheaves of both sides are abstractly isomorphic: we get $\mathcal{O}_X\{1\}$ in degree 0 and $\Omega_{X/A}^1$ in degree 1 (by the transitivity triangle for the left side, and by Theorem 4.10 for the right side). Moreover, the formation of both sides and the map above commutes with p -completed base change (using Corollary 4.11 for the right side). Using this property, we reduce to checking the statement when $I = (p)$. Using Theorem 4.10, it is enough to show that the maps $H^i(\mu)$ for $i = 0, 1$ agree with the maps coming from Construction 4.8 used to produce the maps in Theorem 4.10. For this compatibility, one reduces to the case where $X = \mathbf{A}^1$. By the base change compatibility of the construction, we may then even assume $A = \mathbf{Z}_p$. In this case, we leave it to the reader to check the desired compatibility. Alternately, a complete proof can be found in [ALB19b, Proposition 3.2.1]. \square

The $I = (p)$ case of the preceding discussion is closely related to [DI87, Theorem 3.5] via the de Rham comparison isomorphism (Theorem 15.4); see also [BMS18, §8.2] for a variant in the A_{inf} -cohomology theory, and Remark 15.6 for a variant in the singular case.

4.3. Generalities on computing prismatic cohomology.

Notation 4.15. Let (A, I) be a bounded prism, and let X be a smooth p -adic formal A/I -scheme. When $X := \mathrm{Spf}(R)$ is affine, we also write $(R/A)_{\Delta}$ for $(\mathrm{Spf}(R)/A)_{\Delta}$; a typical object here is seen as a diagram $(B \rightarrow B/IB \leftarrow R)$ where (B, IB) is a prism over (A, I) . In this affine situation, set $\Delta_{R/A} := R\Gamma(\mathrm{Spec}(R), \Delta_{\mathrm{Spec}(R)/A})$ and $\overline{\Delta}_{R/A} := \Delta_{R/A} \otimes_A^L A/I$.

Construction 4.16 (Čech-Alexander complexes for prismatic cohomology). Let R be a p -completely smooth A/I -algebra. Let B be the derived (p, I) -completion of a free δ - A -algebra equipped with an A -algebra surjection $B \rightarrow R$ with kernel J . As A is bounded, the derived (p, I) -completion here agrees with the classical (p, I) -completion by Lemma 3.7. Let B^{\bullet} be the derived (p, I) -completed Čech nerve of $A \rightarrow B$, and let $J^{\bullet} \subset B^{\bullet}$ be the kernel of the augmentation map $B^{\bullet} \rightarrow B \rightarrow R$. As each B^n is a (p, I) -completely free δ - A -algebra, we may apply the functor from Corollary 3.14 to the pair $(B^{\bullet}, J^{\bullet})$, and obtain a cosimplicial object $(C^{\bullet} \rightarrow C^{\bullet}/IC^{\bullet} \leftarrow R)$ in $(R/A)_{\Delta}$ where each $C^i := B^i\{\frac{I^i}{I}\}^{\wedge}$ is (p, I) -completely flat over A . It is easy to see that this cosimplicial object is the Čech nerve of $(C^0 \rightarrow C^0/IC^0 \leftarrow R)$. As B was free, any object in $(R/A)_{\Delta}$ receives a (not

necessarily unique) map from the degree zero term $(C^0 \rightarrow C^0/IC^0 \leftarrow R)$ of this cosimplicial object. In other words, the object $(C^0 \rightarrow C^0/IC^0 \leftarrow R) \in (R/A)_\Delta$ covers the final object of the topos $\mathrm{Shv}((R/A)_\Delta)$. Consequently, $\Delta_{R/A}$ is computed by C^\bullet . Note that the construction of C^\bullet commutes with base change on the prism (A, I) .

In the preceding construction, we may choose B strictly functorially in R ; for instance, set B to be a free derived (p, I) -complete δ - A -algebra on the set $W(R)$, so there is a natural surjective δ - A -algebra map $B \rightarrow W(R)$, and hence a natural surjective map $B \rightarrow R$. Via such a choice, we obtain a strictly functorial chain complex $C^\bullet((R/A)_\Delta, \mathcal{O}_\Delta)$ calculating $\Delta_{R/A}$. This complex, however, does not commute with base change on the prism (A, I) as the choice of B is not base change compatible.

Remark 4.17 (A base change compatible Čech-Alexander complex for the affine line). Let us explain how to make Construction 4.16 explicit for $R = A/I\langle X \rangle$ being the p -adic completion of a polynomial ring. In this case, for the ring B from Construction 4.16, we may simply take the derived (p, I) -completion of $A\{X\}$, with the map $B \rightarrow R$ determined by sending X to X and $\delta^i(X)$ to 0 for $i > 0$. The resulting cosimplicial δ - A -algebra C^\bullet commutes with base change on the prism (A, I) .

Using the complexes from Construction 4.16, we prove two stability properties for prismatic cohomology that will be useful for the proof of the Hodge-Tate comparison. We note that both are easy consequences of the Hodge-Tate comparison, and in particular the first is a special case of Corollary 4.11:

Lemma 4.18 (Base change). *Let R be a p -completely smooth A/I -algebra. Let $(A, I) \rightarrow (A', I')$ be a map of bounded prisms such that $A \rightarrow A'$ has finite (p, I) -complete Tor amplitude. If we write $R' := R \hat{\otimes}_A A'$ for the base change, then the natural map induces an isomorphism $\Delta_{R/A} \hat{\otimes}_A^L A' \cong \Delta_{R'/A'}$, and similarly for $\bar{\Delta}_{R/A}$.*

Proof. By inspection and Corollary 3.14, the formation of the complex in Construction 4.16 commutes with (p, I) -completed base change along $A \rightarrow A'$ in a weak sense, i.e., the cosimplicial ring $C^\bullet((R/A)_\Delta, \mathcal{O}_\Delta) \hat{\otimes}_A^L A'$ obtained as the *termwise* derived (p, I) -completed base change of the cosimplicial ring $C^\bullet((R/A)_\Delta, \mathcal{O}_\Delta)$ computes $\Delta_{R'/A'}$. It now remains to observe that since $A \rightarrow A'$ has finite (p, I) -complete Tor amplitude, the functor of (p, I) -completed base change along $A \rightarrow A'$ commutes with totalizations of cosimplicial (p, I) -complete A -modules by Lemma 4.20. The result for $\bar{\Delta}$ follows by reduction modulo I . \square

Second, it localizes for the étale topology.

Lemma 4.19 (Étale localization). *Let $R \rightarrow S$ be a p -completely étale map of p -completely smooth A/I -algebras. Then the natural map $\bar{\Delta}_{R/A} \hat{\otimes}_R^L S \rightarrow \bar{\Delta}_{S/A}$ is an isomorphism.*

Proof. Let us begin with a general observation. There is an obvious functor $(S/A)_\Delta \rightarrow (R/A)_\Delta$ given by restriction of scalars along $R \rightarrow S$. This functor has a right adjoint $F : (R/A)_\Delta \rightarrow (S/A)_\Delta$ described as follows: given a prism $(B \rightarrow B/IB \leftarrow R) \in (R/A)_\Delta$, the induced p -completely étale map $B/IB \rightarrow B/IB \hat{\otimes}_R^L S$ deforms uniquely to a (p, I) -completely étale δ -map $B \rightarrow B_S$ by Lemma 2.18, giving an object $F(B \rightarrow B/IB \leftarrow R) := (B_S \rightarrow B_S/IB_S \simeq B/IB \hat{\otimes}_R^L S \leftarrow S)$ of

$(S/A)_{\Delta}$. As F is a right adjoint, it preserves products and carries weakly final objects (i.e., those objects that admit a map from any other object) to weakly final objects.

Now consider the cosimplicial object $(C^{\bullet} \rightarrow C^{\bullet}/IC^{\bullet} \leftarrow R)$ in $(R/A)_{\Delta}$ given by Construction 4.16, so C^{\bullet}/IC^{\bullet} computes $\overline{\Delta}_{R/A}$. By construction, this cosimplicial object is the Čech nerve of its zeroth term $(C^0 \rightarrow C^0/IC^0 \leftarrow R)$, and this zeroth term is a weakly final object. Consider the corresponding cosimplicial object $(D^{\bullet} \rightarrow D^{\bullet}/ID^{\bullet} \leftarrow S) := F(C^{\bullet} \rightarrow C^{\bullet}/IC^{\bullet} \leftarrow R)$ in $(S/A)_{\Delta}$. As F preserves both products and weakly final objects, it follows that $\overline{\Delta}_{S/A}$ is computed by D^{\bullet}/ID^{\bullet} . Our task is thus to show that the natural map $C^{\bullet}/IC^{\bullet} \widehat{\otimes}_R^L S \rightarrow D^{\bullet}/ID^{\bullet}$ is an isomorphism. This is true at the level of terms by construction of the functor F , so the claim follows from Lemma 4.20. \square

The following purely algebraic lemma was used above.

Lemma 4.20. *Let C be a commutative ring equipped with a finitely generated ideal J . Let $C \rightarrow C'$ be a map of commutative rings that has finite J -complete Tor amplitude. Then the J -completed base change functor $-\widehat{\otimes}_C^L C'$ commutes with totalizations in $D^{\geq 0}$.*

Proof. Let M^{\bullet} be a cosimplicial object in $D^{\geq 0}$ with totalization M . We must show that

$$\mathrm{Tot}(M^{\bullet}) \widehat{\otimes}_C^L C' \simeq \mathrm{Tot}(M^{\bullet} \widehat{\otimes}_C^L C')$$

via the natural map. If instead M^{\bullet} one uses the n -truncation $M^{\leq n}$ of M , then the corresponding assertion is clear. We are thus reduced to showing the following: there exists some $c \geq 0$ such that if $K \in D^{\geq n}(C)$, then $K \widehat{\otimes}_C^L C' \in D^{\geq n-c}(C')$ for all $n \geq 0$. Choose generators $x_1, \dots, x_r \in J$. By derived Nakayama, it is enough to find some $c \geq 0$ such that the functor

$$K \mapsto (K \otimes_C^L C') \otimes_{C'}^L \mathrm{Kos}(C'; x_1, \dots, x_r)$$

takes $D^{\geq n}$ to $D^{\geq n-c}$. But we can write

$$(K \otimes_C^L C') \otimes_{C'}^L \mathrm{Kos}(C'; x_1, \dots, x_r) \simeq (K \otimes_C^L \mathrm{Kos}(C; x_1, \dots, x_r)) \otimes_{\mathrm{Kos}(C; x_1, \dots, x_r)}^L \mathrm{Kos}(C'; x_1, \dots, x_r).$$

Now the operation $K \mapsto K \otimes_C^L \mathrm{Kos}(C; x_1, \dots, x_r)$ takes $D^{\geq n}$ to $D^{\geq n-r}$, while the operation $N \mapsto N \otimes_{\mathrm{Kos}(C; x_1, \dots, x_r)}^L \mathrm{Kos}(C'; x_1, \dots, x_r)$ takes $D^{\geq m}$ to $D^{\geq m-a}$ for some fixed a by assumption on the Tor amplitude. It follows that taking $c = a + r$ solves the problem. \square

5. THE CHARACTERISTIC p CASE: THE HODGE-TATE AND CRYSTALLINE COMPARISONS

In this section, we prove the crystalline comparison for prismatic cohomology; the key tool is the relationship between divided power envelopes and δ -structures (Corollary 2.38). Using the Cartier isomorphism, we deduce the Hodge-Tate comparison over crystalline prisms (Corollary 5.4).

Notation 5.1. Let $(A, (p))$ be a crystalline prism and let $I \subset A$ be a pd-ideal with $p \in I$. In particular the Frobenius $A/p \rightarrow A/p$ factors over the quotient $A/p \rightarrow A/I$, inducing a map $\psi : A/I \rightarrow A/p$. Let R be a smooth A/I -algebra and let $R^{(1)} = R \otimes_{A/I, \psi} A/p$.

Theorem 5.2 (The crystalline comparison in characteristic p). *Write $R\Gamma_{\text{crys}}(R/A)$ for the crystalline cohomology of $\text{Spf}(R)$ relative to the pd-thickening $\text{Spec}(A/I) \subset \text{Spf}(A)$. Then there is a canonical isomorphism $\Delta_{R^{(1)}/A} \simeq R\Gamma_{\text{crys}}(R/A)$ of E_∞ - A -algebras compatibly with the Frobenius.*

Remark 5.3. We will often apply the theorem when $R = \tilde{R} \otimes_{A/p} A/I$ for some smooth A/p -algebra \tilde{R} . In that case $R^{(1)}$ is the pullback of \tilde{R} under ϕ , and the base change compatibility of prismatic cohomology proved below translates the theorem into

$$\phi_A^* \Delta_{\tilde{R}/A} \cong R\Gamma_{\text{crys}}(\tilde{R}/A) .$$

Proof. Let us first explain how to construct a map $\Delta_{R^{(1)}/A} \rightarrow R\Gamma_{\text{crys}}(R/A)$ of E_∞ - A -algebras compatibly with the Frobenius. Let \mathcal{C} be the category of A -algebra surjections $B \rightarrow R$ where B is a δ - A -algebra that is p -completely ind-smooth as an A -algebra. For any $(B \rightarrow R) \in \mathcal{C}$, we can compute the complex $R\Gamma_{\text{crys}}(R/A)$ by D^\bullet , where $(D^\bullet \rightarrow R)$ is the p -completed pd-envelope of $(B^\bullet \rightarrow R)$, where B^\bullet is the p -completed Čech nerve of $A \rightarrow B$. Note that Corollary 2.38 and a Zariski localization argument show that the δ -structure on B^\bullet passes to yield a unique δ -structure on D^\bullet . Combining the resulting Frobenius lift on D^\bullet with the existence of divided powers on the kernel of $D^\bullet \rightarrow R$ gives a commutative diagram

$$\begin{array}{ccc} D^\bullet & \xrightarrow{\phi_D} & \phi_* D^\bullet \\ \downarrow & & \downarrow \\ R & \longrightarrow & \phi_* D^\bullet / p \end{array}$$

of cosimplicial A -algebras, where R is regarded as a constant cosimplicial ring. This gives a cosimplicial object $(D^\bullet \rightarrow D^\bullet/p \leftarrow R^{(1)})$ of $(R^{(1)}/A)_\Delta$, and thus yields a comparison map

$$\Delta_{R^{(1)}/A} \rightarrow D^\bullet \simeq R\Gamma_{\text{crys}}(R/A),$$

as promised. As \mathcal{C} is sifted, this comparison map is independent of the choice made.

We must show that this is an equivalence. We are now allowed to make choices, so pick a lift \tilde{R} of R to A/p . Choose an A -algebra surjection $B \rightarrow \tilde{R}$ where B is a p -completely free δ - A -algebra (i.e., B is the p -completion of a free δ - A -algebra on a set of generators). We can form the Čech nerve B^\bullet as before, and obtain the cosimplicial ring $C^\bullet = B^\bullet\{\frac{L^\bullet}{p}\}$ as in Construction 4.16 that computes $\Delta_{\tilde{R}/A}$. Note that then $\phi_A^* C^\bullet$ computes $\Delta_{R^{(1)}/A}$ (beware that a priori we do not know

that ϕ_A^* commutes with totalizations, but we will not need this). Our goal is to explain why this cosimplicial ring also computes $R\Gamma_{\text{crys}}(R/A)$. The maps $A \rightarrow B^\bullet \rightarrow C^\bullet$ give a map

$$\phi_A^* C^\bullet \rightarrow \phi_B^* C^\bullet$$

of cosimplicial rings. As B is a p -completely free δ - A -algebra, the natural map $A \rightarrow B^\bullet$ is a cosimplicial homotopy equivalence, so the map above is one as well. On the other hand, we can identify the second term as

$$\phi_B^* C^\bullet = \phi_B^* (B^\bullet \{ \frac{I^\bullet}{p} \}) = B^\bullet \{ \frac{\phi(I^\bullet)}{p} \} \cong D_{B^\bullet}(I^\bullet) = D^\bullet,$$

where the isomorphism comes from Corollary 2.38. The last term computes $R\Gamma_{\text{crys}}(R/A)$, and one checks that the identification constructed in this fashion is compatible with the previous recipe for defining the map. \square

Corollary 5.4 (The Hodge-Tate comparison in characteristic p). *For any smooth A/p -algebra S , the Hodge-Tate comparison*

$$H^i(\overline{\mathbb{A}}_{S/A}) \cong \Omega_{S/(A/p)}^i \{-i\}$$

holds true.

Proof. By Lemma 4.19, we can assume that $S = A/p[X_1, \dots, X_n]$ is a polynomial algebra, and then by Lemma 4.18 we can reduce to the case $A = \mathbf{Z}_p$. Then $S = R^{(1)}$ for the \mathbf{F}_p -algebra $R = \mathbf{F}_p[X_1, \dots, X_n]$, and the result follows from Theorem 5.2 and the Cartier isomorphism. \square

6. THE MIXED CHARACTERISTIC CASE OF THE HODGE-TATE COMPARISON

To prove the Hodge-Tate comparison in general, we shall need the following construction to pass from general bounded prisms (A, I) to crystalline ones.

Construction 6.1 (Mapping the universal oriented prism to a crystalline one). Let $(A, (d))$ be an oriented prism with the following properties: the Frobenius on A/p is flat, and the ring $A/(d)$ is p -torsionfree. For example, the universal oriented prism from Example 3.4 has these properties. Let $B := A\{\frac{\phi(d)}{p}\}^\wedge$ be the p -complete simplicial commutative δ - A -algebra obtained by freely adjoining $\phi(d)/p$ to A ; as $\phi(d) = pu$ for a unit $u \in \pi_0(B)$ by Lemma 2.24, the A -algebra B is (p, d) -complete. Moreover, by Corollary 2.38, the A -algebra B identifies with the p -completed pd-envelope of $(d) \subset A$ via the natural map $A \rightarrow B$, and is thus discrete and p -torsionfree since d is a nonzerodivisor modulo p . Now consider the map $\alpha : A \rightarrow B$ determined as the composition of the natural map $A \rightarrow B$ with $\phi : A \rightarrow A$. We can regard α as a map of prisms $(A, (d)) \rightarrow (B, (p))$ since $\alpha(d) \in pB$. Modulo p , the map α factors as

$$A/p \xrightarrow{\text{can}} A/(p, d) \xrightarrow{\phi} A/(p, d^p) \xrightarrow{\text{can}} B/p \simeq D_{A/p}((d))$$

where the first map has finite Tor amplitude, the second map is faithfully flat by construction, and the last map is faithfully flat (even free) by the structure of divided power envelopes. Using this description, one verifies the following properties of the base change functor $\widehat{\alpha}^* : D_{\text{comp}}(A) \rightarrow D_{\text{comp}}(B)$ on derived (p, d) -complete objects of the derived category:

- (1) The functor $\widehat{\alpha}^*$ is conservative.
- (2) For any p -completely smooth A/I -algebra R with p -completed base change R_B to B , we have a base change isomorphism $\widehat{\alpha}^* \Delta_{R/A} \simeq \Delta_{R_B/B}$ by Lemma 4.18.

We shall use these properties to deduce general properties of prismatic cohomology from those of crystalline cohomology.

Before proceeding to the proof of the Hodge-Tate comparison, we need to prove Lemma 4.9, ensuring that we have well-defined Hodge-Tate comparison maps. For this, we actually prove the Hodge-Tate comparison for the affine line; the proof below is structured in a way that generalizes to higher dimensions.

Proposition 6.2 (The Hodge-Tate comparison for the affine line). *If R is the p -adic completion $A/I\langle X \rangle$ of the polynomial ring, then the maps $\eta_R^0 : R \rightarrow H^0(\overline{\Delta}_{R/A})$ and $\eta_R^1\{-1\} : \Omega_{R/(A/I)}^1\{-1\} \rightarrow H^1(\overline{\Delta}_{R/A})$ coming from Construction 4.8 (by twisting) are isomorphisms, and $H^i(\overline{\Delta}_{R/A}) = 0$ for $i > 1$. In particular, Lemma 4.9 holds true in general.*

Proof. The last part follows by functoriality. For the rest, choose a map $\eta : R \oplus \Omega_{R/(A/d)}^1\{-1\}[-1] \rightarrow \overline{\Delta}_{R/A}$ in $D(R)$ inducing the maps η_R^i on cohomology for $i = 0, 1$; this is possible because R and $\Omega_{R/(A/d)}^1$ are topologically free over R . We will show that η is an isomorphism.

First, if $I = (p)$, we are done thanks to the Hodge-Tate comparison in characteristic p (Corollary 5.4). Next, if (A, I) is the universal oriented prism $(A_0, (d))$ from Example 3.4, then the claim follows by base change to the $I = (p)$ case using the map α and properties (1) and (2) from Construction 6.1.

Next, consider a general bounded oriented prism $(A, (d))$. The choice of the generator $d \in A$ determines a map $A_0 \rightarrow A$, where $(A_0, (d))$ is the universal oriented prism considered in the previous paragraph. If the map $A_0 \rightarrow A$ had finite (p, I) -complete Tor-amplitude (or we knew base change for prismatic cohomology in general already), we would get the claim via Lemma 4.18. We can however argue as follows. Letting $a : A_0 \rightarrow D_0$ denote the map α from Construction 6.1, we obtain a diagram

$$\begin{array}{ccccc} A_0 & \xrightarrow{\quad} & A & & \\ \downarrow a & & \downarrow b & & \\ \mathbf{Z}_p & \longrightarrow & D_0 & \longrightarrow & E := A \widehat{\otimes}_{A_0}^L D_0 \end{array}$$

of (p, d) -complete simplicial commutative rings, where the square is a pushout square of (p, d) -complete simplicial commutative rings. Note that the bottom left arrow lifts to a map of prisms $(\mathbf{Z}_p, (p)) \rightarrow (D_0, (a(d)))$ as $(p) = (a(d))$ by construction. Write c for the composite map $\mathbf{Z}_p \rightarrow E$. The base change functor \widehat{b}^* is conservative on (p, d) -complete objects in the derived category (since the same holds true for \widehat{a}^* , see property (1) in Construction 6.1), so it suffices to show that $\widehat{b}^*(\eta)$ is an isomorphism. We claim that $\widehat{b}^*\overline{\Delta}_{R/A} \simeq \widehat{c}^*\overline{\Delta}_{\mathbf{F}_p[X]/\mathbf{Z}_p}$: this is clear at the level of explicit complexes if one uses the complexes from Remark 4.17, and the rest follows from Lemma 4.20 together with the observation that b (resp. c) has finite (p, d) -complete (resp. p -complete) Tor amplitude. We leave it to the reader to check that the resulting identification $\widehat{b}^*\overline{\Delta}_{R/A} \simeq \widehat{c}^*\overline{\Delta}_{\mathbf{F}_p[X]/\mathbf{Z}_p}$ carries the map $\widehat{b}^*\eta$ to a map realizing the Hodge-Tate comparison map on cohomology for the target. In particular, our claim now reduces to the Hodge-Tate comparison for $\overline{\Delta}_{\mathbf{F}_p[X]/\mathbf{Z}_p}$, which was already explained earlier (Corollary 5.4).

Finally, for a general bounded prism (A, I) , we reduce to the case treated in the previous paragraph by making a faithfully flat base change to a bounded oriented prism (Lemma 3.7 (4)). \square

Thus, we now have a well-defined Hodge-Tate comparison map from Construction 4.8.

Theorem 6.3 (The Hodge-Tate comparison). *The Hodge-Tate comparison maps*

$$\Omega_{X/(A/I)}^i \rightarrow H^i(\overline{\Delta}_{X/A})\{i\}$$

of sheaves on $X_{\text{ét}}$ from Construction 4.8 are isomorphisms.

Proof. Since the comparison map is globally defined, it is enough to prove the statement for affines. Thus, we shall check that for any p -completely smooth A/I -algebra R , the analogous map

$$\Omega_{R/(A/I)}^i \rightarrow H^i(\overline{\Delta}_{R/A})\{i\}$$

is an isomorphism of R -modules. As both differential forms and prismatic cohomology localize in the étale topology (Lemma 4.19), it is enough to prove the statement when $R = A/I\langle X_1, \dots, X_n \rangle$ is the p -adic completion of a polynomial ring. In this case, we choose a map

$$\eta : \bigoplus_{i=0}^n \Omega_{R/(A/I)}^i \{-i\}[-i] \rightarrow \overline{\Delta}_{R/A}$$

in the derived category $D(R)$ lifting the Hodge-Tate comparison maps (up to twists) on cohomology, and proceed exactly as explained in Lemma 6.2 for $n = 1$. \square

Using base change for prismatic cohomology, we can now also prove the de Rham comparison, at least under the technical assumption that $W(A/I)$ is p -torsion free. This is satisfied, for example, if A/I is p -torsion free or if $I = (p)$ and A/p is reduced. This technical assumption will be removed later (Corollary 15.4).

Theorem 6.4 (The de Rham comparison). *Let X be a smooth formal A/I -scheme and assume that $W(A/I)$ is p -torsion free. Then there is a natural isomorphism $\Delta_{X/A} \widehat{\otimes}_{A,\phi}^L A/I \simeq \Omega_{X/(A/I)}^*$ of commutative algebras in $D(X_{\text{ét}}, A/I)$, where the completion is p -adic.*

Proof. As before, it is enough to solve the problem in the affine case. Thus, for a p -completely smooth A/I -algebra R , we must construct a functorial isomorphism $\Delta_{R/A} \widehat{\otimes}_{A,\phi}^L A/I \simeq \Omega_{R/(A/I)}^*$ of commutative algebras in $D(A/I)$. We shall prove a stronger statement that involves relating both sides to crystalline cohomology.

To formulate this stronger statement, note that the kernel $J \subset W(A/I)$ of the canonical composite surjection $W(A/I) \rightarrow A/I \rightarrow A/(I, p)$ is a pd-ideal. By generalities on crystalline cohomology, the commutative algebra object $R\Gamma_{\text{crys}}((R/p)/W(A/I)) \in D(W(A/I))$ lifts the de Rham complex $\Omega_{R/(A/I)}^* \in D(A/I)$. On the other hand, the natural map $A \rightarrow A/I$ refines uniquely to a δ -map $A \rightarrow W(A/I)$ by Remark 2.7. It is therefore enough to construct a functorial isomorphism

$$\Delta_{R/A} \widehat{\otimes}_{A,\psi}^L W(A/I) \simeq R\Gamma_{\text{crys}}((R/p)/W(A/I))$$

of commutative algebras in $D(W(A/I))$, where $\psi : A \rightarrow W(A/I)$ is obtained by composing the canonical map $A \rightarrow W(A/I)$ with the Frobenius. We shall deduce this from the crystalline comparison.

Note that the composite $\psi : A \rightarrow W(A/I)$ carries I into (p) : the canonical map $A \rightarrow W(A/I)$ carries I into $VW(A/I)$, so ψ carries I into $FVW(R) = (p)$. By our p -torsionfreeness assumption on $W(A/I)$, the map ψ thus refines to a map $(A, I) \rightarrow (W(A/I), (p))$ of prisms. Base change for prismatic cohomology then gives an isomorphism

$$\Delta_{R/A} \widehat{\otimes}_{A,\psi}^L W(A/I) \cong \Delta_{R'/W(A/I)} .$$

for the base change $R' = R \widehat{\otimes}_{A/I, \psi} W(A/I)/p$. It thus suffices to identify the right side above with $R\Gamma_{\text{crys}}((R/p)/W(A/I))$. But this follows by applying Theorem 5.2 to the pd-ideal $J \subset W(A/I)$, and noting that R' is the base change of R/p along $W(A/I)/J \simeq A/(p, I) \rightarrow W(A/I)/p$ induced by the Frobenius on $W(A/I)$. \square

7. SEMIPERFECTOID RINGS AND PERFECTION IN MIXED CHARACTERISTIC

In this section, we study semiperfectoid rings. In §7.1, we construct an initial object of the absolute prismatic site $(S)_{\Delta}$ of a semiperfectoid ring S essentially by hand, use it to construct a “perfectoidization” S_{perfd} of S , and formulate the main theorem of this section: the map $S \rightarrow S_{\text{perfd}}$ is surjective (Theorem 7.4). To prove this theorem, we need a better way to access the aforementioned initial object; this is provided by the theory of derived prismatic cohomology, which is the subject of §7.2. In §7.3, these ideas are put to use: we give a new proof of a key flatness result of André from [And18a], and use it to prove the promised surjectivity of $S \rightarrow S_{\text{perfd}}$.

7.1. Universal prisms for semiperfectoid rings.

Notation 7.1. Fix a semiperfectoid ring S , i.e. S is a derived p -complete ring that can be written as a quotient of a perfectoid ring R . We recall that S is called *quasiregular* if S has bounded p^∞ -torsion, and the cotangent complex $L_{S/\mathbf{Z}_p}[-1]$ is p -completely flat over S (see [BMS19, §4.4]). In that case S is classically p -adically complete.

Proposition 7.2. *The category of prisms (A, I) equipped with a map $S \rightarrow A/I$ admits an initial object $(\Delta_S^{\text{init}}, I)$ and $I = (d)$ is principal.*

In terms of Remark 4.6, the category in question is the absolute prismatic site $(\text{Spf}(S))_{\Delta}$.

Proof. Choose a surjective map $R \rightarrow S$ from a perfectoid ring. By Lemma 4.7, the category of prisms (A, I) with a map $R \rightarrow A/I$ admits the initial object $(A_{\text{inf}}(R), \ker \theta)$, and we recall that $\ker \theta = (d)$ is always principal. Let $J \subset A_{\text{inf}}(R)$ be the kernel of the surjection $A_{\text{inf}}(R) \xrightarrow{\theta} R \rightarrow S$. Given any prism (C, K) with a map $S \rightarrow C/K$, we get a unique map $(A_{\text{inf}}(R), (d)) \rightarrow (C, K)$ making $A_{\text{inf}}(R)/d = R \rightarrow S \rightarrow C/K$ commute, and $K = dC$ by Lemma 3.5. As d is a nonzerodivisor in C , there are unique elements $\frac{f}{d} \in C$ for each $f \in J$. Let

$$B = A_{\text{inf}}(R)\{x_f | f \in J\} / (dx_f - f | f \in J)_{\delta}.$$

The preceding arguments show that there is a unique map of δ -rings $B \rightarrow C$. As C is d -torsionfree, this extends to a unique map $B/B[d^\infty]_{\delta} \rightarrow C$ (where $B[d^\infty]_{\delta}$ is the smallest δ -stable ideal of B containing $B[d^\infty]$), and this extends further to the derived (p, d) -completion of $B/B[d^\infty]_{\delta}$. This may however contain new d -torsion elements; but after a transfinite induction (using that completion commutes with ω_1 -filtered colimits), we arrive at a derived (p, d) -complete δ - $A_{\text{inf}}(R)$ -algebra Δ_S^{init} such that d is a nonzerodivisor. We still have a unique map $(\Delta_S^{\text{init}}, (d)) \rightarrow (C, K)$. Moreover, Δ_S^{init} is a prism as d is distinguished. Finally, the map

$$R \cong A_{\text{inf}}(R)/d \rightarrow \Delta_S^{\text{init}}/d$$

factors over the quotient S of R as all elements of J become divisible by d in Δ_S^{init} . Thus, $(\Delta_S^{\text{init}}, (d))$ with the map $S \rightarrow \Delta_S^{\text{init}}/d$ is the desired initial object. \square

In general, Δ_S^{init} is not particularly well-behaved; for example it is not bounded in general. After perfection, this issue disappears.

Corollary 7.3. *There is a universal perfectoid ring S_{perfd} equipped with a map $S \rightarrow S_{\text{perfd}}$. In fact, S_{perfd} is the p -adic completion of $(\Delta_S^{\text{init}})_{\text{perfd}}/I$.*

Proof. Note that the category of perfectoid rings S_{perfd} with a map $S \rightarrow S_{\text{perfd}}$ is equivalent to the category of perfect prisms (A, I) with a map $R \rightarrow A/I$. Thus, the claim follows from Lemma 3.9. \square

One goal in this section is prove the following theorem.

Theorem 7.4. *The map $S \rightarrow S_{\text{perfd}}$ is surjective.*

Remark 7.5 (Zariski closed sets are strongly Zariski closed). In [Sch15, Section II.2], the second author defined notions of “Zariski closed” and “strongly Zariski closed” subsets of an affinoid perfectoid space $X = \text{Spa}(R, R^+)$. Here R is a perfectoid Tate ring, in particular there is some topologically nilpotent unit $\varpi \in R$, and $R^+ \subset R$ is an open and integrally closed subring of R° . Then strongly Zariski closed subsets are in bijection with quotients $R \rightarrow R'$ to perfectoid Tate rings R' , while Zariski closed subsets are those closed subsets of X that are the vanishing locus of some ideal $I \subset R$. It was claimed in [Sch15, Remark II.2.4] that not any Zariski closed subset is strongly Zariski closed. The theorem implies that this is false, and that the notions in fact agree. Indeed, let $I \subset R$ be any ideal, and let S be the ϖ -adic completion of $R^+/I \cap R^+$. Then S is a semiperfectoid ring, and so by the theorem $R' = S_{\text{perfd}}[1/\varpi]$ is a perfectoid Tate ring that is a quotient of R and such that any map from R to a perfectoid ring that vanishes on I factors uniquely over R' . This implies that the vanishing locus of I is exactly the strongly Zariski closed subset $\text{Spa}(R', R'^+)$, where $R'^+ \subset R'$ is the minimal open and integrally closed subring that contains the image of R^+ .

For the proof of Theorem 7.4, we will need a different definition of Δ_S^{init} which makes it possible to use the Hodge-Tate comparison result that we proved for smooth algebras (Theorem 6.3) in order to compute Δ_S^{init} .

7.2. Derived prismatic cohomology. In this section, we explain how to extend the notion of prismatic cohomology to all p -complete rings via Quillen’s formalism of non-abelian left derived functors (aka left Kan extensions), as recast in the ∞ -categorical language in [Lur09, §5.5.9] (see also [BLM18, §9.2] for an exposition closer to our context). Write $\mathcal{D}(A)$ for the derived ∞ -category of a commutative ring A .

Construction 7.6 (Constructing derived prismatic cohomology). Fix a bounded prism (A, I) . Consider the functor $R \mapsto \Delta_{R/A}$ on p -completely smooth A/I -algebras R valued in the category of commutative algebras in the ∞ -category of (p, I) -complete objects in $\mathcal{D}(A)$ equipped with a ϕ_A -semilinear endomorphism ϕ . Write $R \mapsto \Delta_{R/A}$ for the left Kan extension of this functor to all derived p -complete simplicial commutative A/I -algebras, so $\Delta_{R/A}$ is a derived (p, I) -complete commutative algebra in $\mathcal{D}(A)$ equipped with an endomorphism $\phi : \Delta_{R/A} \rightarrow \phi_{A,*}\Delta_{R/A}$; we call this theory *derived prismatic cohomology*.

The Hodge-Tate comparison shows that $\overline{\Delta}_{R/A} := \Delta_{R/A} \otimes_A^L A/I$ admits an increasing \mathbf{N} -indexed filtration with gr_i given by sending R to the derived p -completion of $\wedge^i L_{R/(A/I)}\{-i\}[-i]$; we refer to this as the *Hodge-Tate comparison* for $\overline{\Delta}_{R/A}$. Let us record some immediate consequences:

- (1) (The value on smooth algebras) If R is p -completely smooth, then $\Delta_{R/A}$ is unchanged, so derived prismatic cohomology extends prismatic cohomology to all p -complete A/I -algebras.
- (2) (The étale sheaf property) The Hodge-Tate comparison for derived prismatic cohomology shows that $R \mapsto \Delta_{R/A}$ is a sheaf for the p -completely étale topology on the category of

derived p -complete A/I -algebras. In particular, if X is a p -adic formal scheme, then we can naturally define its derived prismatic cohomology $\Delta_{X/A}$ as a commutative algebra in the ∞ -category of derived (p, I) -complete objects in $\mathcal{D}(X_{\text{ét}}, A)$ (equipped with a Frobenius).

- (3) (The quasisyntomic sheaf property) Assume that (A, I) is a perfect prism. Then the assignment $R \mapsto \Delta_{R/A}$ forms a sheaf for the quasisyntomic topology on the category of quasisyntomic A/I -algebras (see [BMS19, Example 5.11] for a variant involving derived de Rham cohomology in characteristic p).
- (4) (Base change behaviour) The formation of $\Delta_{R/A}$ commutes with base change in the sense that for any map of bounded prisms $(A, I) \rightarrow (B, J)$, letting R_B be the derived p -completion of $R \otimes_A^L B$, one has

$$\Delta_{R_B/B} \cong \Delta_{R/A} \widehat{\otimes}_A^L B,$$

where the completion is the derived (p, J) -completion.

- (5) (Colimit preservation) The functor $\Delta_{-/A}$ from p -complete simplicial commutative A/I -algebras to (p, I) -complete E_∞ -algebras in $\mathcal{D}(A)$ is symmetric monoidal. In fact, it commutes with all colimits (as the same holds true for the associated graded of the derived Hodge-Tate cohomology $\overline{\Delta}_{-/A}$ functor).

A key advantage of derived prismatic cohomology is that there is a large supply of rings for which this theory is concentrated in degree 0 (Example 7.9). In such situations, this theory is also closely related to an initial object of the prismatic site.

Lemma 7.7 (Derived prismatic cohomology when it is discrete). *Assume (A, I) is a bounded prism, and R is a derived p -complete simplicial A/I -algebra such that $\overline{\Delta}_{R/A}$ is concentrated in degree 0.*

- (1) *The ϕ_A -linear Frobenius ϕ_R on $\Delta_{R/A}$ naturally lifts to a δ - A -structure on the ring $\Delta_{R/A}$.*
- (2) *The pair $(\Delta_{R/A}, I\Delta_{R/A})$ gives a prism over (A, I) equipped with a map $R \rightarrow \overline{\Delta}_{R/A} = \Delta_{R/A}/I\Delta_{R/A}$.*
- (3) *The category of prisms (B, J) over (A, I) equipped with a map $R \rightarrow B/J$ has an initial object, and $(\Delta_{R/A}, I\Delta_{R/A})$ is weakly initial. In particular, the initial object is the image of some idempotent endomorphism of $(\Delta_{R/A}, I\Delta_{R/A})$.*

In part (3), we conjecture that $(\Delta_{R/A}, I\Delta_{R/A})$ is actually the initial object.

Proof. Using Čech-Alexander complexes and canonical simplicial resolutions, we obtain a functor $R \mapsto F_A(R)$ from derived p -complete A/I -algebras to simplicial cosimplicial derived (p, I) -complete δ - A -algebras computing prismatic cohomology, i.e., we have a functorial identification $\text{colim}_\Delta \lim_{\Delta^{op}} F_A(R) \simeq \Delta_{R/A}$. The δ - A -algebra structure on $F_A(R)$ is classified by a map $F_A(R) \rightarrow W_2(F_A(R))$ of simplicial cosimplicial rings (where the target is defined by pointwise application of $W_2(-)$). This map lies over the map $A \rightarrow W_2(A)$ classifying the δ -structure on A and splits the natural restriction map $W_2(F_A(R)) \rightarrow F_A(R)$. This defines a map

$$\Delta_{R/A} \simeq \text{colim}_\Delta \lim_{\Delta^{op}} F_A(R) \rightarrow \text{colim}_\Delta \lim_{\Delta^{op}} W_2(F_A(R))$$

of E_∞ -rings lying over the map $A \rightarrow W_2(A)$ and splitting the restriction map

$$\operatorname{colim}_\Delta \lim_{\Delta^{op}} W_2(F_A(R)) \rightarrow \operatorname{colim}_\Delta \lim_{\Delta^{op}} F_A(R) \simeq \mathbb{A}_{R/A}.$$

Using the functorial pullback square

$$\begin{array}{ccc} W_2(B) & \xrightarrow{F} & B \\ \downarrow & & \downarrow \\ B & \xrightarrow{\phi} & B \otimes_{\mathbb{Z}}^L \mathbb{F}_p \end{array}$$

of simplicial commutative (and thus of E_∞) rings, as well as the discreteness of $\mathbb{A}_{R/A}$, one can identify $\operatorname{colim}_\Delta \lim_{\Delta^{op}} W_2(F_A(R))$ as $W_2(\mathbb{A}_{R/A})$. Thus, the above map defines a δ - A -algebra structure on $\mathbb{A}_{R/A}$ refining the ϕ_A -linear Frobenius endomorphism ϕ_R , proving part (1).

Part (2) is then immediate from the derived (p, I) -completeness of $\mathbb{A}_{R/A}$ and our assumption that $\overline{\mathbb{A}}_{R/A} = \mathbb{A}_{R/A} \otimes_A^L A/I$ is discrete.

For part (3), we first construct a functorial map $(\mathbb{A}_{R/A}, I\mathbb{A}_{R/A}) \rightarrow (B, J)$ for any prism (B, J) over (A, I) equipped with a map $R \rightarrow B/J$. Fix a resolution $R_\bullet \rightarrow R$ by p -completely ind-smooth A/I -algebras R_i . For each R_i , let $(R_i/A)_\Delta$ be the category of prisms (C, K) over (A, I) equipped with a map $R_i \rightarrow C/K$. Then

$$\mathbb{A}_{R/A} = \operatorname{colim}_\Delta \lim_{C \in (R_\bullet/A)_\Delta} C$$

in the category of (p, I) -complete complexes, as $\mathbb{A}_{R_i/A} = \lim_{C \in (R_i/A)_\Delta} C$ by the theory of Čech-Alexander complexes. But now (B, J) , like any object of $(R/A)_\Delta$, defines compatible objects of all $(R_i/A)_\Delta$, and so restricting the limit to this object gives a functorial map

$$\mathbb{A}_{R/A} = \operatorname{colim}_\Delta \lim_{C \in (R_\bullet/A)_\Delta} C \rightarrow \operatorname{colim}_\Delta B = B.$$

Using the argument from part (1), one shows that this is in fact a map of δ -rings. Now part (3) follows from the following general categorical lemma. \square

Lemma 7.8. *Let \mathcal{C} be an idempotent complete category and $X \in \mathcal{C}$ an object. Let $\mathcal{C}_{X\setminus}$ be the category of objects $Y \in \mathcal{C}$ equipped with a map $X \rightarrow Y$. Assume that the identity on \mathcal{C} factors over the projection $\mathcal{C}_{X\setminus} \rightarrow \mathcal{C}$ via a functor $F : \mathcal{C} \rightarrow \mathcal{C}_{X\setminus}$. Then $F(X)$ is an idempotent endomorphism of X and the corresponding retract of X is an initial object of \mathcal{C} .*

Proof. Let $F(X) = (X \xrightarrow{e} X)$. Applying F to the morphism $(X \xrightarrow{e} X)$ of \mathcal{C} shows that e is an idempotent. Let X' be the corresponding retract of X . Then $F(X') = (X \rightarrow X')$ where $X \rightarrow X'$ is an epimorphism. The functor F is faithful, so for any $Y \in \mathcal{C}$, one has

$$\operatorname{Hom}_{\mathcal{C}}(X', Y) \hookrightarrow \operatorname{Hom}_{\mathcal{C}_{X\setminus}}(X \rightarrow X', X \rightarrow Y).$$

But $X \rightarrow X'$ is an epimorphism, so the right hand side has at most one element, so any Y admits at most one map from X' . On the other hand, there is at least one map $X' \rightarrow Y$ as there is a map $X' \rightarrow X$ and a map $X \rightarrow Y$. Thus, X' is initial, as desired. \square

Example 7.9 (Prismatic cohomology of an lci quotient). Fix a bounded prism (A, I) . Let $R := A/J$, where $J = (I, f_1, \dots, f_r)$ with f_1, \dots, f_r giving a p -completely regular sequence on A/I . Then $\Delta_{R/A}$ identifies with the prismatic envelope $B := A\{\frac{J}{I}\}$ (which identifies with $A\{\frac{f_1}{d}, \dots, \frac{f_r}{d}\}^\wedge$ if $d \in I$ is a generator). To see this, note first that $\Delta_{R/A}$ is concentrated in degree 0 and I -torsionfree thanks to the Hodge-Tate comparison. Thus, Lemma 7.7 applies. By Proposition 3.13, B is initial in the category of prisms over A equipped with a map $R \rightarrow B/IB$. Thus, it remains to see that $\Delta_{R/A}$ is actually also the initial object in this situation.

By base change compatibility of $\Delta_{R/A}$ and prismatic envelopes, we can assume that $I = (d)$ is oriented. Passing to the universal case, we may also assume that A/d is p -torsionfree. It is enough to see that the map $B/IB[\frac{1}{p}] \rightarrow \overline{\Delta}_{R/A}[\frac{1}{p}]$ is surjective, as this then implies that the idempotent endomorphism of $\overline{\Delta}_{R/A}$ is the identity (using that it is torsion-free), and so the idempotent endomorphism of $\Delta_{R/A}$ is an isomorphism, whence $B \cong \Delta_{R/A}$. By the Hodge-Tate comparison, $\overline{\Delta}_{R/A}[\frac{1}{p}]$ is generated as an $R[\frac{1}{p}]$ -algebra by $\text{gr}_1 = L_{R/(A/I)}^\wedge \{-1\}[-1] = J/(J^2 + I)\{-1\}$. It thus suffices to see that the map $B = A\{\frac{J}{I}\} \rightarrow \Delta_{R/A}$ maps $\frac{J}{I}$ surjectively onto $\text{gr}_1 = J/(J^2 + I)\{-1\}$. This is a standard computation.

Proposition 7.10. *Let S be a quasiregular semiperfectoid ring, let R be any perfectoid ring with a map $R \rightarrow S$, and let $(A, I) = (A_{\inf}(R), \ker \theta)$. Then $\overline{\Delta}_{S/A}$ is discrete, and $\Delta_{S/A}$ is the initial prism Δ_S^{init} with a map $S \rightarrow \Delta_S^{\text{init}}/I$. In particular, $\Delta_{S/A} = \Delta_S^{\text{init}}$ does not depend on R . Moreover, the map $S \rightarrow \overline{\Delta}_{S/A}$ is p -completely faithfully flat.*

In the following, we simply write $\Delta_S = \Delta_S^{\text{init}} = \Delta_{S/A}$ for quasiregular semiperfectoid S .

Proof. If S is quasiregular, then $L_{S/R}[-1]$ is p -completely flat, and thus the derived p -completions of all $\wedge^i L_{S/R}[-i]$ are p -completely flat S -modules and in particular discrete. Thus the Hodge-Tate comparison implies that $\overline{\Delta}_{S/A}$ is discrete and p -completely faithfully flat over S . It remains to see that $\Delta_{S/A}$ is initial, or equivalently that the idempotent endomorphism from Lemma 7.7 (3) is the identity.

Replacing R by $R\langle Y_i^{1/p^\infty} | i \in I \rangle$ for some set I and noting that this does not change the cotangent complexes, and thus not $\Delta_{S/R}$, we may assume that $R \rightarrow S$ is surjective. Choose a generating set $\{f_j\}_{j \in J}$ of the kernel of $R \rightarrow S$. Then the resulting map

$$S' = R\langle X_j^{1/p^\infty} | j \in J \rangle / (X_j - f_j | j \in J)^\wedge \rightarrow S : X_j \mapsto 0$$

is surjective and also induces a surjection on cotangent complexes. Thus, by the Hodge-Tate comparison, $\Delta_{S'/R} \rightarrow \Delta_{S/R}$ is surjective, so it suffices to show that the idempotent endomorphism of $\Delta_{S'/A}$ is the identity. But, by truncating the set J of generators, we can express S' as a filtered colimit of regular semiperfectoid rings of the form

$$T = R\langle X_1^{1/p^\infty}, \dots, X_r^{1/p^\infty} \rangle / (X_1 - f_1, \dots, X_r - f_r),$$

so it suffices to prove that the idempotent endomorphism of $\Delta_{T/A}$ is the identity for each such T . Now if $A' = A_{\inf}(R\langle X_1^{1/p^\infty}, \dots, X_r^{1/p^\infty} \rangle)$, then we also have $\Delta_{T/A} \simeq \Delta_{T/A'}$ compatibly with the

idempotent endomorphisms. Example 7.9 then implies that the idempotent endomorphism $\Delta_{T/A'}$ is the identity, proving the claim. \square

7.3. André's lemma.

Proposition 7.11 (Lifting quasisyntomic covers to the prismatic site). *Let (A, I) be a bounded prism. Let R be a quasisyntomic A/I -algebra. Then there exists an object $(B \rightarrow B/IB \leftarrow R) \in (R/A)_{\Delta}$ with $R \rightarrow B/IB$ being p -completely faithfully flat. Consequently:*

- (1) *The map $(A, I) \rightarrow (B, IB)$ is a flat map of prisms (and faithfully flat if $A/I \rightarrow R$ is p -completely faithfully flat).*
- (2) *If A is perfect, then the map $(A, I) \rightarrow (B, IB)_{\text{perf}}$ is flat as well (and faithfully flat if $A/I \rightarrow R$ is p -completely faithfully flat).*

Proof. Let us first construct a quasisyntomic cover $R \rightarrow S$ with S/p relatively semiperfect over $A/(I, p)$. Choose a surjection $A/I\langle x_j | j \in J \rangle \rightarrow R$, where J is some index set, and let S be the p -complete R -algebra obtained by formally extracting all p -power roots of the x_j 's, i.e.,

$$S := A/I\langle x_j^{1/p^\infty} | j \in J \rangle \widehat{\otimes}_{A/I\langle x_j | j \in J \rangle}^L R.$$

It is easy to see that $R \rightarrow S$ is a quasisyntomic cover (so S is quasisyntomic over A/I as well), and that $A/(I, p) \rightarrow S/p$ is relatively semiperfect. In particular, the derived p -completion of the cotangent complex $L_{S/(A/I)}^\wedge[-1]$ is a p -completely flat S -module. The argument in the first half of Proposition 7.10 repeats verbatim shows that $B = \Delta_{S/A}$ gives the required prism (using Lemma 7.7 to get the δ -structure on B). The consequence in (1) follows because each map in the composition $A/I \rightarrow R \rightarrow S \rightarrow B/IB$ is p -completely flat (and p -completely faithfully flat provided $A/I \rightarrow R$ is so), and the consequence in (2) follows by passage to filtered colimits. \square

We give an application.

Theorem 7.12 (André's flatness lemma). *Let R be a perfectoid ring. Then there exists a p -completely faithfully flat map $R \rightarrow S$ of perfectoid rings such that S is absolutely integrally closed (i.e., every monic polynomial has a solution). In particular, every element of S admits a compatible system of p -power roots.*

In fact, the map $R \rightarrow S$ can also be chosen to be ind-syntomic modulo p , see Remark 7.13.

Proof. Write $R = A/I$ for a perfect prism (A, I) . Let \tilde{R} be the p -complete R -algebra obtained by formally adjoining roots of all monic polynomials over R . Then \tilde{R} is a quasisyntomic cover of R , and each monic polynomial over R has a solution in \tilde{R} . Proposition 7.11 then gives a perfect prism (B, IB) over (A, I) equipped with a p -completely faithfully flat map $\tilde{R} \rightarrow B/IB =: R_1$. Transfinitely iterating the construction $R \mapsto R_1$ yields a quasisyntomic cover $R \rightarrow R'$ with R' being perfectoid and absolutely integrally closed. \square

Remark 7.13 (Refined form of André's flatness lemma). The map $R \rightarrow S$ constructed in Theorem 7.12 can be shown to be an ind-syntomic cover modulo p , as we now briefly sketch. Unwinding our construction, it suffices to show the following: for a monic equation $f(Y) \in R[Y]$, if we set

$S := R\langle Y^{1/p^\infty} \rangle / (f(Y))$ over the prism (A, I) , then the resulting map $S \rightarrow T := \overline{\mathbb{A}}_S$ is an ind-syntomic cover modulo p . The “cover” part follows from Proposition 7.10. For ind-syntomicity, by base change, we may assume R is p -torsionfree. Choose $\varpi \in R$ such that $(\varpi^p) = (p)$. By the definition of T as a prismatic envelope and Corollary 2.38, $\phi_A^*(T/\varpi)$ is the pd-envelope of $f(Y)$ in $R/\varpi[Y^{1/p^\infty}]$. The claim now follows by combining the following two facts:

- (1) If A is an \mathbf{F}_p -algebra and $f \in A$ is a nonzerodivisor, then the natural map $A/f^p \rightarrow D_A(f)$ is identified with the natural map $A/f^p \rightarrow A/f^p[g_1, g_2, \dots]/(g_1^p, g_2^p, \dots)$ via $g_i \mapsto \gamma_{ip}(f)$; in particular, this map is ind-syntomic. This is a well-known fact about divided power algebras in characteristic p , and can be proven by reduction to the universal case where $A = \mathbf{F}_p[f]$.
- (2) If $A \rightarrow B = A/I$ is a surjection of \mathbf{F}_p -algebras with nilpotent kernel I , and D is a flat A -algebra with $D/ID \simeq B[g_1, g_2, \dots]/(g_1^p, g_2^p, \dots)$, then D is ind-syntomic over A . To see this, pick $\bar{h}_i \in D$ lifting g_i . Then we get an obvious surjective map $A[h_1, h_2, \dots] \rightarrow D$ by $h_i \mapsto \bar{h}_i$. As $\bar{h}_i^p \in ID$, we can find $\epsilon_i \in IA[h_1, h_2, \dots]$ such that $h_i^p - \epsilon_i$ maps to 0 in D for each i . Setting $D_n = A[h_1, h_2, \dots]/(h_1^p - \epsilon_1, h_2^p - \epsilon_2, \dots, h_n^p - \epsilon_n)$, we get a map $\text{colim}_n D_n \rightarrow D$; this map is an isomorphism as it is a surjection of flat A -algebras that is an isomorphism modulo the nilpotent ideal I . The claim follows as each D_n is ind-syntomic over A .

The observation that $R \rightarrow S$ can be chosen to be ind-syntomic modulo p also follows from [GR, Theorem 16.9.17].

Finally, we can prove Theorem 7.4.

Proof. Recall that we want to prove that for any semiperfectoid ring S , the map $S \rightarrow S_{\text{perfd}}$ is surjective, where S_{perfd} is the universal perfectoid ring with a map from S . Write $S = R/J$ for some perfectoid ring R . Using filtered colimits, we may assume that J is finitely generated, and then by induction we may actually assume that $I = (f)$ is principal. By Theorem 7.12, we can assume that f admits compatible p -power roots $f^{1/p^n} \in R$. But as any perfectoid ring T is reduced, any map $S = R/f \rightarrow T$ automatically factors over the p -completion of $R/(f^{1/p^\infty})$, which is itself perfectoid. Thus, S_{perfd} is the p -completion of $R/(f^{1/p^\infty})$, and in particular the maps $R \rightarrow S \rightarrow S_{\text{perfd}}$ are surjective. \square

8. PERFECTIONS IN MIXED CHARACTERISTIC: GENERAL CASE

The goal of this section is to study a notion of “perfectoidization” for any algebra over a perfectoid ring. In §8.1, we define this notion using derived prismatic cohomology, and then describe it via the perfect prismatic site. These results are then used in §8.2 to reinterpret the perfectoidization as the sheafification of the structure sheaf for the arc-topology (on p -adic formal schemes over a perfectoid ring), which leads to consequences such as a version of excision (Corollary 8.11).

8.1. Perfections via the perfect prismatic site.

Notation 8.1. Fix a perfectoid ring R corresponding to a perfect prism (A, I) . For notational convenience, we fix a distinguished element $d \in I$ (which is automatically a generator).

Definition 8.2 (Perfectoidizations). Fix a derived p -complete simplicial R -algebra S .

- (1) The perfection $\Delta_{S/A, \text{perf}}$ of $\Delta_{S/A}$ is defined as

$$\Delta_{S/A, \text{perf}} := \left(\operatorname{colim}_{\phi_S} \Delta_{S/A} \rightarrow \phi_* \Delta_{S/A} \rightarrow \phi_*^2 \Delta_{S/A} \rightarrow \dots \right)^\wedge \in D(A).$$

This is a derived (p, I) -complete E_∞ - A -algebra equipped with a ϕ_A -semilinear automorphism induced by ϕ_S .

- (2) The perfectoidization S_{perfd} of S is defined as

$$S_{\text{perfd}} := \Delta_{S/A, \text{perf}} \otimes_A^L A/I \in D(R).$$

This is a p -complete E_∞ - S -algebra.

A priori, S_{perfd} depends on the choice of perfectoid base ring A , but it follows from Proposition 8.5 below that it is actually independent of this choice. The same proposition will also show that Definition 8.2 (2) is compatible with Corollary 7.3.

Example 8.3 (Perfectoidizations of characteristic p rings). Assume that S is an R -algebra where $p = 0$. In this case, we claim that S_{perfd} coincides with the usual (direct limit) perfection S_{perf} of S . To see this, as S_{perfd} is independent of the choice of the perfectoid base ring, we may simply take $A = \mathbf{Z}_p$ so $R = \mathbf{F}_p$. But then the Frobenius equivariant map $S \rightarrow \Delta_{S/A}/p$ coming from the derived Hodge-Tate comparison induces an isomorphism on perfections: one checks (by, e.g., reduction to the case $S = \mathbf{F}_p[x]$) that the endomorphism of $\wedge^i L_{S/\mathbf{F}_p}[-i]$ induced by the Frobenius endomorphism of $\Delta_{S/A}$ via the Hodge-Tate comparison is the one induced by the Frobenius on S , and is thus 0 for $i \neq 0$. Thus, we have $S_{\text{perf}} \simeq \operatorname{colim}_{\phi_S} \Delta_{S/A}/p =: S_{\text{perfd}}$, as asserted.

The goal of this section is to describe S_{perfd} via maps from S to perfectoid rings.

Lemma 8.4 (Coconnectivity of perfectoidizations). *For any derived p -complete simplicial R -algebra S , both $\Delta_{S/A, \text{perf}}$ and S_{perfd} lie in $D^{\geq 0}$.*

Proof. We first show the claim for $\Delta_{S/A, \text{perf}}$ using the P^0 operation for E_∞ - \mathbf{F}_p -algebras. As the levelwise Frobenius map on a simplicial cosimplicial \mathbf{F}_p -algebra induces the P^0 -operation on its realization, we learn that $\Delta_{S/A, \text{perf}}/p$ is an E_∞ - \mathbf{F}_p -algebra where P^0 acts invertibly. As P^0 kills

all positive degree homotopy classes [Lur11, Remark 2.2.7], it follows that $\Delta_{S/A,\text{perf}}/p \in D^{\geq 0}$, so $\Delta_{S/A,\text{perf}} \in D^{\geq 0}$ as well.

To deduce that $S_{\text{perfd}} \in D^{\geq 0}$, it is enough to check that d is a nonzerodivisor in $H^0(\Delta_{S/A,\text{perf}})$. As $\Delta_{S/A,\text{perf}}/p \in D^{\geq 0}$, the ring $H^0(\Delta_{S/A,\text{perf}})$ must be p -torsionfree. The reasoning used above also shows that the automorphism ϕ_S of $H^0(\Delta_{S/A,\text{perf}})$ lifts the Frobenius on $H^0(\Delta_{S/A,\text{perf}})/p \subset H^0(\Delta_{S/A,\text{perf}}/p)$. It follows that $H^0(\Delta_{S/A,\text{perf}})$ is a perfect p -complete δ -ring, and ϕ_S is the unique Frobenius lift on such a ring. In particular, the map $A \rightarrow H^0(\Delta_{S/A,\text{perf}})$ is a map of δ -rings, so it carries d to a distinguished element. The claim now follows from Lemma 2.34. \square

Fix a derived p -complete simplicial R -algebra S as above. Our next goal is to describe $\Delta_{S/A,\text{perf}}$ in terms of the perfect prismatic site $(S/A)_{\Delta}^{\text{perf}}$ from Remark 4.5; this is also simply the site of perfectoid R -algebras R' with a map $S \rightarrow R'$ of R -algebras, and does not depend on the choice of A or R since Lemma 4.7 gives that $(S/A)_{\Delta}^{\text{perf}} = (S)_{\Delta}^{\text{perf}}$.

By Lemma 7.7, for $(B, J) \in (S/A)_{\Delta}^{\text{perf}}$, we have a functorial map

$$\Delta_{S/A} \rightarrow B,$$

which after passing to the perfection on the left and the limit over all B on the right induces a comparison map

$$\Delta_{S/A,\text{perf}} \rightarrow R\Gamma((S/A)_{\Delta}^{\text{perf}}, \mathcal{O}_{\Delta}).$$

Proposition 8.5. *For any choice of topology on $(S/A)_{\Delta}^{\text{perf}}$ between the flat and the chaotic topology, the map*

$$\Delta_{S/A,\text{perf}} \rightarrow R\Gamma((S/A)_{\Delta}^{\text{perf}}, \mathcal{O}_{\Delta})$$

is an equivalence. In particular, the perfectoidization

$$S_{\text{perfd}} \cong R\Gamma((S/A)_{\Delta}^{\text{perf}}, \overline{\mathcal{O}}_{\Delta}) \cong \lim_{S \rightarrow R'} R'$$

is the derived limit of R' over all maps from S to perfectoid rings R' , and does not depend on the choice of A .

Proposition 8.5 implies that when S is semiperfectoid, S_{perfd} agrees with the perfectoidization from Corollary 7.3, and that in general $\Delta_{S/A,\text{perf}}$ and S_{perfd} depend only on $\pi_0 S$ (and not on A).

Proof. First we check independence of the choice of topology. We claim first that the site $(S/A)_{\Delta}^{\text{perf}}$ admits a weakly final object. We can evidently assume that S is discrete. It suffices to prove that there is a map $S \rightarrow \tilde{S}$ to a semiperfectoid ring \tilde{S} such that any map from S to a perfectoid ring R' extends to \tilde{S} , for then \tilde{S}_{perfd} is the desired weakly initial object. For this, we inductively take any element $x \in S$ and pass to $S_x = S\langle X, Y \rangle / (x - X^p - pY)$. This adjoins a p -th root of x modulo p , and any map $S \rightarrow R'$ to a perfectoid ring extends to S_x . Repeating this transfinitely gives the desired ring \tilde{S} . Then by Čech-Alexander theory and flat descent, $R\Gamma((S/A)_{\Delta}^{\text{perf}}, \mathcal{O}_{\Delta})$ is computed by the cosimplicial perfectoid S -algebra $(\tilde{S}^{\bullet})_{\text{perfd}}$, where $\tilde{S}^i = (\tilde{S}^{\otimes_S (i+1)})^{\wedge}$ is the p -completed tensor product of $i+1$ copies of \tilde{S} over S , independently of the choice of topology, giving the desired independence.

We now work with the flat topology. First, let us restrict to the case where $\pi_0(S)$ is topologically finitely generated over A/I . For this, it is enough to show that both functors $S \mapsto S_{\text{perfd}}$ and $S \mapsto R\Gamma((S/A)_{\Delta}^{\text{perf}}, \mathcal{O}_{\Delta})$ commute with filtered colimits (when viewed as valued in (p, I) -complete objects in $\mathcal{D}(A)$). This is clear for $S \mapsto S_{\text{perfd}}$ as the analogous property holds true for derived prismatic cohomology. For the functor $S \mapsto R\Gamma((S/A)_{\Delta}^{\text{perf}}, \mathcal{O}_{\Delta})$, we may assume S is discrete (since $(S/A)_{\Delta}^{\text{perf}} = (\pi_0(S)/A)_{\Delta}^{\text{perf}}$). In that case, we can also compute the cohomology by passing to any flat semiperfectoid cover $S \rightarrow \tilde{S}$ and taking the similar Čech-Alexander complex. In particular, we can choose \tilde{S} functorially by extracting all p -power roots of all elements of S . Using such functorial Čech-Alexander complexes, one verifies that $R\Gamma((S/A)_{\Delta}^{\text{perf}}, \mathcal{O}_{\Delta})$ commutes with filtered colimits in S . Thus, we may assume that $\pi_0 S$ is topologically finitely generated over A/I .

Now choose a map $R\langle X_1, \dots, X_r \rangle \rightarrow S$ that is surjective on π_0 . Let \tilde{S} be the p -completion of $R[X_1^{1/p^\infty}, \dots, X_r^{1/p^\infty}] \otimes_{R[X_1, \dots, X_r]} S$; then $\pi_0 \tilde{S}$ is semiperfectoid. Let \tilde{S}^\bullet be the derived p -completion of $\tilde{S}^{\otimes_S(i+1)}$. By Lemma 8.6 below, $\Delta_{S/A}$ can be computed as the limit of $\Delta_{\tilde{S}^\bullet/A}$. By the similar descent for the right-hand side, we can reduce to the case that $\pi_0 S$ is semiperfectoid.

Now assume that $\pi_0 S$ is semiperfectoid. Let R' be the perfectoidization of $\pi_0 S$ in the sense of the last section, i.e. the universal perfectoid ring to which S maps. We need to see that

$$\Delta_{S/A, \text{perf}} \otimes_A^L A/I \rightarrow R'$$

is an isomorphism. Note that the left-hand side is always coconnective by Lemma 8.4. On the other hand, as S is semiperfectoid and so $L_{S/\mathbf{Z}_p}[-1]$ and thus all $\wedge^i L_{S/\mathbf{Z}_p}[-i]$ are connective, it follows from the Hodge-Tate comparison that $\Delta_{S/A}$ and thus also $\Delta_{S/A, \text{perf}}$ are connective. Thus, $\Delta_{S/A, \text{perf}}$ is in fact a perfect δ -ring, and so $S_{\text{perfd}} = \Delta_{S/A, \text{perf}} \otimes_A^L A/I$ is a perfectoid ring. We have maps

$$S \rightarrow S_{\text{perfd}} \rightarrow R'$$

where the composite is surjective by Theorem 7.4. By the universal property of R' , the second map is split surjective. Thus, it suffices to show that $S \rightarrow S_{\text{perfd}}$ is surjective.

As S/p is semiperfect, we may replace R with $R\langle Y_k^{1/p^\infty} \mid k \in K \rangle$ (for suitable index set K), we may assume $R \rightarrow S$ is surjective. Let $\{f_j\}_{j \in J}$ be a set of generators of $R \rightarrow S$. As the formation of $S \mapsto S_{\text{perfd}}$ commutes with p -completely flat base change on R , we may assume by Theorem 7.12 that each f_j admits a compatible system $\{f_j^{1/p^n}\}_{n \geq 0}$ of p -power roots in R . Fixing such a system for each f_j gives a map

$$S' := R\langle X_j^{1/p^\infty} \mid j \in J \rangle / (X_j \mid j \in J)^\wedge \rightarrow S : X_j^{1/p^n} \mapsto f_j^{1/p^n}$$

that is surjective on π_0 and induces a surjection on $\pi_1(L_{-/ \mathbf{Z}_p}/p)$. By the Hodge-Tate comparison theorem, the map $S'_{\text{perfd}} \rightarrow S_{\text{perfd}}$ is also surjective, so it suffices to prove that $S \rightarrow S_{\text{perfd}}$ is surjective when $S = S'$. By passage to filtered colimits, we may also assume J is finite. But then Proposition 7.10 and the construction of R' show that indeed $S_{\text{perfd}} = R'$, as desired. \square

We used the following lemma, which relies on Mathew's notion of descendability from [Mat16]; see also [BS17, §11.2] for a quick review.

Lemma 8.6 (Strong descent for passage to the perfection). *Let $T = R[X_1, \dots, X_n]^\wedge$, and let $T_\infty = R[X_1^{1/p^\infty}, \dots, X_n^{1/p^\infty}]^\wedge$. The natural maps $\Delta_{T/A} \rightarrow \Delta_{T_\infty/A}$ and $\Delta_{T/A, \text{perf}} \rightarrow \Delta_{T_\infty/A, \text{perf}}$ are descendable as maps of commutative algebras in $\mathcal{D}_{\text{comp}}(A)$.*

Recall that $\mathcal{D}_{\text{comp}}(A)$ is the full subcategory of $\mathcal{D}(A)$ spanned by derived (p, I) -complete objects.

Proof. The descendability of $\Delta_{T/A, \text{perf}} \rightarrow \Delta_{T_\infty/A, \text{perf}}$ follows from that of $\Delta_{T/A} \rightarrow \Delta_{T_\infty/A}$ and a general categorical fact: if $B \rightarrow C$ is a descendable map of commutative algebras of index $\leq n$ and both B and C come equipped with compatible endomorphisms ϕ , then the $\text{colim}_\phi B \rightarrow \text{colim}_\phi C$ is descendable of index $\leq 2n$ (see [BS17, Lemma 11.22]).

We now prove the descendability of $\Delta_{T/A} \rightarrow \Delta_{T_\infty/A}$. As the functor $\Delta_{-/A}$ from p -complete simplicial commutative R -algebras to commutative algebras in $\mathcal{D}_{\text{comp}}(A)$ commutes with coproducts, we may assume $n = 1$. Let $F \in \mathcal{D}(\Delta_{T/A})$ denote the fibre of $\Delta_{T/A} \rightarrow \Delta_{T_\infty/A}$. It suffices to show all maps $F^{\otimes 2} \rightarrow \Delta_{T/A}$ in $\mathcal{D}(\Delta_{T/A})$ are null-homotopic (where the tensor product is in $\mathcal{D}(\Delta_{T/A})$). For this, it is enough to show that

$$\text{RHom}_{\Delta_{T/A}}(F^{\otimes 2}, \Delta_{T/A}) \in D^{<0}.$$

By completeness, it suffices to check the same assertion after base change along $A \rightarrow A/(p, \phi^{-1}(d))$. In particular, we may assume R is a perfect ring of characteristic p , so $T = R[X]$ and $T_\infty = R[X^{1/p^\infty}]$ is its perfection. Using the de Rham comparison and the Cartier isomorphism, we are reduced to checking the following:

- (*) Let $\Omega := T \oplus \Omega_{T/R}^1[-1]$, regarded as a T -cdga that is a split square zero extension of T by $\Omega_{T/R}^1[-1]$. Let F be the fibre of the canonical map $\Omega \rightarrow T \rightarrow T_\infty$. Then

$$\text{RHom}_\Omega(F^{\otimes 2}, \Omega) \in D^{<0},$$

where the tensor product is in $\mathcal{D}(\Omega)$.

The fibre of $\Omega \rightarrow T$ is $\Omega_{T/R}^1[-1]$, which is isomorphic to $T[-1]$ as a dg-module over Ω via the choice of the generator $dX \in \Omega_{T/R}^1$. Also, T_∞ is free when regarded as a T -module with one of the generators being $1 \in T_\infty$, so the fibre of $T \rightarrow T_\infty$ identifies with $T^{\oplus I}[-1]$ where I a set. Combining, we find that the fibre F of $\Omega \rightarrow T_\infty$ is isomorphic to an Ω -module of the form $T^{\oplus J}[-1]$ for some set J . As $\text{RHom}(-, -)$ converts coproducts in the first factor into products, it is enough to check that $\text{RHom}_\Omega((T[-1])^{\otimes 2}, \Omega) \in D^{<0}$ (where the tensor product is in $\mathcal{D}(\Omega)$). This is a standard calculation using the resolution of the dg-module $T[-1]$ over the dg-algebra Ω given by

$$(\dots \rightarrow \Omega[-3] \rightarrow \Omega[-2] \rightarrow \Omega[-1]) \xrightarrow{\sim} T[-1],$$

where all transition maps are determined by $dX \in \Omega_{T/R}^1$. □

8.2. arc-descent. In this section, we use Proposition 8.5 to deduce some very strong descent results for the association $S \mapsto S_{\text{perfd}}$, formulated in terms of the arc-topology from [BM18]⁴.

⁴We are trying to achieve optimal results here and thus use the arc-topology. Getting similar results for the slightly weaker v -topology would be enough for the applications below, and could avoid reference to [BM18].

Definition 8.7 (The arc-topology of p -adic formal schemes). Consider the category \mathbf{fSch} of p -adic formal schemes X . Endow \mathbf{fSch} with the structure of a site by declaring a map $Y \rightarrow X$ of qcqs p -adic formal schemes to be an arc-cover if for all p -adically complete valuation rings V of rank 1 with a map $\mathrm{Spf} V \rightarrow X$ there is a faithfully flat extension $V \subset W$ of p -adically complete valuation rings of rank 1 and a lift to a map $\mathrm{Spf} W \rightarrow Y$, and taking the topology generated by arc-covers and open covers. We will refer to this as the arc-site on \mathbf{fSch} .

Let $\mathrm{Perfd} \subset \mathbf{fSch}$ be the full subcategory of p -adic formal schemes that are locally of the form $\mathrm{Spf} R$ for some perfectoid ring R , with the induced topology.

Perfectoids form a basis for the arc-topology:

Lemma 8.8. *Any $X \in \mathbf{fSch}$ admits a arc-cover by some $Y \in \mathrm{Perfd}$.*

Proof. We may assume $X = \mathrm{Spf} R$ is affine. Choose a p -completely faithfully flat map $R \rightarrow \tilde{R}$ such that \tilde{R} is semiperfectoid, so $\mathrm{Spf} \tilde{R} \rightarrow \mathrm{Spf} R$ is an arc-cover. Moreover, the map $\mathrm{Spf} \tilde{R}_{\mathrm{perfd}} \rightarrow \mathrm{Spf} \tilde{R}$ is also an arc-cover as any map $\tilde{R} \rightarrow V$ to a p -adically complete valuation ring factors over $\tilde{R}_{\mathrm{perfd}}$: one can assume that V is perfectoid by replacing it by the p -completion of an absolute integral closure, and the universal property of $\tilde{R} \rightarrow \tilde{R}_{\mathrm{perfd}}$ gives the desired extension. The composite $\mathrm{Spf} \tilde{R}_{\mathrm{perfd}} \rightarrow \mathrm{Spf} R$ is then the desired arc-cover. \square

The structure sheaf on \mathbf{fSch} is obviously not an arc-sheaf: nilpotent elements of a p -complete ring are not detected by mapping into valuation rings. On the other hand, it turns out that such problems disappear on the basis $\mathrm{Perfd} \subset \mathbf{fSch}$:

Proposition 8.9. *The presheaf \mathcal{O} on Perfd is a sheaf, and if $X = \mathrm{Spf} R$ is affine with R perfectoid, then $H_{\mathrm{arc}}^i(X, \mathcal{O}) = 0$ for $i > 0$.*

For any p -adic formal scheme T , the functor $X \mapsto \mathrm{Hom}(X, T)$ is an arc-sheaf on Perfd .

Proof. By Zariski descent, one can reduce to case of affine X . For the first statement, it suffices to prove that if $R \rightarrow S$ is a map of perfectoid rings such that $\mathrm{Spf} S \rightarrow \mathrm{Spf} R$ is an arc-cover, the complex

$$0 \rightarrow R \rightarrow S \rightarrow S \hat{\otimes}_R S \rightarrow \dots$$

is acyclic. It is enough to prove that the complex

$$0 \rightarrow R^\flat \rightarrow S^\flat \rightarrow S^\flat \hat{\otimes}_{R^\flat} S^\flat \rightarrow \dots$$

is acyclic where the completion is now d -adic, where d generates $\ker(A_{\mathrm{inf}}(R) \rightarrow R)$. Indeed, this implies that the same is true after applying Witt vectors to this sequence, and then also for the quotient modulo d .

Now we claim that $\mathrm{Spec}(S)^\flat \sqcup \mathrm{Spec}(R)^\flat[\frac{1}{d}] \rightarrow \mathrm{Spec}(R)^\flat$ is an arc-cover in the sense of [BM18, Definition 1.2]. Take any map $R^\flat \rightarrow V$ to a rank 1-valuation ring; we can assume that V is perfect. If the image of d in V is invertible, then it lifts to a map $R^\flat[\frac{1}{d}] \rightarrow V$. Otherwise, we may replace V by its d -adic completion. Then the map $R^\flat \rightarrow V$ is equivalent to the map $R \rightarrow V^\sharp$ to a p -adically complete valuation ring, which by assumption can be lifted to a map $S \rightarrow W$ for some p -adically complete extension W of V^\sharp which we may assume to be perfectoid. Then $S^\flat \rightarrow W^\flat$ gives the desired lift.

By v -descent, [BS17, Theorem 4.1], and the characterization [BM18, Theorem 1.6] of arc-descent in terms of v -descent and a property for valuation rings that in the present case is given by [BS17, Lemma 6.3], for any arc-cover $\mathrm{Spec}(B) \rightarrow \mathrm{Spec}(A)$ of perfect schemes, the complex

$$0 \rightarrow A \rightarrow B \rightarrow B \otimes_A B \rightarrow \dots$$

is exact. Applying this to $A = R^\flat$, $B = S^\flat \times R^\flat[\frac{1}{d}]$, and passing to (derived) d -adic completions, all terms involving $R^\flat[\frac{1}{d}]$ disappear, and we see that indeed

$$0 \rightarrow R^\flat \rightarrow S^\flat \rightarrow S^\flat \widehat{\otimes}_{R^\flat} S^\flat \rightarrow \dots$$

is exact.

If $T = \mathrm{Spf} S$ is affine, it is immediate from the preceding that $\mathrm{Hom}(X, T)$ is a sheaf on Perfd . In general, assume that $Y \rightarrow X$ is an arc-cover and we have given a map $Y \rightarrow T$ such that the two induced maps $Y \times_X Y \rightarrow T$ agree. It suffices to see that $|Y| \rightarrow |T|$ factors over a continuous map $|X| \rightarrow |T|$, as then one can localize on T and assume that T is affine. As the topological space in question depends only on the special fibre, we can assume that T is of characteristic p , and then X and Y are perfect schemes. By Zariski descent, it suffices to solve the problem when X and Y are affine. We may then assume that T is quasicompact, and by an argument of Gabber, cf. [Bha16, Remark 4.6], [Sta, Tag 03K0], we can assume that T is also quasiseparated. Then T can be written as an inverse limit of finite type \mathbf{F}_p -schemes along affine transition maps by [TT90, Appendix C], so we can assume that T is of finite type. In that case we can also assume that X and Y are perfection of schemes of finite type over \mathbf{F}_p . But then $X \rightarrow Y$ is an arc-cover only if it is an h -cover, in which case $|X| \rightarrow |Y|$ is a quotient map, giving the desired continuous map $|Y| \rightarrow |T|$. \square

Corollary 8.10. *For any p -complete ring S , one has*

$$S_{\mathrm{perfd}} = R\Gamma_{\mathrm{arc}}(\mathrm{Spf} S, \mathcal{O}) .$$

In particular, the association $S \mapsto S_{\mathrm{perfd}}$ satisfies descent for the arc-topology.

Proof. Let $\mathrm{affPerfd} \subset \mathrm{Perfd}$ be the subcategory of $\mathrm{Spf} R$ with R perfectoid. As $\mathrm{affPerfd} \subset \mathrm{fSch}$ is a basis (by proof of Lemma 8.8), one has

$$R\Gamma_{\mathrm{arc}}(\mathrm{Spf} S, \mathcal{O}) = R\Gamma_{\mathrm{arc}}(\mathrm{affPerfd}/\mathrm{Spf} S, \mathcal{O}) .$$

On the other hand, the projection from $\mathrm{affPerfd}$ with its arc-topology to $\mathrm{affPerfd}$ with the chaotic topology has trivial higher direct images for \mathcal{O} by Proposition 8.9, giving

$$R\Gamma_{\mathrm{arc}}(\mathrm{Spf} S, \mathcal{O}) = \lim_{S \rightarrow R'} R' = S_{\mathrm{perfd}} ,$$

as desired. \square

Corollary 8.11. *Let $S \rightarrow S'$ be a map of derived p -complete rings and assume that for some derived p -complete ideal $I \subset S$, the map $S \rightarrow S'$ induces an isomorphism of arc-sheaves outside of I in the sense that for any p -complete valuation ring V of rank 1 with a map $S \rightarrow V$ that does not kill I ,*

there is a unique extension to a map $S' \rightarrow V$.⁵ Then

$$\begin{array}{ccc} S_{\text{perfd}} & \longrightarrow & S'_{\text{perfd}} \\ \downarrow & & \downarrow \\ (S/I)_{\text{perfd}} & \longrightarrow & (S'/I)_{\text{perfd}} \end{array}$$

is a pullback square of E_∞ - S -algebras (in particular, in the derived category of S -modules).

We note that even when S and S' are perfectoid, the result is interesting, and in particular ensures that the kernel and cokernel of $S \rightarrow S'$ are killed by $\ker(S \rightarrow (S/I)_{\text{perfd}})$.

Proof. Replacing S' by $S' \times S/I$, we may assume that $\text{Spf} S' \rightarrow \text{Spf} S$ is an arc-cover. Now we follow the proof of [BS17, Theorem 2.9] (cf. [Voe96, Lemma 3.6]). Let \mathcal{F}_T be the sheaf $X \mapsto \text{Hom}(X, \text{Spf} T)$ on Perfd for any derived p -complete ring T . By Corollary 8.10, it suffices to see that

$$\mathcal{F}_S = \mathcal{F}_{S'} \sqcup_{\mathcal{F}_{S'/I}} \mathcal{F}_{S/I}$$

as sheaves of spaces on the site Perfd . But as $\mathcal{F}_{S'/I} \hookrightarrow \mathcal{F}_{S'}$, this pushout is still discrete, so we can compute it in the category of sheaves of sets. In other words, we need to see that for any sheaf of sets F on Perfd , we have

$$F(S) = F(S') \times_{F(S'/I)} F(S),$$

where F is extended to fSch by descent. For this, we use descent along $\text{Spf} S' \rightarrow \text{Spf} S$. Note that

$$\text{Spf} S' \sqcup \text{Spf}(S'/I) \times_{\text{Spf}(S/I)} \text{Spf}(S'/I) \rightarrow \text{Spf} S' \times_{\text{Spf} S} \text{Spf} S'$$

is an arc-cover by the assumption on I . Unraveling, this gives the result. \square

On the other hand, a main advantage of the definition of S_{perfd} in terms of derived prismatic cohomology is that it allows us to prove the following result.

Proposition 8.12. *The functor $S \mapsto S_{\text{perfd}}$ from derived p -complete simplicial rings admitting a map from a perfectoid ring, to derived p -complete E_∞ -rings is symmetric monoidal, i.e. for any diagram $S_1 \leftarrow S_3 \rightarrow S_2$, the induced map*

$$S_{1,\text{perfd}} \hat{\otimes}_{S_{3,\text{perfd}}}^L S_{2,\text{perfd}} \rightarrow (S_1 \hat{\otimes}_{S_3}^L S_2)_{\text{perfd}}$$

is an equivalence.

Proof. Pick a perfectoid ring A mapping to S_3 . Then derived prismatic cohomology $\mathbb{A}_{-/A}$ is symmetric monoidal by the Hodge-Tate comparison, and this passes to the perfection, and then to the quotient modulo I . \square

Corollary 8.13. *Say S is a derived p -complete simplicial ring admitting a map from a perfectoid ring. Assume that S_{perfd} is connective (or, equivalently by Lemma 8.4, concentrated in degree 0). Then S_{perfd} is a perfectoid ring, and $S \rightarrow S_{\text{perfd}}$ is the universal map from S to a perfectoid ring.*

⁵This holds, for example, if $S \rightarrow S'$ is an integral map and $\text{Spec}(S') \rightarrow \text{Spec}(S)$ is an isomorphism outside $V(I)$.

Proof. If S_{perfd} is discrete, then so is $\mathbb{A}_{S/A, \text{perf}}$ by derived Nakayama. But then $\mathbb{A}_{S/A, \text{perf}}$ is a perfect p -complete δ - A -algebra by the proof of Lemma 7.7. The pair $(\mathbb{A}_{S/A, \text{perf}}, I\mathbb{A}_{S/A, \text{perf}})$ is then necessarily a perfect prism (Lemma 2.34), so S_{perfd} is a perfectoid ring. To prove the universality of $S \rightarrow S_{\text{perfd}}$, we must show that any map $S_{\text{perfd}} \rightarrow S'$ with S' perfectoid is uniquely determined by the composition $S \rightarrow S_{\text{perfd}} \rightarrow S'$; equivalently, we must check that any map $\eta : S_{\text{perfd}} \widehat{\otimes}_S^L S_{\text{perfd}} \rightarrow S'$ with S' perfectoid factors (necessarily uniquely) over the multiplication map $S_{\text{perfd}} \widehat{\otimes}_S^L S_{\text{perfd}} \rightarrow S_{\text{perfd}}$. As S' is perfectoid, such a map η factors over $S_{\text{perfd}} \widehat{\otimes}_S^L S_{\text{perfd}} \rightarrow (S_{\text{perfd}} \widehat{\otimes}_S^L S_{\text{perfd}})_{\text{perfd}}$ by applying the functor $(-)_{\text{perfd}}$ since $S' \simeq S'_{\text{perfd}}$. Now $(S_{\text{perfd}} \widehat{\otimes}_S^L S_{\text{perfd}})_{\text{perfd}} \simeq S_{\text{perfd}}$ via the multiplication map thanks to Proposition 8.12, so we are done. \square

9. THE ÉTALE COMPARISON THEOREM

In this section, we explain how to recover étale cohomology of the generic fibre from prismatic cohomology for arbitrary rings.

Theorem 9.1 (The étale comparison theorem). *Fix a perfectoid ring R corresponding to a perfect prism $(A, (d))$, and a p -adic formal scheme X over R . Write $X_\eta := X \times_{\mathrm{Spf}(R)} \mathrm{Spa}(R[1/p], R)$ for the adic generic fibre of X , and let $\mu : X_{\eta, \acute{e}t} \rightarrow X_{\acute{e}t}$ denote the “nearby cycles” map. There is a canonical identification*

$$R\mu_* \mathbf{Z}/p^n \simeq (\Delta_{X/A}[1/d]/p^n)^{\phi=1}.$$

In particular, if $X = \mathrm{Spf}(S)$ is affine, then there is a canonical identification

$$R\Gamma(\mathrm{Spec}(S[1/p]), \mathbf{Z}/p^n) \simeq (\Delta_{S/A}[1/d]/p^n)^{\phi=1} \quad (3)$$

for each $n \geq 0$.

Note that there are no restrictions on the singularities of X above. We shall apply this in Theorem 9.4 to relate the p -adic Tate twists introduced in [BMS19] to the usual Tate twists on the generic fibre for perfectoid rings. The proof of Theorem 9.1 will rely on a slightly surprising compatibility of the formation of ϕ -fixed points with completed colimits.

Lemma 9.2 (Commuting fixed points with completed colimits). *Let B be an \mathbf{F}_p -algebra equipped with an element $t \in B$. Let $\mathcal{D}(B[F])$ be the ∞ -category of pairs (M, ϕ) where $M \in \mathcal{D}(B)$ and $\phi : M \rightarrow \phi_* M$ is a map. Let $\mathcal{D}_{\mathrm{comp}}(B[F]) \subset \mathcal{D}(B[F])$ be the full subcategory spanned by pairs (M, ϕ) with M derived t -complete. Then the functors $\mathcal{D}_{\mathrm{comp}}(B[F]) \rightarrow \mathcal{D}(\mathbf{F}_p)$ given by $M \mapsto M^{\phi=1}$ and $M \mapsto (M[1/t])^{\phi=1}$ commute with colimits.*

Proof. In this proof, all colimits refer to colimits in the underlying category $\mathcal{D}(B)$. Let $\{(M_i, \phi_i)\}$ be a diagram in $\mathcal{D}_{\mathrm{comp}}(B[F])$. The fiber of the map $\mathrm{colim}_i M_i \rightarrow \widehat{\mathrm{colim}_i M_i}$ is uniquely t -divisible, and thus identifies with the fiber of the map $\mathrm{colim}_i M_i[1/t] \rightarrow (\widehat{\mathrm{colim}_i M_i})[1/t]$. As the lemma can be reformulated as the statement that $F^{\phi=1} = 0$, it suffices to prove the statement for the functor $M \mapsto M^{\phi=1}$. For this, we claim that for any $(N, \phi) \in \mathcal{D}_{\mathrm{comp}}(B[F])$, we have $N^{\phi=1} \simeq (N/t)^{\phi=1}$. First, this makes sense because we have $\phi(t) = t^p \subset tB$, so for any $N \in \mathcal{D}(B[F])$, there is an induced ϕ -structure on N/t compatible with the one on N . Secondly, if N is derived t -complete, then the fibre of $N \rightarrow N/t$ is complete when endowed with the t -adic filtration, and the ϕ -action on the fibre is topologically nilpotent with respect to this filtration as $\phi(t) = t^p \subset t^2 B$, so $(-)^{\phi=1}$ must vanish on the fibre, giving $N^{\phi=1} \simeq (N/t)^{\phi=1}$, as asserted. The lemma now follows because both functors in the composition

$$\mathcal{D}_{\mathrm{comp}}(B[F]) \xrightarrow{N \mapsto N/t} \mathcal{D}(B[F]) \xrightarrow{(-)^{\phi=1}} \mathcal{D}(\mathbf{F}_p)$$

commute with all colimits. \square

Proof of Theorem 9.1. Consider the following two functors on affine formal schemes $X = \mathrm{Spf} S \in \mathrm{fSch}_{/\mathrm{Spf} R}$. The first takes X to $R\Gamma(\mathrm{Spec}(S[\frac{1}{p}]), \mathbf{Z}/p^n)$, while the other takes X to $(\Delta_{S/A}[1/d]/p^n)^{\phi=1}$. We claim that both are arc-sheaves. Granting this, we immediately get a comparison map in the direction $R\Gamma(\mathrm{Spec}(S[\frac{1}{p}]), \mathbf{Z}/p^n) \rightarrow (\Delta_{S/A}[1/d]/p^n)^{\phi=1}$.

For the first, this is [BM18, Corollary 6.17]. For the other, Lemma 9.2 implies that the map

$$(\Delta_{S/A}[1/d]/p^n)^{\phi=1} \rightarrow (\Delta_{S/A,\text{perf}}[1/d]/p^n)^{\phi=1}$$

is a quasi-isomorphism: It suffices to check this when $n = 1$, and then $\Delta_{S/A,\text{perf}}/p$ is the completed filtered colimit of

$$\Delta_{S/A}/p \xrightarrow{\phi_S} \Delta_{S/A}/p \xrightarrow{\phi_S} \dots$$

As each map ϕ_S trivially induces an equivalence on applying $((-)[\frac{1}{d}])^{\phi=1}$, Lemma 9.2 gives the claim. But $\Delta_{S/A,\text{perf}}$ is an arc-sheaf as $S \mapsto S_{\text{perfd}}$ is by Corollary 8.10.

In other words, we are allowed to localize in the arc-topology. At this point, there are two ways to finish the proof. In the first approach, we can assume S is perfectoid thanks to arc-localization. One can then identify $\Delta_{S/A}[1/d]/p^n = W(S^b)[1/d]/p^n$, and then by Artin-Schreier-Witt we have

$$R\Gamma(\text{Spec}(S^b[\frac{1}{d}]), \mathbf{Z}/p^n) = (\Delta_{S/A}[1/d]/p^n)^{\phi=1}.$$

It remains to prove that there is a canonical quasi-isomorphism

$$R\Gamma(\text{Spec}(S[\frac{1}{p}]), \mathbf{Z}/p^n) \simeq R\Gamma(\text{Spec}(S^b[\frac{1}{d}]), \mathbf{Z}/p^n).$$

This follows from the comparison between the étale cohomology of $\text{Spec}(S[\frac{1}{p}])$ and $\text{Spa}(S[\frac{1}{p}], S)$ (and similarly for S^b), cf. [Hub96, Corollary 3.2.2], and [Sch12, Theorem 1.11].

In the second approach, one can apply further arc-descent to reduce to a case like $S = \prod R_i$ being a product of absolutely integrally closed valuation rings of rank ≤ 1 , where the claim can be verified by hand (as $H^i = 0$ for $i > 0$ and the H^0 becomes explicit); we leave this as an exercise to the reader. \square

Remark 9.3. There is also a variant of Theorem 9.1 without inverting p or d : if X is a p -adic formal scheme R -scheme, then there is a canonical identification

$$\mathbf{Z}/p^n \simeq (\Delta_{X/A}/p^n)^{\phi=1}$$

of étale sheaves on X for all $n \geq 0$; we leave this to the reader.

Recall that for any integer $n \geq 0$, we have defined p -adic Tate twists $\mathbf{Z}_p(n)$ as sheaves on the quasisyntomic site in [BMS19, §7.4]; the notion is reviewed more thoroughly in §12. For a perfectoid ring R , these admit a concrete description in terms of the corresponding perfect prism $(A, (d))$:

$$\mathbf{Z}_p(n)(R) \simeq \left(\phi^{-1}(d)^n A \xrightarrow{\frac{\phi}{d^n} - 1} A \right). \quad (4)$$

One can show that the right side is independent of the choice of generator d up to quasi-isomorphism. Our goal is to describe these complexes in terms involving only the special or generic fibres.

Theorem 9.4. *Let R be a perfectoid ring. For any integer $n \geq 1$, there is a canonical identification*

$$\mathbf{Z}_p(n)(R) \simeq R\Gamma(\text{Spec}(R[1/p]), \mathbf{Z}_p(n)).$$

For $n = 0$, there is a canonical identification

$$\mathbf{Z}_p(0)(R) \simeq R\Gamma(\text{Spec}(R), \mathbf{Z}_p) \simeq R\Gamma(\text{Spec}((R/p)_{\text{perf}}), \mathbf{Z}_p).$$

Proof. The statement for $n = 0$ at the end follows from Remark 9.3 (and Gabber’s affine analog of proper base change to pass from R to R/p); we leave the details to the reader. Assume from now that $n \geq 1$. Write $F_n(R) = \mathbf{Z}_p(n)(R)$ and $G_n(R) = R\Gamma(\mathrm{Spec}(R[1/p]), \mathbf{Z}_p(n))$; these are p -complete arc-sheaves on the category of perfectoid rings. By [BMS19, Proposition 7.17], we know that $\mathbf{Z}_p(1) = \lim_n \mu_{p^n}$ as arc-sheaves on perfectoid rings (or even as sheaves on the full quasisyntomic site), so there is a canonical map $\alpha_1 : F_1 \rightarrow G_1$. By multiplicativity, this induces a map

$$\alpha_n : F_n \rightarrow G_n$$

for all $n \geq 1$. It is enough to show that $\alpha_n/p = \alpha_n \otimes_{\mathbf{Z}_p}^L \mathbf{Z}/p$ is an isomorphism.

It is slightly tricky⁶ to identify α_n explicitly. The argument should be as follows: As both sides are arc-sheaves, we can in particular assume that there is a map $\mathbf{Z}_p[\zeta_{p^\infty}] \rightarrow R$, inducing an element $q = [\epsilon] \in A_{\mathrm{inf}}(R)$, and the distinguished element $d = [p]_q$ generates $\ker(A_{\mathrm{inf}}(R) \rightarrow R)$. Set $\mu = q - 1 \in A_{\mathrm{inf}}(R)$. Unwinding identifications and using the étale comparison theorem, the map α_n/p should be given by the map of complexes

$$\alpha_n/p : F_n(R)/p := \left(d^{\frac{n}{p}} R^b \xrightarrow{\frac{\phi}{d^n} - 1} R^b \right) \rightarrow G_n(R)/p := \left(R^b[1/d] \xrightarrow{\phi - 1} R^b[1/d] \right)$$

induced by multiplication by μ^{-n} on the terms. In other words, it is given by the evident map of complexes

$$\left(\mu^{-n/p} R^b \xrightarrow{\phi - 1} \mu^{-n} R^b \right) \rightarrow \left(R^b[1/\mu] \xrightarrow{\phi - 1} R^b[1/\mu] \right).$$

In any case, this prescription defines a comparison map

$$\alpha'_n : F_n(R)/p \rightarrow G_n(R)/p$$

on perfectoid $\mathbf{Z}_p[\zeta_{p^\infty}]$ -algebras. The map α'_n is a quasi-isomorphism: It is the colimit over the maps of complexes

$$\left(\mu^{-np^{i-1}} R^b \xrightarrow{\phi - 1} \mu^{-np^i} R^b \right) \rightarrow \left(\mu^{-np^i} R^b \xrightarrow{\phi - 1} \mu^{-np^{i+1}} R^b \right)$$

induced by the identity; these are homotopic to the maps induced by ϕ , which are isomorphisms of complexes.

It remains to show that also α_n/p is a quasi-isomorphism. We use that both F_n and G_n are p -complete arc-sheaves, so we can assume that R is a product of p -complete valuation rings of rank ≤ 1 with algebraically closed fraction field. Also, if R is any perfect \mathbf{F}_p -algebra, then both F_n and G_n vanish for $n \geq 1$: this is obvious for G_n , and thus follows for F_n by using α'_n . We can thus ignore the p -torsion factors of R , so we have reduced to checking the result when $R = \prod_j V_j$ is a product of p -complete p -torsionfree valuation rings V_j of rank 1 with algebraically closed fraction field; we fix a map $\mathbf{Z}_p[\zeta_{p^\infty}] \rightarrow R$.

⁶The map α_1 comes from a map $\tilde{\alpha}_1 : \mathbf{Z}_p(1)(R) \rightarrow A_{\mathrm{inf}}(R)\{1\}$ by observing that the image lies in a suitable Frobenius eigenspace; the Breuil-Kisin twist has been suppressed above as it can be ignored using our choice of $d \in I$. The map $\tilde{\alpha}_1$ comes from the cyclotomic trace map relating algebraic K -theory and THH, and is rather inexplicit. Nevertheless, it has been identified in [ALB19a] as a q -logarithm when working over the q -de Rham prism (which applies for any quasiregular semiperfectoid ring R that receives a map from $\mathbf{Z}_p^{\mathrm{cycl}}$). In [BL], this will be extended to all prisms: there exists a natural “prismatic logarithm” $\mathbf{Z}_p(1)(A/I) \rightarrow A\{1\}$ over any prism (A, I) .

Next, we claim that $F_n(R)/p \simeq \prod_i F_n(V_i)/p$ and similarly for $G_n(-)/p$. This is clear for $F_n(-)/p$ as $A_{\text{inf}}(-)$ has the same property, and follows for $G_n(-)/p$ by using α'_n . In the final case $R = V = \mathcal{O}_C$ being the ring of integers in a complete algebraically closed extension C of \mathbb{Q}_p , we note that $G_n(-)/p = R\Gamma(\text{Spec}(C), \mathbf{F}_p(n)) \cong \mathbf{F}_p[0]$ for all n , and thus also $F_n(-)/p = \mathbf{F}_p[0]$ for all n . That α_n/p is an isomorphism now follows from the result for $n = 1$ and multiplicativity. \square

The following is a non-obvious consequence from a purely algebraic perspective:

Corollary 9.5. *For a perfectoid ring R , the groups $\text{Pic}(R)$ and $\text{Pic}(R[1/p])$ are uniquely p -divisible.*

Proof. By deformation theory, we have $\text{Pic}(R) \cong \text{Pic}(A_{\text{inf}}(R)) \cong \text{Pic}(R^\flat)$, so the claim follows for $\text{Pic}(R)$ as any perfect ring has a uniquely p -divisible Picard group.

For $\text{Pic}(R[1/p])$, Theorem 9.4 shows that $H^2(\text{Spec}(R[1/p]), \mu_p) \cong H^2(\text{Spec}(R), \mu_p)$, which vanishes by Equation (4), so the Kummer sequence shows that $\text{Pic}(R[1/p])$ is p -divisible. On the other hand, Theorem 9.4 also gives the equality $H^1(\text{Spec}(R[1/p]), \mu_p) \cong H^1(\text{Spec}(R), \mu_p)$, which implies that $\text{Pic}(R[1/p])[p] \leftarrow \text{Pic}(R)[p]$ is surjective by the Kummer sequence; as we have seen that the source is zero, it follows that $\text{Pic}(R[1/p])$ has no p -torsion, and is thus uniquely p -divisible. \square

10. THE ALMOST PURITY THEOREM

The goal of this section is to use the perfectoidization functor $S \mapsto S_{\text{perfd}}$ from Section 8 to prove a general version of the almost purity theorem that includes André's perfectoid Abhyankar lemma [And18b]; this goal is accomplished in §10.2. To accomplish this task, we need a notion of almost mathematics that deals with ramification with respect to any (finitely generated) ideal in a perfectoid ring; this is possible thanks to Theorem 7.4, and is the subject of §10.1.

10.1. Almost mathematics with respect to any ideal. Let R be a perfectoid ring and let $J \subset R$ be an ideal with perfectoidization $J_{\text{perfd}} = \ker(R \rightarrow (R/J)_{\text{perfd}})$. In this section, we study almost mathematics for p -complete R -modules with respect to the ideal J_{perfd} .

Definition 10.1. A derived p -complete R -module M is J -almost zero if $J_{\text{perfd}}M = 0$.

When $J = (p)$, we have $J_{\text{perfd}} = \sqrt{pR}$, and the above notion then agrees with the usual notion of “almost zero” for modules over a perfectoid ring.

Proposition 10.2. *The subcategory of J -almost zero derived p -complete R -modules is stable under kernels, cokernels and extensions in the category of derived p -complete R -modules, and forms a \otimes -ideal. It is equivalent to the category of derived p -complete $(R/J)_{\text{perfd}}$ -modules.*

Proof. Consider the “forgetful” functor from the category of derived p -complete $(R/J)_{\text{perfd}}$ -modules to the category of derived p -complete R -modules. To prove the proposition, it suffices to prove that this functor is fully faithful and preserves Ext^1 . We will in fact prove that it preserves $R\text{Hom}$. Thus let M and N be two derived p -complete $(R/J)_{\text{perfd}}$ -modules. Then

$$R\text{Hom}_R(M, N) = R\text{Hom}_{(R/J)_{\text{perfd}}}(M \widehat{\otimes}_R^L (R/J)_{\text{perfd}}, N) .$$

Thus it is enough to see that $M \rightarrow M \widehat{\otimes}_R^L (R/J)_{\text{perfd}}$ is an isomorphism. As M is an $(R/J)_{\text{perfd}}$ -module, it suffices to prove this when $M = (R/J)_{\text{perfd}}$, which is part of the next lemma. \square

Lemma 10.3. *The multiplication maps*

$$J_{\text{perfd}} \widehat{\otimes}_R^L J_{\text{perfd}} \rightarrow J_{\text{perfd}} , \quad (R/J)_{\text{perfd}} \widehat{\otimes}_R^L (R/J)_{\text{perfd}} \rightarrow (R/J)_{\text{perfd}}$$

are quasi-isomorphisms, and

$$J_{\text{perfd}} \widehat{\otimes}_R^L (R/J)_{\text{perfd}} = 0 .$$

Proof. It is clear that the first two maps are isomorphism precisely when $J_{\text{perfd}} \widehat{\otimes}_R^L (R/J)_{\text{perfd}} = 0$, so it suffices to prove that

$$(R/J)_{\text{perfd}} \widehat{\otimes}_R^L (R/J)_{\text{perfd}} \rightarrow (R/J)_{\text{perfd}}$$

is a quasi-isomorphism. This is clearly true on H^0 , so it suffices to prove more generally that for any maps $S \leftarrow R \rightarrow T$ of perfectoid rings, the ring $S \widehat{\otimes}_R^L T$ is concentrated in degree 0 and perfectoid. Let $(B, IB) \leftarrow (A, I) \rightarrow (C, IC)$ be the diagram of perfect prisms corresponding to $S \leftarrow R \rightarrow T$. Then $D := B \widehat{\otimes}_A^L C$ is concentrated in degree 0 and a perfect δ -ring, as this can be checked after (derived) reduction modulo p , where it follows from Tor-independence of perfect rings, cf. e.g. [BS17, Lemma

3.16]. In particular, (D, ID) is a perfect prism. In particular, ID defines a Cartier divisor and we have $D/ID = S \widehat{\otimes}_R^L T$, giving the desired statement. \square

Proposition 10.4. *A derived p -complete complex M of R -modules is J -almost zero, i.e. $H^i(M)$ is J -almost zero for all i , if and only if*

$$J_{\text{perfd}} \widehat{\otimes}_R^L M = 0 .$$

The category of all such is equivalent to the category $D_{p\text{-comp}}((R/J)_{\text{perfd}})$ of derived p -complete complexes of $(R/J)_{\text{perfd}}$ -modules, and is a thick tensor ideal in the category of all derived p -complete complexes of R -modules.

Proof. We have already proved that $D_{p\text{-comp}}((R/J)_{\text{perfd}}) \rightarrow D_{p\text{-comp}}(R)$ is fully faithful. Clearly, any object in the image is J -almost zero; conversely, writing any object as a limit and colimit of its truncations, and using that the inclusion commutes with all limits and colimits, shows that any J -almost zero complex is in the image. For all objects M in the image,

$$J_{\text{perfd}} \widehat{\otimes}_R^L M = J_{\text{perfd}} \widehat{\otimes}_R^L (R/J)_{\text{perfd}} \otimes_{(R/J)_{\text{perfd}}} M = 0 .$$

Conversely, if $J_{\text{perfd}} \widehat{\otimes}_R^L M = 0$, then $M \rightarrow M \widehat{\otimes}_R^L (R/J)_{\text{perfd}}$ is a quasi-isomorphism and so M is J -almost zero. \square

The following connectivity criterion in J -almost mathematics shall be useful later.

Lemma 10.5. *A p -complete complex $M \in D(R)$ is connective if and only if $M \widehat{\otimes}_R^L R/J_{\text{perfd}}$ is connective and M is J -almost connective (i.e., $H^i(M)$ is J -almost zero for $i > 0$).*

Proof. The “only if” direction is clear as $(-) \widehat{\otimes}_R^L R/J_{\text{perfd}}$ preserves connectivity. For the converse, fix a derived p -complete complex $M \in D(R)$. We have a triangle

$$J_{\text{perfd}} \widehat{\otimes}_R^L M \rightarrow M \rightarrow (R/J_{\text{perfd}}) \widehat{\otimes}_R^L M,$$

so it suffices to see that if M is J -almost connective then $J_{\text{perfd}} \widehat{\otimes}_R^L M$ is connective (the converse is also true, and clear). But in the triangle

$$J_{\text{perfd}} \widehat{\otimes}_R^L \tau^{\leq 0} M \rightarrow J_{\text{perfd}} \widehat{\otimes}_R^L M \rightarrow J_{\text{perfd}} \widehat{\otimes}_R^L \tau^{> 0} M$$

the last term vanishes by Proposition 10.4 and the first term is connective. \square

The following approach to Galois coverings in the J -almost category shall prove useful.

Definition 10.6. A p -complete R -algebra S equipped with an action of a finite group G is a J -almost G -Galois extension if the maps

$$R \rightarrow R\Gamma(G, S) , \quad S \widehat{\otimes}_R^L S \rightarrow \prod_G S$$

are J -almost isomorphisms, i.e. the cones are J -almost zero.

Remark 10.7. The conditions imply similar statements modulo p^n , which then by passing to H^0 imply that

$$R/p^n \rightarrow (S/p^n)^G, \quad S/p^n \otimes_{R/p^n} S/p^n \rightarrow \prod_G S/p^n$$

are J -almost isomorphisms.

Note that for each n , $(R/p^n, J_{\text{perfd},n})$ defines an almost setting (called a *basic setup* in [GR03, §2.1.1]), where $J_{\text{perfd},n}$ is the image of J_{perfd} in R/p^n (i.e., $J_{\text{perfd},n}$ is an idempotent ideal of R/p^n). If $J = (g)$ is generated by one element, then it satisfies the hypothesis of [GR03, §2.5.15] that $J_{\text{perfd},n} \otimes_{R/p^n} J_{\text{perfd},n}$ is a flat R/p^n -module: Indeed, when g admits p -power roots, this is given by $\text{colim}_{g^{1/p^i-1}/p^{i-1}} R/p^n$.⁷

Proposition 10.8 ([And18b, Proposition 1.9.1]). *Let S be a p -complete R -algebra with G -action that is a J -almost G -Galois cover. Then $R/p^n \rightarrow S/p^n$ is almost finite projective and almost unramified with respect to $(R/p^n, J_{\text{perfd},n})$. More generally, if for a subgroup $H \subset G$ the map $S' = S^H \rightarrow S$ is a J -almost H -Galois cover, then $R/p^n \rightarrow S'/p^n$ is almost finite projective and almost unramified.*

10.2. The almost purity theorem. Our goal is to prove the following version of the almost purity theorem (and André's perfectoid Abhyankar lemma) over a perfectoid ring R , handling ramification along arbitrary closed subsets of $\text{Spec}(R)$.

Theorem 10.9. *Let R be a perfectoid ring, $J \subset R$ a finitely generated ideal and $J_{\text{perfd}} = \ker(R \rightarrow (R/J)_{\text{perfd}})$. Let S be a finitely presented finite R -algebra such that $\text{Spec}(S) \rightarrow \text{Spec}(R)$ is finite étale outside $V(J)$. Then S_{perfd} is discrete and a perfectoid ring, and the map $S \rightarrow S_{\text{perfd}}$ is an isomorphism away from $V(J)$. Moreover, for $n > 0$, the map $R/p^n \rightarrow S_{\text{perfd}}/p^n$ is almost finite projective and almost unramified with respect to the almost setting $(R/p^n, J_{\text{perfd},n})$.*

In fact, if S admits a G -action for some finite group G such that $\text{Spec}(S) \rightarrow \text{Spec}(R)$ is a G -Galois cover outside $V(J)$, then $R \rightarrow S_{\text{perfd}}$ is a J -almost G -Galois cover in the sense of Definition 10.6.

We first prove Theorem 10.9 only in the J -almost setting. This J -almost version then implies a general discreteness statement (Theorem 10.11) for perfectoidizations of integral algebras over perfectoid rings, which then yields Theorem 10.9 in general. The key inputs in the argument are:

⁷For this, we have to see that

$$\text{colim}_{g^{1/p^i-1}/p^{i-1}} R/p^n \rightarrow J_{\text{perfd}}/p^n$$

is an isomorphism. Surjectivity is clear. For injectivity, assume that $a \in R$ is an element such that when considered as an element in the i -th copy of R/p^n it maps to 0 in J_{perfd}/p^n . This means that $g^{1/p^i}a = p^n g^{1/p^k}b$ for some k (which we can assume to be larger than i) and $b \in R$. In particular

$$g^{1/p^k}(g^{1/p^i-1/p^k}a - p^n b) = 0.$$

As R is reduced, this implies that also

$$g^{1/p^{k+1}}(g^{1/p^i-1/p^k}a - p^n b) = 0,$$

or in other words

$$g^{1/p^i-1/p^k+1/p^{k+1}}a = p^n g^{1/p^{k+1}}b \in p^n R,$$

which means that the image of a in $\text{colim}_{g^{1/p^i-1}/p^{i+1}} R/p^n$ is equal to 0.

- (1) The arc-descent properties of $S \mapsto S_{\text{perfd}}$, especially Corollary 8.11.
- (2) André's lemma (Theorem 7.12).

Remark 10.10. Our proof of Theorem 10.9 yields, in particular, a new proof of the almost purity theorem: our arguments do not make any use of adic spaces (in particular, not of perfectoid spaces).

Proof of Theorem 10.9 in the J -almost setting. With hypotheses and notation as in Theorem 10.9, our goal is to show Theorem 10.9 holds true in J -almost setting, i.e., to show:

- (*) S_{perfd} is J -almost connective, i.e. $J_{\text{perfd}} H^i(S_{\text{perfd}}) = 0$ for $i > 0$, and $S \rightarrow S_{\text{perfd}}$ is an isomorphism away from $V(J)$. Moreover, for all $n > 0$ the map

$$R/p^n \rightarrow H^0(S_{\text{perfd}})/p^n$$

is almost finite projective and almost unramified with respect to the almost setting $(R/p^n, J_{\text{perfd},n})$.

In fact, if S admits a G -action for some finite group G such that $\text{Spec}(S) \rightarrow \text{Spec}(R)$ is a G -Galois cover outside $V(J)$, then $R \rightarrow H^0(S_{\text{perfd}})$ is a J -almost G -Galois cover in the sense of Definition 10.6.

By Corollary 8.11, we are allowed to replace S by an S -algebra S' that is integral over S and such that $\text{Spec}(S') \rightarrow \text{Spec}(S)$ is an isomorphism outside $V(J)$ (as then $S_{\text{perfd}} \rightarrow S'_{\text{perfd}}$ is a J -almost isomorphism). In particular, this implies that S_{perfd} is independent of the choice of S inducing a given finite étale cover of $\text{Spec}(R) \setminus V(J)$, up to J -almost isomorphism.

Assume first that S admits an action of G such that $\text{Spec}(S) \rightarrow \text{Spec}(R)$ is a G -Galois extension outside $V(J)$. We want to prove that $R \rightarrow S_{\text{perfd}}$ is a J -almost G -Galois cover. By Theorem 7.12, we can assume that R is absolutely integrally closed. This implies in particular that any finite étale cover of any Zariski localization is Zariski locally split, so we can find generators f_1, \dots, f_r of J such that $\text{Spec}(S) \rightarrow \text{Spec}(R)$ admits a splitting outside $V(f_i)$ for $i = 1, \dots, r$. As being a J -almost isomorphism is equivalent to being an (f_i) -almost isomorphism for all $i = 1, \dots, r$, we can assume that $J = (f)$ is a principal ideal such that $\text{Spec}(S) \rightarrow \text{Spec}(R)$ splits over $\text{Spec}(R)[\frac{1}{f}]$.⁸ But recall that we were allowed to replace S by an integral S -algebra S' such that $\text{Spec}(S') \rightarrow \text{Spec}(S)$ is an isomorphism outside $V(J)$. This allows us to replace S by the trivial G -torsor, where the result is clear.

Now assume more generally that $\text{Spec}(S) \rightarrow \text{Spec}(R)$ has constant degree r outside $V(J)$. In that case, we can find a Σ_r -Galois extension of $X \setminus V(J)$ of which $\text{Spec}(S) \setminus V(J)$ is the quotient by Σ_{r-1} , cf. e.g. [And18b, Lemma 1.9.2]. By Proposition 10.8, we can reduce to the case of a G -Galois extension.

In general, there is an open and closed decomposition of $\text{Spec}(R) \setminus V(J) = \bigsqcup_{i=1}^n U_i$ over which the degree of $\text{Spec}(S) \rightarrow \text{Spec}(R)$ is constant. For each i , let $(R_i)_{\text{perfd}}$ be the perfectoidization of the image R_i of R in $H^0(U_i, \mathcal{O})$. Then $R \rightarrow \prod_{i=1}^n (R_i)_{\text{perfd}}$ satisfies the hypothesis of Corollary 8.11 (any map $R \rightarrow V$ that does not kill J gives a map $\text{Spec}(V) \rightarrow \text{Spec}(R) \setminus V(J) = \bigsqcup_{i=1}^n U_i$, so factors over exactly one of the U_i , inducing a map $R_i \rightarrow V$, which then extends uniquely to the perfectoidization of R_i). Thus, we may replace R by $(R_i)_{\text{perfd}}$ for some i , and hence assume that $\text{Spec}(S) \rightarrow \text{Spec}(R)$ has constant degree outside $V(J)$. \square

⁸Even if J was principal to start with, say $J = (p)$ as in the usual almost purity theorem, this step may change the principal ideal, and requires general elements.

Theorem 10.11. *Let R be a perfectoid ring corresponding to a perfect prism (A, I) . Let $R \rightarrow S$ be the p -completion of an integral map. Then $\Delta_{S/A, \text{perf}}$ is discrete and a perfect p -complete δ - A -algebra. Consequently, S_{perfd} is discrete and a perfectoid ring.*

Proof. By passage to filtered colimits, we may assume $R \rightarrow S$ is finitely presented finite map⁹. By Corollary 8.13, it is enough to prove S_{perfd} is connective. Choose a sequence $g_1, \dots, g_n \in R$ as in Lemma 10.12. We shall prove the connectivity of S_{perfd} by induction on n . If $n = 1$, the conditions in Lemma 10.12 ensure that the map $R \rightarrow S_{\text{red}}$ is finite étale, and thus S_{red} is perfectoid. But $S \rightarrow S_{\text{red}}$ is an isomorphism on arc-sheafification, so $S_{\text{perfd}} \cong S_{\text{red, perfd}} \cong S_{\text{red}}$ is discrete by Corollary 8.10.

Now assume $n > 1$. The inductive hypothesis applied to $(R/g_1 R)_{\text{perfd}} \rightarrow S \otimes_R (R/g_1 R)_{\text{perfd}}$ and Proposition 8.12 ensure that $(S/g_1 S)_{\text{perfd}}$ is connective. The criterion in Lemma 10.5 reduces us to checking that S_{perfd} is g_1 -almost connective. The conditions in Lemma 10.12 imply the following:

- (1) Either $p = 0$ in $R[1/g_1]$ or $p \in R[1/g_1]^*$.
- (2) The map $R[1/g_1] \rightarrow S_{\text{red}}[1/g_1]$ factors as $R[1/g_1] \rightarrow T_1 \rightarrow S_{\text{red}}[1/g_1]$, where the first map is finite étale and the second map is a universal homeomorphism. If $p \in R[1/g_1]^*$, then the second map is actually an isomorphism.

We claim that $T_1 \rightarrow S_{\text{red}}[1/g_1]$ is an isomorphism in general. By the last sentence of (2) above, we may assume $p = 0$ in $R[1/g_1]$, so $R[1/g_1]$ is a perfect \mathbf{F}_p -algebra. But then T_1 is perfect as well, so $T_1 \rightarrow S_{\text{red}}[1/g_1]$ is a universal homeomorphism from a perfect ring into a reduced ring; any such map must be an isomorphism, so $T_1 \cong S_{\text{red}}[1/g_1]$. Summarizing, the map $R[1/g_1] \rightarrow S[1/g_1] \rightarrow S_{\text{red}}[1/g_1]$ is a finite étale cover.

If S' denotes the integral closure of R in $S_{\text{red}}[1/g_1]$, then the R -finiteness of S gives a map $S \rightarrow S'$ of integral R -algebras that is an isomorphism of arc-sheaves outside (g_1) . By Corollary 8.11, the map $S_{\text{perfd}} \rightarrow S'_{\text{perfd}}$ is a g_1 -almost isomorphism. As we only want to show g_1 -almost connectivity of S_{perfd} , it suffices to check the g_1 -almost connectivity of S'_{perfd} . But this follows from Theorem 10.9 by approximating S' by finitely presented finite R -algebras inducing the given finite étale cover $R[1/g_1] \rightarrow S_{\text{red}}[1/g_1]$ on inverting g_1 . \square

We needed the following lemma, stating roughly that any finitely presented finite map of qcqs schemes becomes finite étale over a constructible stratification of the target, at least up to universal homeomorphisms.

Lemma 10.12. *Let $R \rightarrow S$ be a finitely presented finite map of commutative $\mathbf{Z}_{(p)}$ -algebras. Then there exists a sequence of elements $g_1, \dots, g_n \in R$ such that, if we set $R_i = R/(g_1, \dots, g_{i-1})_{\text{red}}[1/g_i]$ and $S_i = S/(g_1, \dots, g_{i-1})_{\text{red}}[1/g_i]$ for $i = 1, \dots, n$, then the following hold true:*

- (1) *The ideal (g_1, \dots, g_n) is the unit ideal of R ; equivalently, $\cup_i \text{Spec}(R_i) = \text{Spec}(R)$.*
- (2) *The map $R_i \rightarrow S_i$ factors as $R_i \rightarrow T_i \rightarrow S_i$, where $R_i \rightarrow T_i$ is finite étale, and $T_i \rightarrow S_i$ is a universal homeomorphism. Moreover, the map $T_i[1/p] \rightarrow S_i[1/p]$ is an isomorphism.*

⁹Note that any such S is finitely presented as an R -module: any generator of S as an R -algebra satisfies a monic polynomial, so any finite presentation of S as an R -algebra immediately gives a finite presentation of S as an R -module. In particular, such an S is already p -complete.

(3) In each R_i , we either have $p = 0$ or $p \in R_i^*$.

Proof. It suffices to find $g_1, \dots, g_n \in R$ satisfying (1) and (2): we can then stratify further if necessary to also achieve (3); explicitly, replacing (g_1, \dots, g_n) with $(pg_1, g_1, pg_2, g_2, \dots, pg_n, g_n)$ does the trick. For (1) and (2), by noetherian approximation, we may assume R is a finitely presented $\mathbf{Z}_{(p)}$ -algebra. Say $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ are the minimal primes of R , and let $K = \prod_i K_i$ be the corresponding product of residue fields of R , so K is the total ring of fractions of R_{red} . As $R \rightarrow S$ is finite, the induced map $K \rightarrow (S \otimes_R K)_{\text{red}}$ is then the product of maps of the form $K_i \rightarrow L_i$, where each $L_i = (S \otimes_R K_i)_{\text{red}}$ is a finite reduced K_i -algebra. But then each L_i is a finite product of finite field extensions of K_i , so the map $K_i \rightarrow L_i$ factors as $K_i \rightarrow M_i \rightarrow L_i$, where the first map is finite étale, while the second map is a universal homeomorphism and an isomorphism if p is invertible in K_i . By an approximation argument, we can then find some $g_1 \in R$ invertible on K such that $R_{\text{red}}[1/g_1] \rightarrow S_{\text{red}}[1/g_1]$ has the form required in (2). This constructs the open stratum, and the rest follows by noetherian induction applied to the map $R/(g_1) \rightarrow S/(g_1)$. \square

Proof of Theorem 10.9. Combine the J -almost version proven above with Theorem 10.11. \square

Remark 10.13. Theorem 10.9 can be used to reprove some known results in commutative algebra relatively quickly. For example, the main theorems of [HM17] follow almost immediately. Let us explain the argument for [HM17, Theorem 1.3]. Fix a complete regular local ring R of mixed characteristic and a finite extension $R \rightarrow S$ contained in a fixed absolute integral closure $R \rightarrow R^+$. Fix some $n > 0$. To prove [HM17, Theorem 1.3], it is enough to show that there exists some $c \in R$ such that for all $i > 0$, the natural transformation

$$\text{Tor}_i^{R/p^n}(S/p^n, -) \rightarrow \text{Tor}_i^{R/p^n}(R^+/p^n, -)$$

is c -almost zero, i.e., the image is annihilated by all p -power roots $c^{1/p^n} \in R^+$ of c . We claim that any $c \in pR$ such that $R[1/c] \rightarrow S[1/c]$ is finite étale will do the job. To see this, choose a faithfully flat extension $R \rightarrow R_\infty$ contained inside the p -adic completion of R^+ with R_∞ perfectoid (see Remark 3.11). Set $T = S \otimes_R R_\infty$, so the finite map $R_\infty \rightarrow T$ is finite étale after inverting c . Theorem 10.9 implies that $R_\infty \rightarrow T_{\text{perfd}}$ is c -almost finite étale. In particular, as $R \rightarrow R_\infty$ is faithfully flat, the functor $\text{Tor}_i^{R/p^n}(T_{\text{perfd}}/p^n, -)$ is c -almost zero (i.e., killed by $(cT_{\text{perfd}})_{\text{perfd}}$) for $i > 0$; here we implicitly use that $c \in pR$ and that the p -torsion in T_{perfd} is p -almost zero (and thus c -almost zero). But the map $S/p^n \rightarrow R^+/p^n$ factors as $S/p^n \rightarrow T/p^n \rightarrow T_{\text{perfd}}/p^n \rightarrow R^+/p^n$, where the last map is obtained from the universal property of perfectoidizations as the p -adic completion of R^+ is perfectoid. This gives a factorization

$$\text{Tor}_i^{R/p^n}(S/p^n, -) \rightarrow \text{Tor}_i^{R/p^n}(T_{\text{perfd}}/p^n, -) \rightarrow \text{Tor}_i^{R/p^n}(R^+/p^n, -),$$

which implies the claim as the middle group is c -almost zero for $i > 0$.

11. THE ÉTALE COHOMOLOGICAL DIMENSION OF PERFECTOID RINGS

Fix a p -torsion free perfectoid ring R corresponding to a perfect prism (A, I) . We apply our general version of the almost purity theorem to prove the following theorem.

Theorem 11.1 (The étale cohomological dimension of algebraic étale sheaves). *The \mathbf{F}_p -étale cohomological dimension of $X = \mathrm{Spec}(R[1/p])$ is ≤ 1 , i.e., for every étale \mathbf{F}_p -sheaf F on X , we have $H^i(X, F) = 0$ for $i > 1$.*

We stress that X is the spectrum, not the adic spectrum, of $R[1/p]$; in particular, F is a filtered colimit of Zariski-constructible sheaves. It is easy to see that the statement is wrong for general sheaves on the adic spectrum of $R[1/p]$.

Proof. It suffices to prove the result when $F = j_! L$, where $j : U \hookrightarrow X$ is the inclusion of a quasicompact open subset of a constructible closed subset of X , and L is an \mathbf{F}_p -local system of constant rank on U . Using the method of the trace [Sta, Tag 03SH], we can find a finite étale cover $V \rightarrow U$ of degree prime-to- p such that $L|_V$ admits a finite filtration whose graded pieces are the constant sheaf \mathbf{F}_p . Let $Y^+ := \mathrm{Spec}(S) \rightarrow X^+ := \mathrm{Spec}(R)$ be a finitely presented finite morphism lifting $V \rightarrow U$. Write $Y = Y^+[1/p]$ and let $j : V \rightarrow Y$ be the resulting open immersion. This data is summarized in the diagram

$$\begin{array}{ccccc} V & \xrightarrow{j} & Y & \longrightarrow & Y^+ \\ \downarrow & & \downarrow & & \downarrow \\ U & \xrightarrow{j} & X & \longrightarrow & X^+ \end{array}$$

where all squares are cartesian. A standard devissage argument now reduces us to showing that $R\Gamma(Y, j_! \mathbf{F}_p) \in D^{\leq 1}$. Let $J \subset S$ be a finitely generated ideal cutting out the closed subset $Y^+ - V \subset Y^+$. By the étale comparison theorem for $\mathrm{Spec}(S)$ and $\mathrm{Spec}(S/J)$, the complex $R\Gamma(Y, j_! \mathbf{F}_p)$ is computed by applying $(-/p[\frac{1}{I}])^{\phi=1}$ to the fibre of

$$\mathbb{A}_{S/A, \mathrm{perf}} \rightarrow \mathbb{A}_{(S/J)/A, \mathrm{perf}},$$

so it is enough to show this fibre is connective. This can be checked after base change along $A \rightarrow A/I$, so we are reduced to checking that the fibre of

$$S_{\mathrm{perfd}} \rightarrow (S/J)_{\mathrm{perfd}}$$

is connective. By Theorem 10.11, both objects above are discrete perfectoid rings, so we must show that the map is surjective. The universal property of perfectoidization shows that $(S/J)_{\mathrm{perfd}} \simeq (S_{\mathrm{perfd}}/JS_{\mathrm{perfd}})_{\mathrm{perfd}}$, so the claim follows from Theorem 7.4. \square

12. THE NYGAARD FILTRATION FOR QUASIREGULAR SEMIPERFECTOID RINGS

In this section, we define and analyze the Nygaard filtration on Δ_S when S is quasiregular semiperfectoid, and use it to endow prismatic cohomology with a Nygaard filtration which will in particular prove the de Rham comparison. The main result and its proof strategy is explained in §12.1; the proof, which is completed in §12.5, depends crucially on an analysis of the Nygaard filtration for a specific quasiregular semiperfectoid ring explained in §12.2.

12.1. Structure of the Nygaard filtration for quasiregular semiperfectoid rings. The Nygaard filtration has a rather direct definition.

Definition 12.1. Let S be a quasiregular semiperfectoid ring with associated prism $(\Delta_S, (d))$. The Nygaard filtration on Δ_S is given by

$$\mathrm{Fil}_N^i \Delta_S = \{x \in \Delta_S \mid \phi(x) \in d^i \Delta_S\} .$$

It is an \mathbf{N} -indexed decreasing multiplicative filtration.

We warn the reader that in general the Nygaard filtration is not separated. The primary goal of this section is to prove the following theorem. Fix any perfectoid ring R (corresponding to a perfect prism $(A, (d))$) mapping to S . Then recall that by the Hodge-Tate comparison, $\overline{\Delta}_S = \Delta_S/d = \Delta_{S/R}/d$ is an S -algebra equipped with the conjugate filtration

$$\mathrm{Fil}_i \overline{\Delta}_S \subset \overline{\Delta}_S$$

which is an \mathbf{N} -indexed increasing exhaustive multiplicative filtration with

$$\mathrm{gr}_i \overline{\Delta}_S = (\wedge^i L_{S/R}[-i])^\wedge .$$

The conjugate filtration depends on the choice of R .

Theorem 12.2. *The image of*

$$\mathrm{Fil}_N^i \Delta_S \xrightarrow{\frac{\phi}{d^i}} \Delta_S \rightarrow \overline{\Delta}_S$$

agrees with $\mathrm{Fil}_i \overline{\Delta}_S$. In particular, there is a natural isomorphism

$$\mathrm{gr}_N^i \Delta_S \cong \mathrm{Fil}_i \overline{\Delta}_S \{i\} .$$

The proof of this theorem in the general case is rather indirect. We will proceed in the following steps:

- (1) The case $S = R\langle X^{1/p^\infty} \rangle / X$ (§12.2).
- (2) The case $S = R\langle X_1^{1/p^\infty}, \dots, X_n^{1/p^\infty} \rangle / (f_1, \dots, f_m)$ for some p -completely regular sequence f_1, \dots, f_m (§12.3).
- (3) Define a Nygaard filtration on $\Delta_{B/A}$ for any p -completely smooth R -algebra B by quasisyntomic descent (§12.4).
- (4) Define a derived Nygaard filtration on $\Delta_{S/A}$ via left Kan extension from smooth algebras, show that it agrees with the Nygaard filtration and finish the proof (§12.5).

12.2. The Nygaard filtration in a special case. In this section, we consider the semiperfectoid ring

$$S = (\mathbf{Z}_p[\zeta_{p^\infty}, X^{1/p^\infty}]/X)^\wedge.$$

We consider it as an algebra over the perfectoid ring $R = \mathbf{Z}_p[\zeta_{p^\infty}]^\wedge$. The corresponding perfect prism is given by the pair (A, I) , where A is the $(p, q-1)$ -adic completion of $\mathbf{Z}_p[q^{1/p^\infty}]$ and $I = [p]_q = 1 + q + \dots + q^{p-1}$; here $q^{1/p^n} \in A$ maps to $\zeta_{p^{n+1}} \in R = A/I$. From the definition of $\Delta_S = \Delta_S^{\text{init}}$ as a prismatic envelope it follows that

$$\Delta_S = A\langle Y^{1/p^\infty} \rangle \left\{ \frac{Y^p}{[p]_q} \right\}^\wedge,$$

where we normalize the coordinates so that the map $S \rightarrow \overline{\Delta}_S$ takes X^{1/p^n} to $Y^{1/p^{n-1}}$. (These normalizations will make the following formulas appear nicer.) The δ -structure on Δ_S is determined by $\delta(q) = \delta(Y) = 0$.

It turns out that in this situation, there is an explicit description of Δ_S as a q -divided power algebra. Recall that for any integer $n \geq 0$, one sets

$$[n]_q = \frac{q^n - 1}{q - 1} = 1 + q + \dots + q^{n-1}, \quad [n]_q! = \prod_{i=1}^n [i]_q.$$

Lemma 12.3. *We have $\frac{Y^n}{[n]_q!} \in \Delta_S$ for all integers $n \geq 0$. The resulting A -module map*

$$\bigoplus_{i \in \mathbf{N}[1/p]} A \cdot \frac{Y^i}{[[i]]_q!} \rightarrow \Delta_S \tag{5}$$

identifies Δ_S as the $(p, q-1)$ -adic completion of the left side.

Proof. Lemma 12.4 below implies that $\frac{Y^n}{[n]_q!} \in \Delta_S$ for all $n \geq 0$. To check that the map is an isomorphism after completion, we may take the base change along the map $A \rightarrow A/(q-1)$ of δ -rings; after this base change, $\Delta_S = A\langle Y^{1/p^\infty} \rangle \left\{ \frac{Y^p}{[p]_q} \right\}$ becomes

$$A/(q-1)\langle Y^{1/p^\infty} \rangle \left\{ \frac{Y^p}{p} \right\}$$

which by Corollary 2.38 agrees with the divided power envelope of (Y) in $A/(q-1)\langle Y^{1/p^\infty} \rangle$ which indeed has the desired description. \square

The following lemma will be reused below, so it is formulated in larger generality than necessary right now.

Lemma 12.4. *Let D be a $(p, q-1)$ -completely flat $\mathbf{Z}_p[[q-1]]$ -algebra. Assume we are given a map $\phi : D \rightarrow D$ and an element $x \in D$ such that the following hold true:*

(1) *We have $\phi(q) = q^p$ and $\phi(x) = x^p$.*

(2) *We have $\phi(x) \in [p]_q D$.*

Then $x^n \in [n]_q! D$ for all $n \geq 0$.

Proof. As $[i]_q$ is invertible for i coprime to p , it suffices to prove the following statement:

(*) Given $m \geq 0$, if $x^m \in [m]_q!D$, then $x^{mp} \in [mp]_q!D$.

Using Lemma 12.5 (1), it suffices to show that $\phi([m]_q!) \cdot [p]_q^m \mid x^{mp}$ under the above assumption on x . Now $\phi([m]_q!)$ is a nonzerodivisor modulo $[p]_q$: this holds true in $\mathbf{Z}_p[[q-1]]$ as the polynomial $\phi([m]_q!) = \prod_{i=1}^m \frac{q^{ip}-1}{q^p-1}$ does not vanish at the primitive p -th roots of unity in $\overline{\mathbf{Q}_p}$, and thus also in D by flatness. Thus, (*) reduces to showing

(*)' Given $m \geq 0$, if $x^m \in [m]_q!D$, then x^{mp} is divisible by both $[p]_q^m$ and $\phi([m]_q!)$.

Now $x^m \in [m]_q!D$ implies that $x^{mp} = \phi(x^m) \in \phi([m]_q!)D$, which proves half of (*'). For the other half, we simply observe that $x^{mp} = (x^p)^m \in [p]_q^m D$ by our assumption $x^p \in [p]_q D$. \square

Next, we record two multiplicative identities in $\mathbf{Z}_p[[q-1]]$, one of which was already used above.

Lemma 12.5. *In the ring $\mathbf{Z}_p[[q-1]]$, we have the following multiplicative identities:*

- (1) *For any $m \in \mathbf{N}$, we have $[mp]_q! = u \cdot \phi([m]_q!) \cdot [p]_q^m$ with u a unit.*
- (2) *For any $i \in \mathbf{N}[1/p]$, we have $[i]_p! = [[ip]]_q! \cdot v$ with v a unit.*

Proof. For (1), note that, up to multiplication by units, $[mp]_q!$ equals $\prod_{i=1}^m [ip]_q$: for any integer $k \geq 0$, the polynomial $[k]_q$ is invertible if k is not divisible by p . Moreover, we also have an equality

$$\prod_{i=1}^m [ip]_q! = \prod_{i=1}^m \frac{q^{ip}-1}{q-1} = \prod_{i=1}^m \left(\frac{q^{ip}-1}{q^p-1} \cdot \frac{q^p-1}{q-1} \right) = \phi([m]_q!) \cdot [p]_q^m.$$

The desired identity now easily follows.

For (2), write $i = [i] + \epsilon$ for $0 \leq \epsilon < 1$ in $\mathbf{N}[1/p]$. Then $[ip] = [i]p + [\epsilon p]$. As $0 \leq \epsilon < 1$ in $\mathbf{N}[1/p]$, we have $0 \leq [\epsilon p] < p$ in \mathbf{N} . But then any integer k with $[i]p < k \leq [ip]$ is coprime to p , so the corresponding polynomial $[k]_q$ is invertible; this easily implies the claim. \square

Now we can describe the Nygaard filtration explicitly.

Lemma 12.6. *Theorem 12.2 holds true for S . More precisely:*

- (1) *The Nygaard filtration $\mathrm{Fil}_N^n \Delta_S$ identifies with the $(p, q-1)$ -adic completion of the A -submodule*

$$\bigoplus_{i \in \mathbf{N}[1/p]} [p]_{q^{1/p}}^{n-[i]} \cdot A \cdot \frac{Y^i}{[[i]]_q!} \subset \bigoplus_{i \in \mathbf{N}[1/p]} A \cdot \frac{Y^i}{[[i]]_q!},$$

under the isomorphism in (5), and we follow the convention that $[p]_{q^{1/p}}^{n-[i]} = 1$ for non-positive exponents (i.e., when $i \geq n+1$).

- (2) *The image of*

$$\frac{\phi}{[p]_q^n} : \mathrm{Fil}_N^n \Delta_S \rightarrow \overline{\Delta}_S = \Delta_S/[p]_q = \left(\bigoplus_{i \in \mathbf{N}[1/p]} \mathbf{Z}_p[\zeta_{p^\infty}] \cdot \frac{Y^i}{[[i]]_q!} \right)^\wedge$$

is given by the summands with $i < p(n+1)$.

- (3) *The conjugate filtration $\mathrm{Fil}_n \overline{\Delta}_S \subset \overline{\Delta}_S$ is also given by the summands with $i < p(n+1)$.*

Proof. For (1), we note that ϕ is a graded map in the above $\mathbf{N}[1/p]$ -grading, multiplying the grading by p . In particular, it follows that $\mathrm{Fil}_N^n \Delta_S$ is also graded. Using both parts of Lemma 12.5, we can write

$$\phi\left(\frac{Y^i}{[[i]]_q!}\right) = \frac{Y^{ip}}{\phi([i]_q!)} = \frac{Y^{ip}}{[[ip]]_q!} \cdot [p]_q^{[i]} \cdot u \quad (6)$$

for a unit u . It follows from this formula that $[p]_{q^{1/p}}^{n-[i]} \cdot \frac{Y^i}{[[i]]_q!} \in \mathrm{Fil}_N^n \Delta_S$ for all i (under the convention that $[p]_{q^{1/p}}^{n-[i]} = 1$ for non-positive exponents). The same formula (and the fact that ϕ is bijective on A) also shows that no smaller multiple of $\frac{Y^i}{[[i]]_q!}$ can lie in $\mathrm{Fil}_N^n \Delta_S$, giving (1).

For part (2), we use the above formula to see the following:

- For $i < n+1$, the map ϕ maps $(\mathrm{Fil}_N^n \Delta_S)_{deg=i}$ isomorphically onto $[p]_q^n \cdot (\Delta_S)_{deg=ip}$.
- For $i \geq n+1$, the map ϕ maps $(\mathrm{Fil}_N^n \Delta_S)_{deg=i}$ into $[p]_q^{n+1} \cdot (\Delta_S)_{deg=ip}$.

This immediately gives (2).

For part (3), recall that the natural map

$$S := (\mathbf{Z}_p[\zeta_{p^\infty}, X^{1/p^\infty}]/X)^\wedge \rightarrow \overline{\Delta}_S$$

carries X^{1/p^n} to $Y^{1/p^{n-1}}$. As $[p]_q \mid Y^i$ in $\overline{\Delta}_S$ for $i \geq p$, it follows that the image of the above map is given by the summands with $i < p$, giving the claim in part (3) for $n = 0$. For $n = 1$, we use the end of Example 7.9 to see that $\mathrm{gr}^1 \overline{\Delta}_S$ is generated by $\frac{Y^p}{[p]_q}$. The higher graded pieces are then generated by the divided powers of $\frac{Y^p}{[p]_q}$ by the multiplicativity of the conjugate filtration, which by Lemma 12.5 (1) indeed agree up to units with $\frac{Y^{pi}}{[pi]_q!}$. This gives the desired comparison. \square

12.3. The Nygaard filtration for finitely presented semiperfectoid rings. In this subsection we prove the following result.

Proposition 12.7. *Theorem 12.2 holds true for $S = R\langle X_1^{1/p^\infty}, \dots, X_n^{1/p^\infty} \rangle / (f_1, \dots, f_m)$ where f_1, \dots, f_m is a p -completely regular sequence.*

Proof. We may replace R by $R\langle X_1^{1/p^\infty}, \dots, X_n^{1/p^\infty} \rangle$ and assume that $n = 0$. By André's lemma, Theorem 7.12, we may assume that all f_i admit a compatible sequence of p -power roots. We note that to prove the proposition, it suffices to exhibit some (a priori possibly different) filtration $\mathrm{Fil}_M^i \Delta_S \subset \Delta_S$ such that ϕ is divisible by d^i on $\mathrm{Fil}_M^i \Delta_S$ and the induced map

$$\mathrm{gr}_M^i \Delta_S \xrightarrow{\frac{\phi}{d^i}} \overline{\Delta}_S$$

is injective with image $\mathrm{Fil}_i \overline{\Delta}_S$. Indeed, these conditions imply by induction on i that $\mathrm{Fil}_M^i \Delta_S = \mathrm{Fil}_N^i \Delta_S$, which then has the desired property. In particular, we can reduce to

$$S' = R\langle X_1^{1/p^\infty}, \dots, X_m^{1/p^\infty} \rangle / (X_1, \dots, X_m)$$

as there is a map $R' = R\langle X_1^{1/p^\infty}, \dots, X_m^{1/p^\infty} \rangle \rightarrow R$ (given by the f_i and their roots) such that $S = S' \widehat{\otimes}_{R'}^L R$ and thus

$$\Delta_S = \Delta_{S'} \widehat{\otimes}_{\Delta_{R'}}^L \Delta_R$$

and we can set

$$\mathrm{Fil}_M^i \Delta_S = \mathrm{Fil}_N^i \Delta_{S'} \widehat{\otimes}_{\Delta_{R'}}^L \Delta_R.$$

Moreover, by the Künneth formula, we can reduce to the case $m = 1$, and we can also assume that $R = \mathbf{Z}_p[\zeta_{p^\infty}]^\wedge$ (by André's lemma and base change). Thus, the proposition follows from Lemma 12.6. \square

12.4. The Nygaard filtration on prismatic cohomology. Fix a perfectoid ring R corresponding to a perfect prism (A, I) . We want to endow $\Delta_{B/A}$ with a functorial Nygaard filtration for any p -completely smooth $R = A/I$ -algebra B . Although we will later be able to give a better definition, we use the following direct recipe.

For any surjection $R\langle X_1, \dots, X_n \rangle \rightarrow B$, the ring

$$\tilde{B} = R\langle X_1^{1/p^\infty}, \dots, X_n^{1/p^\infty} \rangle \otimes_{R\langle X_1, \dots, X_n \rangle} B$$

is semiperfectoid and Zariski locally of the form considered in the previous subsection, so by localization Theorem 12.2 holds true for \tilde{B} . The same applies to all terms of the Čech nerve \tilde{B}^\bullet of $B \rightarrow \tilde{B}$. The cosimplicial δ -ring

$$\Delta_{\tilde{B}^\bullet}$$

computes $\Delta_{B/A}$ (for example, by the Hodge-Tate comparison and quasisyntomic descent).

Definition 12.8. The Nygaard filtration

$$\mathrm{Fil}_N^i \Delta_{B/A} \rightarrow \Delta_{B/A}$$

is the totalization of $\mathrm{Fil}_N^i \Delta_{\tilde{B}^\bullet} \subset \Delta_{\tilde{B}^\bullet}$.

Proposition 12.9. *There is a natural map*

$$“\frac{\phi}{d^i}” : \mathrm{Fil}_N^i \Delta_{B/A} \rightarrow \Delta_{B/A}$$

that projects to an isomorphism of $\mathrm{gr}_N^i \Delta_{B/A} \cong \tau^{\leq i} \Delta_{B/A}$.

Proof. This follows from Proposition 12.7 by passing to totalizations of the cosimplicial objects. \square

It follows that the Nygaard filtration on $\Delta_{B/A}$ is independent of the choice of the surjection $R\langle X_1, \dots, X_n \rangle \rightarrow B$: adding extra variables to the X_i , one gets a comparison map between the two induced Nygaard filtrations, and it induces isomorphisms on $\mathrm{gr}_N^i \Delta_{B/A}$, so by descending induction on i on all $\mathrm{Fil}_N^i \Delta_{B/A}$.

12.5. End of proof. Finally, we can finish the proof of Theorem 12.2. First, by left Kan extension of $B \mapsto \mathrm{Fil}_N^i \Delta_{B/A}$ we can define a “derived Nygaard filtration” $\mathrm{Fil}_{N'}^i \Delta_{B/A}$ on $\Delta_{B/A}$ for any derived p -complete simplicial R -algebra, with $\mathrm{gr}_{N'}^i \Delta_{B/A} \cong \mathrm{Fil}_i \overline{\Delta}_{B/A}$. In case $B = S$ is quasisregular semiperfectoid, these properties imply that $\mathrm{Fil}_{N'}^i \Delta_{B/A}$ is concentrated in cohomological degrees 0 and -1 for all $i \geq 0$.

Proposition 12.10. *For all $i \geq 0$, the complex $\mathrm{Fil}_{N'}^i \Delta_S$ sits in degree 0 and defines a filtration of Δ_S that agrees with $\mathrm{Fil}_N^i \Delta_S$, and Theorem 12.2 holds true for S .*

Proof. We argue by induction on i , so assume that $\mathrm{Fil}_{N'}^i = \mathrm{Fil}_N^i$ holds true for some i . This is clearly true for $i = 0$, giving the inductive start. First, we check that then Theorem 12.2 holds true in degree i , i.e. the image of

$$\frac{\phi}{d^i} : \mathrm{Fil}_N^i \Delta_S \rightarrow \overline{\Delta}_S$$

is given by $\mathrm{Fil}_i \overline{\Delta}_S$. Using $\mathrm{Fil}_N^i = \mathrm{Fil}_{N'}^i$, one sees that the image is at most $\mathrm{Fil}_i \overline{\Delta}_S$. On the other hand, by picking a surjection $S' = R\langle X_i^{1/p^\infty}, Y_j^{1/p^\infty} \rangle / (Y_j) \rightarrow S$ that also induces a surjection on cotangent complexes and using Proposition 12.7 (and passage to filtered colimits), one sees that the image is at least $\mathrm{Fil}_i \overline{\Delta}_S$. Thus, Theorem 12.2 holds true in degree i .

We also see that $\mathrm{Fil}_{N'}^i \Delta_S \rightarrow \mathrm{gr}_{N'}^i \Delta_S$ is surjective, and hence its derived kernel $\mathrm{Fil}_{N'}^{i+1} \Delta_S$ is still concentrated in degree 0. Moreover, it agrees with the kernel of

$$\frac{\phi}{d^i} : \mathrm{Fil}_N^i \Delta_S \rightarrow \overline{\Delta}_S$$

which by definition is $\mathrm{Fil}_N^{i+1} \Delta_S$. Thus, $\mathrm{Fil}_{N'}^{i+1} \Delta_S = \mathrm{Fil}_N^{i+1} \Delta_S$, as desired. \square

13. COMPARISON WITH [BMS19]

The goal of this section is to compare the constructions of this paper with those in [BMS19]. Recall that for a quasiregular semiperfectoid S , we defined in [BMS19] the ring $\widehat{\Delta}_S = \pi_0 \mathrm{TC}^-(S; \mathbf{Z}_p) = \pi_0 \mathrm{TP}(S; \mathbf{Z}_p)$ together with a Frobenius semilinear endomorphism ϕ_S (induced by the cyclotomic Frobenius map) and a commutative diagram

$$\begin{array}{ccc} \widehat{\Delta}_S & \xrightarrow{\phi_S} & \widehat{\Delta}_S \\ \downarrow a_R & & \downarrow \mathrm{can} \\ S & \xrightarrow{\eta_S} & \widehat{\Delta}_S/d \end{array} \quad (7)$$

of commutative algebras. By construction, $\widehat{\Delta}_S$ is complete for a filtration $\mathrm{Fil}_N^i \widehat{\Delta}_S \subset \widehat{\Delta}_S$ also termed the Nygaard filtration in [BMS19]. Our comparison theorem is the following:

Theorem 13.1. *There is a functorial (in S) δ -ring structure on $\widehat{\Delta}_S$ refining the endomorphism ϕ . The induced map $\Delta_S = \Delta_S^{\mathrm{init}} \rightarrow \widehat{\Delta}_S$ identifies $\widehat{\Delta}_S$ with the Nygaard completion of Δ_S , compatibly with Nygaard filtrations.*

Note that the δ -ring structure on $\widehat{\Delta}_S$ is uniquely determined when $\widehat{\Delta}_S$ is p -torsionfree, which happens for example if S is p -torsionfree. In that case, the first part of the theorem simply says that the endomorphism ϕ on $\widehat{\Delta}_S$ lifts the Frobenius. Our proof of this fact is very indirect and we do not know of a good conceptual reason for this in terms of topological Hochschild homology.

Proof. First we check the claim, independent of the theory of [BMS19], that the δ -ring structure on Δ_S extends uniquely to a continuous δ -ring structure on its Nygaard completion. This follows from the following lemma.

Lemma 13.2. *Let S be any quasiregular semiperfectoid ring.*

(1) *One has*

$$\delta(\mathrm{Fil}_N^i \Delta_S) \subset \mathrm{Fil}_N^{pi} \Delta_S + (d, p)^{i-1} \Delta_S .$$

(2) *The completion of Δ_S with respect to the sequence of ideals $\mathrm{Fil}_N^i \Delta_S$ agrees with its completion with respect to the sequence of ideals $\mathrm{Fil}_N^i \Delta_S + (d, p)^j \Delta_S$.*

Proof. In part (1), pick a perfectoid ring R mapping to S . There is a surjection $S' = R\langle X_i^{1/p^\infty}, Y_j^{1/p^\infty} \rangle / (X_i) \rightarrow S$ inducing a surjection on cotangent complexes and thus on Δ_S and all $\mathrm{Fil}_N^i \Delta_S$ by the explicit description. Thus we can assume $S = S'$. Replacing R by $R\langle Y_j^{1/p^\infty} \rangle$ we can assume there are no Y_j 's. By a filtered colimit argument one can reduce to the case that there are only finitely many X_i . Next, we want to use the Künneth formula to reduce to the case of a single X . For this, we need to remark that it is enough to check the claim on a set of generators for the ideal $\mathrm{Fil}_N^i \Delta_S$ by the addition formula for δ , and that if $i = j + k$ and $x \in \mathrm{Fil}_N^j \Delta_S$, $y \in \mathrm{Fil}_N^k \Delta_S$ satisfy

$$\delta(x) = x_1 + x_2 \in \mathrm{Fil}_N^{pj} \Delta_S + (d, p)^{j-1} \Delta_S , \quad \delta(y) = y_1 + y_2 \in \mathrm{Fil}_N^{pk} \Delta_S + (d, p)^{k-1} \Delta_S ,$$

then

$$\delta(xy) = x^p \delta(y) + y^p \delta(x) + p \delta(x) \delta(y) = x^p y_1 + x^p y_2 + y^p x_1 + y^p x_2 + p x_1 y_1 + p x_1 y_2 + p x_2 y_1 + p x_2 y_2 ,$$

where

$$\begin{aligned} p x_2 y_2 &\in (d, p)^{i-1} \Delta_S , \\ x^p y_1, y^p x_1, p x_1 y_1 &\in \mathrm{Fil}_N^{pi} \Delta_S , \\ (x^p + p x_1) y_2 &= (\phi(x) - p x_2) y_2 \in (d, p)^{i-1} \Delta_S , \\ (y^p + p y_1) x_2 &= (\phi(y) - p y_2) x_2 \in (d, p)^{i-1} \Delta_S , \end{aligned}$$

so indeed $\delta(xy) \in \mathrm{Fil}_N^{pi} \Delta_S + (d, p)^{i-1} \Delta_S$. Thus, finally, we can reduce to the case of one variable. Using Lemma 12.6 and the previous considerations, it is enough to check the claim for $x = [p]_{q^{1/p}} \in \mathrm{Fil}_N^1 \Delta_S$ and $x = \frac{Y^i}{[i]_q!} \in \mathrm{Fil}_N^i \Delta_S$ (as these generate the Nygaard filtration multiplicatively). In the first case $i = 1$ and the claim is trivial as $(d, p)^{i-1} \Delta_S = \Delta_S$. In the other case $\delta(x)$ is homogeneous of degree pi , and all such elements lie in $\mathrm{Fil}_N^{pi} \Delta_S$, as desired.

For part (2), it is enough to show that all $\mathrm{gr}_N^i \Delta_S$ are classically (d, p) -adically complete. Under the ϕ -linear identification with $\mathrm{Fil}_i \overline{\Delta}_S$, it is enough to prove that the latter are classically $(\phi(d), p)$ -adically complete. But $\phi(d)$ agrees up to units with p modulo d , so it suffices to prove that the latter are classically p -adically complete. This follows from them being derived p -complete and p -completely flat over S . \square

Now we start the proof of the theorem. Fix a perfectoid ring R corresponding to a perfect prism $(A, (d))$, and restrict to R -algebras S . By André's lemma, we may assume that there is a compatible systems of p -power roots of unity in R , and so we can take $R = \mathbf{Z}_p[\zeta_{p^\infty}]^\wedge$. By descent to p -completely smooth R -algebras and left Kan extension, we defined in [BMS19, Construction 7.12] a functor $S \mapsto \widehat{\Delta}_{S/A}^{\mathrm{nc}}$ (there simply denoted by $\Delta_{S/A}$ as will be justified a posteriori by the theorem; here nc stands for "non-completed"). This can be endowed with a Nygaard filtration $\mathrm{Fil}_N^i \widehat{\Delta}_{S/A}^{\mathrm{nc}}$ by the same procedure, and then $\widehat{\Delta}_{S/A}$ is the Nygaard completion for quasisyntomic R -algebras S , in particular if S is quasiregular semiperfectoid. By the Segal conjecture for smooth algebras, [BMS19, Corollary 9.12], one has a Hodge-Tate comparison for $\widehat{\Delta}_{S/A}/d$ in the smooth case, and thus also in general. Moreover, the same result and the identification of $\mathrm{gr}_N^i \widehat{\Delta}_{S/A}^{\mathrm{nc}}$ with $\mathrm{gr}^i \mathrm{THH}(S; \mathbf{Z}_p)$ implies that for quasiregular semiperfectoid S the Nygaard filtration is given by

$$\mathrm{Fil}_N^i \widehat{\Delta}_{S/A}^{\mathrm{nc}} = \{x \in \widehat{\Delta}_{S/A}^{\mathrm{nc}} \mid \phi(x) \in d^i \widehat{\Delta}_{S/A}^{\mathrm{nc}}\}$$

and the image of

$$\frac{\phi}{d^i} : \mathrm{Fil}_N^i \widehat{\Delta}_{S/A}^{\mathrm{nc}} \rightarrow \widehat{\Delta}_{S/A}^{\mathrm{nc}}/d$$

is given by the conjugate filtration $\mathrm{Fil}_i \widehat{\Delta}_{S/A}^{\mathrm{nc}}/d$ coming from the Hodge-Tate comparison. Moreover, the graded pieces $\mathrm{gr}_i \widehat{\Delta}_{S/A}^{\mathrm{nc}}/d$ are given by the p -completion of $\wedge^i L_{S/R}[-i]$. In other words, structurally $\widehat{\Delta}_{S/A}^{\mathrm{nc}}$ has exactly the same properties as Δ_S .

Our task now is to show that $\widehat{\Delta}_{S/A}^{\text{nc}} = \Delta_S$ compatibly with ϕ , as $\widehat{\Delta}_{S/A}$ is the Nygaard completion of the left, which then inherits its functorial δ -ring structure by the lemma. As both sides are defined via left Kan extensions from p -completely smooth algebras, and in the p -completely smooth case via descent, it suffices to prove the result for the quasiregular semiperfectoid algebras S required in this descent. In other words, as in §12.4, we can assume $S = R\langle X_1^{1/p^\infty}, \dots, X_n^{1/p^\infty} \rangle / (f_1, \dots, f_m)$ where f_1, \dots, f_m is a p -completely regular sequence relative to R . In particular, in this case $\widehat{\Delta}_{S/A}^{\text{nc}}$ is p -torsion free, so the δ -ring structure is unique if it exists, and then the map $\Delta_S \rightarrow \widehat{\Delta}_{S/A}^{\text{nc}}$ is unique. Following the steps in the proof Proposition 12.7, we can assume that all f_i admit compatible p -power roots in R , and then to $S = R\langle X^{1/p^\infty} \rangle / (X)$ by Künneth and base change.

In this case, we can apply Lemma 12.4 to $D = \widehat{\Delta}_{S/A}^{\text{nc}}$. Using the explicit description of Δ_S , this gives a ϕ -equivariant map

$$\Delta_S \rightarrow \widehat{\Delta}_{S/A}^{\text{nc}}$$

in this case. This sits in a commutative diagram

$$\begin{array}{ccc} \Delta_S & \longrightarrow & \widehat{\Delta}_{S/A}^{\text{nc}} \\ \downarrow & & \downarrow \\ \Delta_{S/p} & \xrightarrow{\cong} & \widehat{\Delta}_{(S/p)/A}^{\text{nc}}, \end{array}$$

where the lower isomorphism comes from [BMS19, Theorem 8.17] (plus descent and left Kan extension to pass to the non-completed version of $\widehat{\Delta}_{(S/p)/\mathbf{F}_p}^{\text{nc}}$, and independence of the perfectoid base). Moreover the upper map is the derived p -completed base extension of $\Delta_S \rightarrow \widehat{\Delta}_{S/A}^{\text{nc}}$ along $A = A_{\text{inf}}(R) \rightarrow A_{\text{crys}}(R/p)$ (as both Δ and $\widehat{\Delta}^{\text{nc}}$ satisfy base change and using [BMS19, Theorem 8.17] again). By derived Nakayama, it suffices to see that $\Delta_S \rightarrow \widehat{\Delta}_{S/A}^{\text{nc}}$ is an isomorphism after base change along $A_{\text{inf}}(R) \rightarrow A_{\text{inf}}(R)/(p, d)$, but this map factors over $A_{\text{crys}}(R/p)$, so the result follows. \square

14. p -ADIC TATE TWISTS AND THE ODD VANISHING CONJECTURE FOR K -THEORY

In [BMS19], for each $n \geq 0$, we defined a sheaf $\mathbf{Z}_p(n)$ of complexes on the category of p -complete quasisyntomic rings. It was given by the formula

$$\mathbf{Z}_p(n)(S) := \mathrm{fib}(\mathcal{N}^{\geq n} \widehat{\Delta}_S\{n\} \xrightarrow{1-\phi_n} \widehat{\Delta}_S\{n\}).$$

Here $\widehat{\Delta}_S$ is a version of the Nygaard completed prismatic cohomology of S defined using topological Hochschild homology and quasisyntomic descent in [BMS19]; when S lives over a perfectoid ring, this theory coincides with the Nygaard completion of prismatic cohomology as defined in this paper, with $\mathcal{N}^{\geq \cdot} \widehat{\Delta}_S$ being the Nygaard filtration (Theorem 13.1). Our goal in this section is to prove the following structural property of this sheaf, conjectured in [BMS19, Conjecture 7.18]:

Theorem 14.1. *For each $n \geq 0$, the sheaf $\mathbf{Z}_p(n)$ is discrete and p -torsionfree.*

It was shown in [BMS19, Theorem 1.12 (5)] that (p -complete) topological cyclic homology of quasisyntomic rings admits a complete descending \mathbf{N} -indexed filtration with graded pieces given by the sheaves $\mathbf{Z}_p(n)$'s. The paper [CMM18] identifies K -theory with topological cyclic homology for a large class of rings. Theorem 14.1 then has the following consequence:

Corollary 14.2. *Locally on the quasisyntomic site, the functor $K(-; \mathbf{Z}_p)$ is concentrated in even degrees, i.e., $\pi_n K(-; \mathbf{Z}_p)$ vanishes for n odd after quasisyntomic sheafification.*

Another corollary of the method of the proof is the following, showing a slightly more precise form of Corollary 14.2 in a certain situation; notably, this shows that $\pi_* K(\mathcal{O}_C/p^n; \mathbf{Z}_p)$ is concentrated in even degrees for $n \geq 0$, where C/\mathbf{Q}_p is a complete and algebraically closed extension. We thank Martin Speirs for raising this question.

Corollary 14.3. *Say R is a perfectoid $\mathbf{Z}_p[\zeta_{p^\infty}]^\wedge$ -algebra. Fix a regular sequence $f_1, \dots, f_r \in R$ such that each f_i admits a compatible system of p -power roots; let $S = R/(f_1, \dots, f_r)$. The map $\pi_* K(R; \mathbf{Z}_p) \rightarrow \pi_* K(S; \mathbf{Z}_p)$ is surjective in odd degrees. In particular, if $\pi_* K(R; \mathbf{Z}_p)$ is concentrated in even degrees, the same holds true for $\pi_* K(S; \mathbf{Z}_p)$.*

The proof of Theorem 14.1 depends on two inputs. First, we use André's lemma (Theorem 7.12) to restrict attention to a particular nice class of quasisyntomic rings. Secondly, we use the explicit description of prismatic cohomology and its Nygaard filtration for this class of rings (coming from §12.2) to make calculations. Let us begin with the latter.

Lemma 14.4. *Let R be a perfectoid $\mathbf{Z}_p[\zeta_{p^\infty}]^\wedge$ -algebra. Set $R' = R\langle X_1^{1/p^\infty}, \dots, X_r^{1/p^\infty} \rangle$ and $S = R'/(X_1, \dots, X_r)$. The natural map $\mathbf{Z}_p(n)(R') \rightarrow \mathbf{Z}_p(n)(S)$ is surjective on H^1 .*

Proof. Let $A = A_{\mathrm{inf}}(R)$, so $(A, [p]_q)$ is the perfect prism corresponding to R , and we have

$$A_{\mathrm{inf}}(R') = \widehat{\bigoplus_{i \in \mathbf{N}[1/p]^r} A \cdot X^i}, \quad \text{where} \quad X^i = \prod_{j=1}^r X_j^{i_j}. \quad (8)$$

In this description, the Nygaard filtration is given by

$$\mathrm{Fil}_N^n A_{\mathrm{inf}}(R') = [p]_{q^{1/p}}^n A_{\mathrm{inf}}(R').$$

Write

$$\eta_{R'} : A_{\text{inf}}(R') \twoheadrightarrow \text{coker}(\phi_n - 1 : \text{Fil}_N^n A_{\text{inf}}(R') \rightarrow A_{\text{inf}}(R')) =: H^1(\mathbf{Z}_p(n)(R'))$$

for the canonical map.

Write $D = \mathbb{A}_{S/A}$, so

$$D = A_{\text{inf}}(R') \left\{ \frac{X_1^p}{[p]_q}, \dots, \frac{X_r^p}{[p]_q} \right\}^\wedge \simeq \widehat{\bigoplus_{\underline{i} \in \mathbf{N}[1/p]^r} A \cdot \frac{X^{\underline{i}}}{[[\underline{i}]]_q!}}, \quad \text{where } [[\underline{i}]]_q! = \prod_{j=1}^r [[i_j]]_q!. \quad (9)$$

Taking products of the calculation in (6), we find

$$\phi\left(\frac{X^{\underline{i}}}{[[\underline{i}]]_q!}\right) = \frac{X^{\underline{i}p}}{[[\underline{i}p]]_q!} \cdot [p]_q^{\sum_j [i_j]} \cdot u,$$

where u is a unit. Using this description and the $[p]_q$ -torsionfreeness of A , one checks that the Nygaard filtration is given by

$$\text{Fil}_N^n D = \widehat{\bigoplus_{\underline{i} \in \mathbf{N}[1/p]^r} [p]_{q^{1/p}}^{n - \sum_j [i_j]} \cdot A \cdot \frac{X^{\underline{i}}}{[[\underline{i}]]_q!}}, \quad (10)$$

where we declare $[p]_q^a = 1$ for $a \leq 0$. Again, we write

$$\eta_S : D \twoheadrightarrow \text{coker}(\phi_n - 1 : \text{Fil}_N^n D \rightarrow D) =: H^1(\mathbf{Z}_p(n)(S))$$

for the canonical map.

Using these explicit descriptions, we shall prove the lemma by analyzing the surjective map η_S on various graded pieces of D . Before delving into the specifics, let us explain the structure of the argument. The key is to observe (by comparing (8) and (9)) that the map $A_{\text{inf}}(R') \rightarrow D$ is bijective on components of sufficiently small degree \underline{i} ; in fact, it suffices to assume that $i_j < 1$ for all j since that forces the denominator appearing in (9) to be 1. Thus, the lemma would follow once we knew that $H^1(\mathbf{Z}_p(n)(S))$ is already spanned by the image of components of D with sufficiently small degree under the η_S . We shall verify this to be the case by using the explicit description of the Nygaard filtration given in (10).

To carry out this strategy, it is convenient to use the following notation. Given a $(p, [p]_q)$ -complete $A_{\text{inf}}(R)$ -module M equipped with a $(p, [p]_q)$ -complete $\mathbf{N}[1/p]^r$ -grading

$$M := \widehat{\bigoplus_{\underline{i} \in \mathbf{N}[1/p]^r} M_{\underline{i}}}$$

(such as A or D or $\text{Fil}_N^n D$) and any $\alpha \in \mathbf{N}$, we write

$$M_{\leq \alpha, \sqcup} := \widehat{\bigoplus_{\underline{i} \in \mathbf{N}[1/p]^r, \sum_j [i_j] \leq \alpha} M_{\underline{i}}},$$

and similarly for $M_{\geq \alpha, \sqcup}$. For future use, we observe that

$$M \simeq M_{\leq \alpha, \sqcup} \oplus M_{\geq \alpha+1, \sqcup} \quad \text{and} \quad M_{\leq 0, \sqcup} = \widehat{\bigoplus_{\underline{i} \in \mathbf{N}[1/p]^r \cap [0,1]^r} M_{\underline{i}}}$$

with these conventions.

We now begin the proof. First, we claim that

$$\eta_S(D_{\geq n+1, \sqcup}) = 0.$$

To see this, we must check that $D_{\geq n+1, \sqcup} \subset D$ lies in the image of $\phi_n - 1 : \mathrm{Fil}_N^n D \rightarrow D$. The description (10) shows that

$$\phi_n \left((\mathrm{Fil}_N^n D)_{\geq n+1, \sqcup} \right) \subset [p]_q D_{\geq n+1, \sqcup}.$$

On the other hand, the identity map induces an isomorphism

$$(\mathrm{Fil}_N^n D)_{\geq n+1, \sqcup} = D_{\geq n+1, \sqcup},$$

again by (10). Combining these, it follows that the map

$$(\mathrm{Fil}_N^n D)_{\geq n+1, \sqcup} \xrightarrow{\phi_n - 1} D_{\geq n+1, \sqcup}$$

is surjective: explicitly, for any $x \in D_{\geq n+1, \sqcup}$, the infinite sum

$$-(x + \phi_n(x) + \phi_n^2(x) + \dots)$$

converges to an element $y \in D_{\geq n+1, \sqcup} = (\mathrm{Fil}_N^n D)_{\geq n+1, \sqcup}$ that is a lift of x under $\phi_n - 1$. This proves that η_S kills $D_{\geq n+1, \sqcup}$, so

$$D_{\leq n, \sqcup} \subset D \xrightarrow{\eta_S} H^1(\mathbf{Z}_p(n)(S))$$

is surjective.

Next, for any $\underline{i} \in \mathbf{N}[1/p]^r$ with $\sum_j [i_j] \leq n$, the map $\phi_n : \mathrm{Fil}_N^n D \rightarrow D$ induces an isomorphism

$$\phi_n : (\mathrm{Fil}_N^n D)_{\frac{1}{p}, \underline{i}} \simeq D_{\underline{i}}$$

by (10). As $\eta_S(x) = \eta_S(\phi_n(x))$ for any $x \in D$, we learn that

$$\eta_S(D_{\underline{i}}) \subset \eta_S(D_{\frac{1}{p}, \underline{i}}).$$

Iterating this observation shows that we can divide the degree by arbitrarily many powers of p . In particular, after dividing finitely many times, we learn that the map

$$D_{\leq 0, \sqcup} \subset D \xrightarrow{\eta_S} H^1(\mathbf{Z}_p(n)(S))$$

is surjective.

Next, we observe that the map

$$A_{\mathrm{inf}}(R')_{\leq 0, \sqcup} \rightarrow D_{\leq 0, \sqcup}$$

is surjective. Indeed, the degree \underline{i} terms of (8) and (9) coincide since the degrees \underline{i} occurring here satisfy $i_j < 1$ for all j . By functoriality, we conclude

$$H^1(\mathbf{Z}_p(n)(R')) \rightarrow H^1(\mathbf{Z}_p(n)(S))$$

is also surjective. □

The following lemma will be quite useful in making reductions.

Lemma 14.5. *Fix a perfectoid ring R . If $S' \rightarrow S$ is a surjection of quasiregular semiperfectoid R -algebras such that the p -completion of $L_{S'/R} \rightarrow L_{S/R}$ is also surjective on $H^{-1}(-)$, then $\Delta_{S'} \rightarrow \Delta_S$ is surjective.*

Proof. This follows from the Hodge-Tate comparison. \square

We can finally put everything together to prove Theorem 14.1.

Proof of Theorem 14.1. By [BMS19, Remark 7.20] and Theorem 13.1, it suffices to prove discreteness. Moreover, as we work locally on the quasisyntomic site, we may restrict attention to quasiregular semiperfectoid \mathcal{O}_C -algebras, where C/\mathbf{Q} is an algebraically closed nonarchimedean field. In this setting, the Breuil-Kisin twists and the Nygaard completion may be ignored whilst calculating $\mathbf{Z}_p(n)$: more precisely, the proof of [BMS19, Lemma 7.22] shows that

$$\mathbf{Z}_p(n)(S) \simeq \text{fib}(\text{Fil}_N^n \Delta_S \xrightarrow{1-\phi_n} \Delta_S).$$

Consider the following assertion:

$(*)_S$ Given a quasiregular semiperfectoid \mathcal{O}_C -algebra S and an element $\alpha \in H^1(\mathbf{Z}_p(n)(S))$, there exists a quasi-syntomic cover $S \rightarrow S'$ such that α maps to 0 in $H^1(\mathbf{Z}_p(n)(S'))$.

Our goal is to prove $(*)_S$ for all S . Let us first prove this when $S = R$ is a perfectoid \mathcal{O}_C -algebra. Theorem 9.4 and Artin-Schreier theory show that any class in $H^1(\mathbf{Z}_p(0)(-))$ can be annihilated by a pro-(finite étale) cover of R , which settles the $n = 0$ case of $(*)_R$. If $n > 0$, then Theorem 9.4 (and a trivialization of $\mathbf{Z}_p(1) \cong \mathbf{Z}_p(n)$ on the generic fibre) reduce us to the $n = 1$ case. By Kummer theory (and [BMS19, Proposition 7.17] to identify $\mathbf{Z}_p(1)$ as the quasisyntomic sheaf $\lim_n \mu_{p^n}$), for any perfectoid ring R , the group $H^1(\mathbf{Z}_p(1)(R))$ is H^0 of the derived p -completion of R^* ; here we use that $\text{Pic}(R)$ is uniquely p -divisible by Corollary 9.5. André's lemma (Theorem 7.12) gives a cover $R \rightarrow R'$ with $(R')^*$ being p -divisible, whence $H^1(\mathbf{Z}_p(1)(R')) = 0$, so we are done.

Next, we verify $(*)_S$ for a specific example. Set

$$S_0 = R\langle x_1^{1/p^\infty}, \dots, x_r^{1/p^\infty} \rangle / (x_1, \dots, x_r),$$

with R being a perfectoid \mathcal{O}_C -algebra. We claim that $(*)_{S_0}$ holds: indeed, Lemma 14.4 implies that $(*)_{S_0}$ follows from $(*)_{R\langle x_1^{1/p^\infty}, \dots, x_r^{1/p^\infty} \rangle}$, which was already shown above.

Finally, we handle the general case. Fix a quasiregular semiperfectoid S , presented as a quotient R'/I with R' a perfectoid \mathcal{O}_C -algebra. Fix a set $\{x_t \in I\}_{t \in T}$ of generators of I . By André's lemma, we may replace R' by a quasisyntomic cover if necessary to assume that each x_t admits a compatible system of p -power roots. On fixing such a system, we obtain an evident surjection $S' := (R'[\{x_t^{1/p^\infty}\}_{t \in T}]/(x_t))^\wedge \rightarrow S$. The induced map on p -complete cotangent complexes is also surjective on H^{-1} , so we can then assume $S = S'$ by Lemma 14.5. Filtering T by its finite subsets then reduces us to the ring S_0 considered in the previous paragraph, so we are done. \square

Proof of Corollary 14.3. Via [BMS19, Theorem 1.12 (5)] and [CMM18], it is enough to show that for all $n \geq 0$, applying $H^1(\mathbf{Z}_p(n)(-))$ to $R \rightarrow S$ gives a surjective map. Choose a p -power compatible

system of roots $\{f_i^{1/p^n}\}_{n \geq 0}$ for each i . Via this choice, we obtain a commutative square

$$\begin{array}{ccc} R\langle x_1^{1/p^\infty}, \dots, x_r^{1/p^\infty} \rangle & \longrightarrow & R\langle x_1^{1/p^\infty}, \dots, x_r^{1/p^\infty} \rangle / (x_1, \dots, x_r) \\ \downarrow & & \downarrow \\ R & \longrightarrow & S := R/(f_1, \dots, f_r) \end{array}$$

where the top horizontal map is the obvious one, and left vertical map is determined by $x_i^{1/p^n} \mapsto f_i^{1/p^n}$ for all $n \geq 0$ and $i \in 1, \dots, r$. The top horizontal map is surjective on $H^1(\mathbf{Z}_p(n)(-))$ by Lemma 14.4, while the vertical maps are surjective on $H^1(\mathbf{Z}_p(n)(-))$ by Lemma 14.5. The commutativity implies the same for the bottom horizontal map, as wanted. \square

15. THE NYGAARD FILTRATION: RELATIVE CASE

In §15.1, we explain how to prove a version of the results of §12 relative to any bounded base prism (A, I) . This yields, in particular, the Nygaard filtration on the prismatic complex on a smooth formal A/I -scheme, thus proving Theorem 1.15. In §15.2, we apply these results to compare the theory constructed in this paper with the Breuil-Kisin type theory from [BMS19].

15.1. Constructing the Nygaard filtration on relative prismatic cohomology. Let S be any quasisyntomic A/I -algebra. Recall that this means that S is a p -completely flat A/I -algebra such that $L_{S/(A/I)}$ has p -adic Tor amplitude in $[-1, 0]$, and we always assume that S is p -adically complete.

Definition 15.1. A quasisyntomic A/I -algebra S is large if there is a surjection $A/I\langle X_i^{1/p^\infty} \mid i \in I \rangle \rightarrow S$ for some set I .

For large quasisyntomic A/I -algebras $L_{S/(A/I)}[-1]$ is a p -completely flat S -module as $\Omega_{S/(A/I)}^1$ vanishes after p -completion. Note that large quasisyntomic algebras form a basis for the quasisyntomic site of A/I as one can always extract compatible sequences of p -power roots of elements. The standard examples are $A/I\langle X_1^{1/p^\infty}, \dots, X_n^{1/p^\infty} \rangle / (f_1, \dots, f_m)$ for some p -completely regular sequence (f_1, \dots, f_m) relative to A/I .

Theorem 15.2. *Let S be a large quasisyntomic A/I -algebra.*

- (1) *The derived prismatic cohomology $\Delta_{S/A}$ is concentrated in degree 0 and a (p, I) -completely flat δ - A -algebra. It is the initial object of $(S/A)_\Delta$.*

Let

$$\phi_{S/A} : \Delta_{S/A}^{(1)} := \Delta_{S/A} \widehat{\otimes}_{A, \phi}^L A \rightarrow \Delta_{S/A}$$

denote the relative Frobenius, and

$$\mathrm{Fil}_N^i \Delta_{S/A}^{(1)} = \{x \in \Delta_{S/A}^{(1)} \mid \phi_{S/A}(x) \in I^i \Delta_{S/A}\}.$$

- (2) *The image of*

$$\phi_{S/A} : \mathrm{gr}_N^i \Delta_{S/A}^{(1)} \hookrightarrow \overline{\Delta}_{S/A}\{i\}$$

is given by $\mathrm{Fil}_i \overline{\Delta}_{S/A}\{i\}$.

- (3) *The formation of Fil_N^i commutes with base change in A .*

Proof. In part (1), note that $\Delta_{S/A}$ is concentrated in degree 0 by the Hodge-Tate comparison and the assumption that $L_{S/(A/I)}[-1]$ is p -completely flat, and we also see that it is (p, I) -completely flat. To see that $\Delta_{S/A}$ is initial, it is enough by Lemma 7.7 to show that a functorial idempotent endomorphism of $\Delta_{S/A}$ is the identity, which follows from the proof of Proposition 7.10.

Part (3) is immediate from the explicit description, but in fact one can see that if part (2) is true for the A/I -algebra S , then it is also true for any base change of S along a map $(A, I) \rightarrow (B, J)$ of bounded prisms. Indeed, one can define a putative Nygaard filtration on $\Delta_{S \widehat{\otimes}_A B/B} = \Delta_{S/A} \widehat{\otimes}_A B$ via base change, and it follow satisfy (2), which implies that it has to be the Nygaard filtration.

The proof of part (2) follows exactly the outline of the previous section. In particular, by descent to smooth algebras and left Kan extension, it suffices to treat the case

$$S = A/I\langle X_1^{1/p^\infty}, \dots, X_n^{1/p^\infty} \rangle / (f_1, \dots, f_m)$$

for some p -completely regular sequence (f_1, \dots, f_m) relative to A/I . By Proposition 7.11, we can after a flat base change in A assume that all f_i admit compatible p -power roots f_i^{1/p^j} . Arguing as in the proof of Proposition 12.7 reduces us to the case

$$S = A/I\langle X^{1/p^\infty} \rangle / (X) .$$

Localizing on A , we may assume that $I = (d)$ is orientable, and then by base change in A we can reduce to the universal oriented prism. In that case the perfection $A \rightarrow A_\infty$ is flat, so we can reduce to the case that A is perfect, where it follows from the previous section. (Note that for all terms in the Čech nerve $A_\infty \hat{\otimes}_A A_\infty$ etc., the Nygaard filtration will simply be the base change from A_∞ by our remark about part (3) above, so one can apply flat descent.) \square

In particular, if X is a smooth formal A/I -scheme, we can define sheaves $\mathbb{A}_{-/A}$ and

$$\mathrm{Fil}_N^i \mathbb{A}_{-/A}^{(1)} \subset \mathbb{A}_{-/A}^{(1)} := \mathbb{A}_{-/A} \hat{\otimes}_{A, \phi}^L A$$

on the quasisyntomic site X_{qsyn} by defining them on the base of large quasisyntomic A/I -algebras S , with the values defined in the theorem. We can now prove Theorem 1.15, whose statement we recall.

Theorem 15.3. *Let (A, I) be a bounded prism and let $X = \mathrm{Spf} R$ be an affine smooth p -adic formal scheme over A/I . There is a canonical isomorphism*

$$R\Gamma_{\mathbb{A}}(X/A) \cong R\Gamma(X_{\mathrm{qsyn}}, \mathbb{A}_{-/A})$$

and we endow prismatic cohomology with the Nygaard filtration

$$\mathrm{Fil}_N^i R\Gamma_{\mathbb{A}}(X/A)^{(1)} = R\Gamma(X_{\mathrm{qsyn}}, \mathrm{Fil}_N^i \mathbb{A}_{-/A}^{(1)}) .$$

Then there are natural isomorphisms

$$\mathrm{gr}_N^i R\Gamma_{\mathbb{A}}(X/A)^{(1)} \cong \tau^{\leq i} \overline{\mathbb{A}}_{R/A} \{i\}$$

for all $i \geq 0$. The Frobenius ϕ on $R\Gamma_{\mathbb{A}}(X/A)$ factors as

$$\phi_A^* R\Gamma_{\mathbb{A}}(X/A) = R\Gamma_{\mathbb{A}}(X/A)^{(1)} \xrightarrow{\tilde{\phi}} L\eta_I R\Gamma_{\mathbb{A}}(X/A) \rightarrow R\Gamma_{\mathbb{A}}(X/A) ,$$

using the décalage functor $L\eta_I$ as e.g. in [BMS18]. The map

$$\tilde{\phi} : \phi_A^* R\Gamma_{\mathbb{A}}(X/A) \rightarrow L\eta_I R\Gamma_{\mathbb{A}}(X/A)$$

is an isomorphism.

Proof. The isomorphism

$$R\Gamma_{\mathbb{A}}(X/A) \cong R\Gamma(X_{\mathrm{qsyn}}, \mathbb{A}_{-/A})$$

follows from the Hodge-Tate comparison and flat descent for the cotangent complex and its wedge powers [BMS19, Theorem 3.1]. The isomorphism

$$\mathrm{gr}_N^i R\Gamma_{\Delta}(X/A)^{(1)} \cong \tau^{\leq i} \overline{\Delta}_{R/A} \{i\}$$

follows via descent from Theorem 15.2 (2). The Frobenius refines to a map of filtered complexes

$$\mathrm{Fil}_N^* R\Gamma_{\Delta}(X/A)^{(1)} \rightarrow I^* R\Gamma_{\Delta}(X/A) .$$

The filtered complex on the left is connective in the Beilinson t -structure by the identification of its graded pieces. Thus [BMS19, Proposition 5.8] implies that the Frobenius lifts to a map

$$\tilde{\phi} : R\Gamma_{\Delta}(X/A)^{(1)} \rightarrow L\eta_I R\Gamma_{\Delta}(X/A) .$$

To see that this is an isomorphism, it suffices by derived Nakayama to check modulo I . Then the right-hand side is given by $\Omega_{R/A}^*$ by the Hodge-Tate comparison and [BMS18, Proposition 6.12]. In particular, both sides commute with base change and satisfy étale localization. We can then reduce to the case of a polynomial algebra and then via base change in A first to an oriented A , then to the universal oriented A , and then by Construction 6.1 to crystalline A , and finally to $A = \mathbf{Z}_p$. Now it follows from Theorem 5.2 and the Cartier isomorphism. \square

In particular, we get the de Rham comparison in general.

Corollary 15.4. *For any bounded prism (A, I) and any smooth formal A/I -scheme X , there is a canonical isomorphism of E_{∞} -algebras in $D(X_{\text{ét}}, A/I)$,*

$$\Delta_{X/A} \widehat{\otimes}_{A, \phi}^L A/I \cong \Omega_{R/(A/I)}^* .$$

Proof. Take the reduction of $\tilde{\phi}$ modulo I and use [BMS18, Proposition 6.12] and the Hodge-Tate comparison. \square

Another application is the following result on the image of ϕ on prismatic cohomology.

Corollary 15.5. *Let (A, I) be a bounded prism and X be a smooth formal A/I -scheme. For any $i \geq 0$, there is a natural map*

$$V_i : \tau^{\leq i} \Delta_{X/A} \otimes_A I^{\otimes i} \rightarrow \tau^{\leq i} \Delta_{X/A}^{(1)}$$

such that ϕV_i is the natural map $\tau^{\leq i} \Delta_{X/A} \otimes_A I^{\otimes i} \rightarrow \tau^{\leq i} \Delta_{X/A}$ and also the composite

$$\tau^{\leq i} \Delta_{X/A}^{(1)} \otimes_A I^{\otimes i} \xrightarrow{\phi \otimes 1} \tau^{\leq i} \Delta_{X/A} \otimes_A I^{\otimes i} \xrightarrow{V_i} \tau^{\leq i} \Delta_{X/A}^{(1)}$$

is the natural map. In particular, V_i induces a map

$$V_i : H^i(X_{\text{ét}}, \Delta_{X/A}^{(1)}) \otimes_A I^{\otimes i} \rightarrow H^i(X_{\text{ét}}, \Delta_{X/A}) = H_{\Delta}^i(X/A)$$

that is an inverse of ϕ up to $I^{\otimes i}$.

Proof. This follows from the isomorphism

$$\tilde{\phi} : \Delta_{X/A}^{(1)} \simeq L\eta_I \Delta_{X/A}$$

and [BMS18, Lemma 6.9]. \square

Remark 15.6. By left Kan extension, Proposition 4.14 combined with Theorem 1.15 (2) (which is a part of Theorem 15.3) imply the following: for any formal A/I -scheme X , there is a canonical identification

$$L_{X/A} \simeq \mathrm{gr}_N^1 \Delta_{X/A}^{(1)}.$$

When $I = (p)$, this identification has been proven (independently) recently by Illusie (to appear).

15.2. Comparison with the Breuil-Kisin type theory from [BMS19]. One of the goals of [BMS19] was to give a construction of Breuil-Kisin-type cohomology theories. Let us verify that the theory defined in [BMS19] agrees with the present construction. Thus, let K be a complete discretely valued extension of \mathbf{Q}_p with perfect residue field with ring of integers \mathcal{O}_K and residue field k , and fix a uniformizer $\pi \in \mathcal{O}_K$. Let $\mathfrak{S} = W(k)[[u]]$ which surjects onto \mathcal{O}_K via $u \mapsto \pi$. Let $I \subset \mathfrak{S}$ be the kernel of this map; then (\mathfrak{S}, I) is a prism. We fix a generator $d \in I$.

In [BMS19], we used relative THH for the base $\mathbf{S}[u]$. The key comparison is now the following.

Proposition 15.7. *For any quasiregular semiperfectoid quasisyntomic \mathcal{O}_K -algebra S , the cyclotomic Frobenius on $\pi_0 \mathrm{TP}(S/\mathbf{S}[u]; \mathbf{Z}_p)$ refines to a δ -ring structure, functorial in S , and identifies with the Nygaard completion $\widehat{\Delta_{S/\mathfrak{S}}^{(1)}}$ of $\Delta_{S/\mathfrak{S}}^{(1)}$, compatibly with Nygaard filtrations.*

Proof. The ring $\pi_0 \mathrm{TP}(S/\mathbf{S}[u]; \mathbf{Z}_p)$ is p -torsion free, so we need to check that the cyclotomic Frobenius is a Frobenius lift. This can be checked after the (p, u) -completed base change along $\mathfrak{S} \rightarrow W(k)[[u^{1/p^\infty}]]$, which gives $\pi_0 \mathrm{TP}(S\langle \pi^{1/p^\infty} \rangle; \mathbf{Z}_p)$ by [BMS19, Corollary 11.8]. Thus, we can apply Theorem 13.1. Now the universal property of $\Delta_{S/\mathfrak{S}}$ and the formal properties of $\pi_0 \mathrm{TP}(S/\mathbf{S}[u]; \mathbf{Z}_p)$ give a map

$$\Delta_{S/\mathfrak{S}}^{(1)} \rightarrow \pi_0 \mathrm{TP}(S/\mathbf{S}[u]; \mathbf{Z}_p).$$

Checking its compatibility with the Nygaard filtration can again be done after base change to $W(k)[[u^{1/p^\infty}]]$ where it follows from Theorem 13.1. In particular, the map extends to the Nygaard completion, and then is an isomorphism, again via reduction to Theorem 13.1. \square

Now note that for i at least the dimension of X , the map

$$\phi_{S/A} : \mathrm{Fil}_N^i \widehat{\Delta_{-/A}^{(1)}} \rightarrow I^i \Delta_{-/A}$$

induces an isomorphism

$$R\Gamma(X_{\mathrm{qsyn}}, \mathrm{Fil}_N^i \widehat{\Delta_{-/A}^{(1)}}) \rightarrow R\Gamma(X_{\mathrm{qsyn}}, \Delta_{-/A}) \otimes_A I^{\otimes i}$$

as both are complete for compatible filtrations (Fil_N^j respectively $I^j \Delta_{-/A}$) and one has isomorphisms on graded pieces by Theorem 15.3. The left-hand side can be expressed in terms of $\pi_0 \mathrm{TP}(-/\mathbf{S}[u]; \mathbf{Z}_p)$ and its Nygaard filtration by the previous proposition. This is how $R\Gamma_{\mathfrak{S}}(X)$ was defined in [BMS19] (cf. [BMS19, Proposition 11.5], noting that the Nygaard filtration is what one finds on higher homotopy groups in TC^-), so we get a canonical isomorphism

$$R\Gamma_{\mathfrak{S}}(X) \cong R\Gamma_{\Delta}(X/\mathfrak{S}),$$

as desired.

16. q -CRYSTALLINE AND q -DE RHAM COHOMOLOGY

In this section, we construct a canonical q -deformation of de Rham cohomology: given a formally smooth \mathbf{Z}_p -scheme X , we construct a ringed site — the q -crystalline site of X — whose cohomology yields a deformation of the de Rham cohomology of X/\mathbf{Z}_p across the map $\mathbf{Z}_p[[q-1]] \xrightarrow{q \rightarrow 1} \mathbf{Z}_p$, and can be computed in local co-ordinates via a q -de Rham complex; this verifies some conjectures from [Sch17]. The main innovation here is the introduction of a q -analog of the notion of divided power thickenings (defined in a co-ordinate free fashion) in the category of δ -rings over $\mathbf{Z}_p[[q-1]]$; this notion is introduced in §16.1, and the basic example is the pair $(\mathbf{Z}_p[[q-1]], (q-1))$. With this ingredient, the q -crystalline site is defined in an evident fashion in §16.2 and the comparison with q -de Rham complexes is the subject of §16.3; our definitions are set up to work over any q -divided power thickening as a base. Along the way, we also check that q -crystalline cohomology is closely related to prismatic cohomology (Theorem 16.17), so the comparison with q -de Rham complexes gives an explicit complex computing prismatic cohomology in many cases.

Notation 16.1. Set $A = \mathbf{Z}_p[[q-1]]$ with δ -structure given by $\delta(q) = 0$, and let $[p]_q = \frac{q^p-1}{q-1} \in A$ be the q -analog of p . Note that $\phi(q-1) = q^p - 1 \in [p]_q A$. We shall often use without comment the congruence $[p]_q = p \pmod{q-1}$ and that $(q-1)^{p-1}$ and p differ by a multiplicative unit in $A/[p]_q \cong \mathbf{Z}_p[\zeta_p]$, where ζ_p is a primitive p -th root of 1. In particular, derived $(p, [p]_q)$ -completion coincides with derived $(p, q-1)$ -completion for any complex of A -modules. Finally, if x is an element of a $[p]_q$ -torsionfree δ - A -algebra D such that $\phi(x) \in [p]_q D$, then we write

$$\gamma(x) := \frac{\phi(x)}{[p]_q} - \delta(x) \in D ;$$

if $q = 1$ in D , then we have $\gamma(x) = \frac{x^p}{p}$ is (up to the unit $(p-1)!$) the usual divided p -th power, and in general we think of $\gamma(x)$ as the “divided $[p]_q$ -th power”.

16.1. q -DIVIDED POWER THICKENINGS.

Definition 16.2 (q -divided power algebras). A q -PD pair is given by a derived $(p, [p]_q)$ -complete δ -pair (D, I) over $(A, (q-1))$ satisfying the following conditions:

- (1) The ideal $I \subset D$ satisfies $\phi(I) \subset [p]_q D$ (so that γ is defined on I) and $\gamma(I) \subset I$.
- (2) The pair $(D, ([p]_q))$ is a bounded prism over $(A, ([p]_q))$, i.e., D is $[p]_q$ -torsionfree, and $D/([p]_q)$ has bounded p^∞ -torsion.
- (3) The ring $D/(q-1)$ is p -torsionfree with finite $(p, [p]_q)$ -complete Tor-amplitude over D .

The corresponding map $D \rightarrow D/I$ is sometimes called a q -PD thickening and the ideal I is sometimes called a q -PD ideal. There is an obvious category of q -PD pairs. If (D, I) is a q -PD pair with $q-1 = 0$ in D , then we call (D, I) a δ -PD pair. The collection of δ -PD pairs forms a full subcategory of the category of all q -PD pairs.

Obviously condition (3) is a technical condition imposed to facilitate some arguments below; also condition (2) might reasonably be weakened to merely asking that $(D, ([p]_q))$ is a prism, i.e. that D is $[p]_q$ -torsionfree. However, as with prismatic cohomology above, it is more convenient to work with bounded prisms. We note that one way to satisfy (2) and (3) is for D to be $(p, [p]_q)$ -completely

flat over A ; another is for $q = 1$ in D and D being p -torsionfree. These will be the cases of interest below, and one can show that if D is noetherian, these are the only possibilities, at least for D local, using the Buchsbaum-Eisenbud criterion.

Remark 16.3 (Characterizing δ -PD pairs). Say D is a δ -ring regarded as an A -algebra via $q = 1$ and $I \subset A$ is an ideal. Then (D, I) is a δ -PD pair if and only if the following hold true:

- (1') The ideal I admits divided powers, i.e., for each $x \in I$, we have $\frac{x^n}{n!} \in I$ for all $n \geq 0$.
- (2') The ring D is p -torsionfree, and both D and I are p -adically complete.

Indeed, it is easy to see that any pair (D, I) satisfying the above conditions is a δ -PD pair. Conversely, if (D, I) is a δ -PD pair, then condition (2') above is automatic. For (1'), we note that condition (1) in Definition 16.2 ensures that $\frac{x^p}{p} \in I$ for all $x \in I$. As $(p-1)!$ is a unit, this means $\frac{x^p}{p!} \in I$ for all $x \in I$. Via the formula $\gamma_k p(x) = u \gamma_k(\gamma_p(x))$ for some unit u , one sees inductively that all divided powers stay in I .

Remark 16.4 (Relating q -PD pairs to δ -PD pairs). The hypotheses are designed to ensure that if (D, I) is a q -PD pair, then $(D/(q-1), ID/(q-1))$ is a δ -PD pair; this construction gives a left adjoint to the inclusion of δ -PD pairs into all q -PD pairs.

Lemma 16.5 (Homological properties of q -PD pairs). *Let (D, I) be a q -PD pair.*

- (1) *The ring D is derived f -complete for every $f \in I$.*
- (2) *The functor $M \mapsto M \widehat{\otimes}_D^L D/(q-1)$ of $(p, [p]_q)$ -completed base change along $D \rightarrow D/(q-1)$ is conservative on $(p, [p]_q)$ -complete complexes.*
- (3) *The functor in (2) commutes with totalizations of cosimplicial $(p, [p]_q)$ -complexes in $D^{\geq 0}$.*

Proof. For (1), we simply note that for any $f \in I$, we have $f^p \in (p, [p]_q)$ since $\phi(f) \in [p]_q D$ by hypothesis, so derived f^p -completeness (and hence derived f -completeness) follows from derived $(p, [p]_q)$ -completeness of D .

For (2), it suffices to show that base change along the composite $D \rightarrow D/(q-1) \rightarrow D/(q-1, p)$ is conservative on derived $(p, [p]_q)$ -complete complexes. If $M \otimes_D^L D/(q-1, p) = 0$, then $M \otimes_D^L \text{Kos}(D; [p]_q, q-1) = 0$ as well, since $\text{Kos}(D; [p]_q, q-1)$ has at most two nonzero cohomology groups, each of which is a $D/(q-1, p)$ -module. But this forces $M = 0$ by derived Nakayama, so the claim follows.

Part (3) follows immediately Definition 16.2 (3) and Lemma 4.20. \square

To check condition (1) from Definition 16.2 in examples, the following remark is useful.

Remark 16.6 (Additivity and multiplicative properties of γ). Let D be a $[p]_q$ -torsionfree δ - A -algebra. Given $x, y \in \phi^{-1}([p]_q D)$, one easily checks that

$$\gamma(x+y) = \gamma(x) + \gamma(y) + \frac{(x+y)^p - x^p - y^p}{p}.$$

Similarly, given $x \in \phi^{-1}([p]_q D)$ and $f \in D$, one checks that

$$\gamma(fx) = \phi(f)\gamma(x) - x^p \delta(f).$$

It follows immediately from the shape of these formulas that if I is any ideal of D , then the subset $J := \{x \in I \mid \phi(x) \in [p]_q D, \gamma(x) \in I\}$ is an ideal of D . In particular, to check $I = J$, it suffices to check that $\phi(x) \in [p]_q D$ and $\gamma(x) \in I$ as x runs through a generating set for I .

The next lemma roughly states that if $\gamma(x)$ makes sense, so does $\gamma(\gamma(x))$.

Lemma 16.7 (Existence of higher q -divided powers). *Let D be a $[p]_q$ -torsionfree δ - A -algebra. The ideal $\phi^{-1}([p]_q D)$ is stable under γ .*

Proof. We must show that if $f \in D$ with $\phi(f) \in [p]_q D$, then $\phi(\gamma(f)) \in [p]_q D$ as well. It suffices to prove this in the universal case $D = A\{f, \frac{\phi(f)}{[p]_q}\}$. In particular, we may assume that D is A -flat by Proposition 3.13. Our goal is to show that

$$\frac{\phi^2(f)}{\phi([p]_q)} \equiv \phi(\delta(f)) \pmod{[p]_q D}$$

Note that $\phi([p]_q)$ equals p in $A/[p]_q A$. In particular, it is a nonzerodivisor in this ring. Flatness implies that $\phi([p]_q)$ equals p and is a nonzerodivisor in $D/[p]_q D$ as well. It thus suffices to check that

$$\phi^2(f) \equiv p\phi(\delta(f)) \pmod{[p]_q D}.$$

Now $\phi(f) = f^p + p\delta(f)$, so $\phi^2(f) = \phi(f)^p + p\phi(\delta(f))$. Our claim follows since $\phi(f) \in [p]_q D$ by assumption. \square

Corollary 16.8 (Smallest and largest q -PD ideals). *Say D is a derived $(p, [p]_q)$ -complete A -algebra satisfying conditions (2) and (3) from Definition 16.2. Then $(q-1)$ is the smallest q -PD ideal in D , and $\phi^{-1}([p]_q D)$ is the largest q -PD ideal in D .*

Proof. We first show that $(q-1) \subset D$ is a q -PD ideal. This ideal is derived $(p, [p]_q)$ -complete as it is the image of the map $D \xrightarrow{q-1} D$ between derived $(p, [p]_q)$ -complete A -modules. For the rest, using Remark 16.6, it suffices to check that $\phi(q-1) \in [p]_q D$ and $\gamma(q-1) \in (q-1)$. The first containment is clear: $\phi(q-1) = q^p - 1 = (q-1)[p]_q \in [p]_q A$. For the second, note that

$$\gamma(q-1) = \frac{\phi(q-1)}{[p]_q} - \delta(q-1) = (q-1) - \delta(q-1).$$

It thus suffices to check that $\delta(q-1) \in (q-1)$. As $D/(q-1)$ is p -torsionfree by hypothesis, it is enough to show that $p\delta(q-1) \in (q-1)$. But we have

$$p\delta(q-1) = \phi(q-1) - (q-1)^p = (q^p - 1) - (q-1)^p = (q-1)([p]_q - (q-1)^{p-1}),$$

which lies in $(q-1)$, so the claim follows.

We now show that $\phi^{-1}([p]_q D)$ is a q -PD ideal. Derived $(p, [p]_q)$ -completeness of the ideal follows as above: $\phi^{-1}([p]_q D)$ is the limit of the diagram $D \xrightarrow{\phi} \phi_* D \xleftarrow{\phi_*([p]_q)} \phi_*(D)$ of derived $(p, [p]_q)$ -complete A -modules. The containment $\phi(\phi^{-1}([p]_q D)) \subset [p]_q D$ is clear, while the containment $\gamma(\phi^{-1}([p]_q D)) \subset \phi^{-1}([p]_q D)$ follows from Lemma 16.7. \square

We now give the most important examples of q -PD pairs for our purposes.

Example 16.9 (Examples of q -PD pairs). The key examples are:

- (1) (The initial object). The pair $(A, (q-1))$ is a q -PD pair. Indeed, conditions (2) and (3) from Definition 16.2 are clear, while (1) follows from Corollary 16.8. Note that the pair $(A, (q-1))$ is the initial object in the category of all q -PD pairs. More generally, the same reasoning shows that if D is a $(p, [p]_q)$ -completely flat A -algebra, then $(D, (q-1))$ is a q -PD pair.
- (2) (A perfect object). Let A_{inf} be the $(p, [p]_q)$ -completed perfection of A , and set $\xi := \phi^{-1}([p]_q) \in A_{\text{inf}}$. Then $(A_{\text{inf}}, (\xi))$ is a q -PD pair. Using Remark 16.6, the only non-trivial statement one must check is that $\gamma(\xi) \in (\xi)$. Unwinding definitions, this amounts to showing that $\delta(\xi) \equiv 1 \pmod{(\xi)}$. As p is a nonzerodivisor on $A_{\text{inf}}/(\xi)$, it suffices to check that $p\delta(\xi) \equiv p \pmod{(\xi)}$. As $\xi^p \equiv 0 \pmod{(\xi)}$, it is enough to show that $\phi(\xi) \equiv p \pmod{(\xi)}$. But

$$\phi(\xi) = [p]_q = \frac{q^p - 1}{q - 1} = 1 + q + \dots + q^{p-1} \equiv p \pmod{(q-1)},$$

so the claim follows as $\xi \mid q-1$.

- (3) (The classical case). If D is a p -torsionfree and p -complete δ -ring equipped with a p -complete ideal I , then (D, I) is a δ -PD pair exactly when each $x \in I$ admits all divided powers in D (Remark 16.3).

Lemma 16.10 (Existence of q -PD envelopes). *Let (D, I) be a q -PD pair. Let P be a $(p, [p]_q)$ -completely flat δ - D -algebra. Let $x_1, \dots, x_r \in P$ be a sequence that is $(p, [p]_q)$ -completely regular relative to D .*

- (1) *The $(p, [p]_q)$ -complete δ - D -algebra*

$$E := P\left\{\frac{\phi(x_1)}{[p]_q}, \dots, \frac{\phi(x_r)}{[p]_q}\right\}^\wedge$$

obtained by freely adjoining $\frac{\phi(x_i)}{[p]_q}$ is $(p, [p]_q)$ -completely flat over D . In particular, it is discrete and $[p]_q$ -torsionfree.

- (2) *Write $J \subset P$ for the ideal generated by I and the x_i 's and let $K \subset E$ be the minimal $(p, [p]_q)$ -complete ideal containing J and stable under the operation γ . Then the natural map $P/J \rightarrow E/K$ is an isomorphism, and in particular we get a natural map $E \rightarrow E/K \cong P/J$. Then (E, K) is a q -PD pair and the induced map $(P, J) \rightarrow (E, K)$ of δ -pairs is the universal map from (P, J) to a q -PD pair.*

In this situation, we often write $D_{J,q}(P) = E$ and call the pair $(D_{J,q}(P), K)$ (or just $D_{J,q}(P)$ if there is no potential for confusion) the q -PD envelope of (P, J) . We then have the following:

- (3) *The functor $(P, J) \mapsto (D_{J,q}(P), K)$ commutes with $(p, [p]_q)$ -completed derived base change along maps $(D, I) \rightarrow (D', I')$ of q -PD pairs. In particular, $D_{J,q}(P) \hat{\otimes}_D^L D/(q-1)$ is the pd -envelope of $J/(q-1) \subset P/(q-1)$.*

Proof. Let \tilde{E} be the $(p, [p]_q)$ -complete simplicial commutative δ - P -algebra obtained by freely adjoining $\frac{\phi(x_i)}{[p]_q}$ to P . We claim that \tilde{E} is $(p, [p]_q)$ -completely flat over D ; this will imply that \tilde{E} is discrete and thus coincides with the ring E above, proving (1). To show $(p, [p]_q)$ -complete flatness

of \tilde{E} over D , it suffices to do so after derived base change along $D \rightarrow D/(q-1)$. After this base change, the claim follows from Lemma 2.43, so we have proven (1).

To prove (2), we first construct a map $E \rightarrow P/J$. To define this map, observe that the previous paragraph shows that $E/(q-1)$ is the p -completely flat $D/(q-1)$ -algebra obtained as the pd-envelope of p -completely flat $D/(q-1)$ -algebra $P/(q-1)$ along the ideal generated by the sequence x_1, \dots, x_r that is p -completely regular relative to $D/(q-1)$. In particular, we have an obvious map $E/(q-1) \rightarrow P/(q-1, x_1, \dots, x_r)$, and hence also an obvious map $E \rightarrow E/(q-1) \rightarrow P/(q-1, x_1, \dots, x_r) \rightarrow P/J$. Let K' denote the kernel of this map. Note that the kernel \overline{K}' of $E/(q-1) \rightarrow P/J$ has divided powers by the previous paragraph (and because $I \cdot D/(q-1)$ has divided powers), and that K' is the preimage of \overline{K}' .

We now prove that (E, K') is a q -PD pair. For this, it suffices to check that $\phi(K') \in [p]_q E$ and that $\gamma(K') \subset K'$. Equivalently, if we set $K'' = K' \cap \phi^{-1}([p]_q E)$, we must show that $K'' = K'$ and that $\gamma(K'') \subset K''$. For the latter, we must check that if $f \in K''$, then $\gamma(f) \in K''$, which is immediate: the containment in $\phi^{-1}([p]_q E)$ follows from Lemma 16.7, while the containment in K' follows as $\gamma(f)$ maps under $E \rightarrow E/(q-1)$ to the element $\frac{f^p}{p}$, which lies in $\overline{K}' \subset E/(q-1)$ as the latter ideal has divided powers (and thus $\gamma(f)$ itself must lie in the inverse image of \overline{K}' , which is K'). It remains to show $K'' = K' = K$. We already have $(I, x_1, \dots, x_r) \subset K''$ and $\gamma(K'') \subset K''$. We shall deduce that $K'' = K' = K$ from Lemma 16.11. To apply this lemma, it suffices to check that K'' is the preimage of its image under $E \rightarrow E/(q-1)$. But this is easy to see: if $x \in K''$ and $y \in E$, then $x + (q-1)y \in K''$ since $\phi(q-1) \in [p]_q D$ and $q-1 \in K$.

This also proves (2) by the definition of K and the identification $K' = K$. Finally, (3) is immediate from the construction of $E \cong \tilde{E}$ and the fact that the hypotheses on the x_i 's commute with base change. \square

Lemma 16.11. *Let B be p -torsionfree \mathbf{Z}_p -algebra equipped with an ideal I that is regular modulo p . Let (D, J) be the pd-pair obtained as the pd-envelope of (B, I) . Then the ideal $J \subset D$ can be described as the smallest ideal of D that contains ID and is stable under the operation $f \mapsto \frac{f^p}{p}$.*

Proof. The regularity hypothesis ensures that D is p -torsionfree. The rest follows by [Sta, Tag 07GS]. \square

16.2. The q -crystalline site. In this section, we fix a q -PD pair (D, I) over $(A, (q-1))$ as well as a p -completely smooth B/I -algebra R .

Definition 16.12 (The q -crystalline site and its cohomology). The q -crystalline site of R relative to D , denoted $(R/D)_{q\text{-crys}}$, is the category of q -PD thickenings of R relative to D , i.e., the category of q -PD pairs (E, J) over (D, I) equipped with a D/I -algebra isomorphism $R \cong E/J$. Let $\mathcal{O}_{q\text{-crys}}$ be the presheaf on $(R/D)_{q\text{-crys}}$ determined by $(E, J) \mapsto E$; this presheaf is naturally valued in δ - B -algebras. The q -crystalline cohomology of R relative to D , denoted $q\Omega_{R/D}$, is defined as $R\Gamma((R/D)_{q\text{-crys}}, \mathcal{O}_{q\text{-crys}})$, viewed as a $(p, [p]_q)$ -complete commutative algebra object in $D(D)$ equipped with a ϕ_D -semilinear endomorphism $\phi_{R/D}$.

Construction 16.13 (Computing q -crystalline cohomology via Čech-Alexander complexes). Choose a δ - D -algebra P that is ind-smooth as a D -algebra as well as a surjection $P \rightarrow R$ of D -algebras; for

example, we may choose P to be the free δ - D -algebra on the set $W(R)$ to obtain a strictly functorial choice. Let P^\bullet be the Čech nerve of $D \rightarrow P$. The kernel of each augmentation $P^{\otimes_D(n+1)} \rightarrow P \rightarrow R$ is generated by I and, locally on $\mathrm{Spec} P$, by a filtered colimit of ideals generated by sequences that are $(p, [p]_q)$ -completely regular relative to D . Consequently, applying Lemma 16.10, we obtain a cosimplicial diagram $D_{J^\bullet, q}(P^\bullet)$ in $(R/D)_{q\text{-crys}}$. When P is chosen to be free as a δ - D -algebra, the limit of this diagram computes $q\Omega_{R/D}$ by Čech theory.

Theorem 16.14 (q -crystalline and crystalline cohomology). *There is a canonical identification*

$$q\Omega_{R/D} \widehat{\otimes}_D^L D/(q-1) \simeq R\Gamma_{\mathrm{crys}}(R/(D/(q-1))).$$

Proof. We use the complex $D_{J^\bullet, q}(P^\bullet)$ from Construction 16.13 to compute $q\Omega_R$. Applying the functor $-\widehat{\otimes}_D^L D/(q-1)$ of p -completed derived base change along $D \rightarrow D/(q-1)$, and using Lemma 16.5 (3), we learn that $q\Omega_D \widehat{\otimes}_D^L D/(q-1)$ is computed by the cosimplicial $D/(q-1)$ -complex $D_{J^\bullet, q}(P^\bullet) \widehat{\otimes}_D^L D/(q-1)$. One then uses Lemma 16.10 (3) to see this base change as a Čech-Alexander complex computing $R\Gamma_{\mathrm{crys}}(R/(D/(q-1)))$, proving the theorem. \square

Remark 16.15 (Computing q -crystalline cohomology via “small” Čech-Alexander complexes). In Construction 16.13, if we take P to be any δ - D -algebra that is $(p, [p]_q)$ -completely smooth as a D -algebra, the resulting complex $D_{J^\bullet, q}(P^\bullet)$ still computes $q\Omega_{R/D}$. Indeed, this follows from Theorem 16.14, Lemma 16.5 (2), and the analogous assertion in crystalline cohomology. In particular, it is often convenient to let P simply be a (p, I) -completely smooth lift of R along $D \rightarrow D/I$.

Theorem 16.16 (Invariance under q -PD thickenings of the base). *Let $(D, J) \rightarrow (D, I)$ be a morphism of q -PD pairs that is the identity on underlying δ -rings. (For example, take $J = (q-1)$.) Let \tilde{R} be a p -completely smooth D/J -algebra lifting R along $D/J \rightarrow D/I$. Then there is a canonical identification $q\Omega_{\tilde{R}/D} \simeq q\Omega_{R/D}$.*

Proof. Let us first observe that if $(E, K) \in (\tilde{R}/D)_{q\text{-crys}}$, then the natural map $E \rightarrow E/K \simeq \tilde{R} \rightarrow R$ is surjective and a q -PD thickening: surjectivity follows from that of $\tilde{R} \rightarrow R$ (which holds true as it does so modulo p), while the rest follows as the kernel can be described as the smallest derived $(p, [p]_q)$ -complete ideal of E generated by K and IE . This construction gives a functor $(\tilde{R}/D)_{q\text{-crys}} \rightarrow (R/D)_{q\text{-crys}}$ that does not change the underlying δ - D -algebra, and thus yields a canonical map $q\Omega_{\tilde{R}/D} \rightarrow q\Omega_{R/D}$. It now follows by inspection of the proof of Lemma 16.10 that the Čech-Alexander complex used to compute $q\Omega_{\tilde{R}/D}$ also computes $q\Omega_{R/D}$. \square

Theorem 16.17 (q -crystalline and prismatic cohomology). *Let R be a p -completely smooth D/I -algebra. Let $R^{(1)}$ be the p -completely smooth $D/([p]_q)$ -algebra defined via p -completed base change along the map $D/I \rightarrow D/([p]_q)$ induced by ϕ_D . Writing $\Delta_{R^{(1)}/D}$ for the prismatic cohomology of $R^{(1)}$ relative to the bounded prism $(D, [p]_q)$, there is a canonical isomorphism $\Delta_{R^{(1)}/D} \simeq q\Omega_{R/D}$.*

Proof. Let us first describe a canonical map $\Delta_{R^{(1)}/D} \rightarrow q\Omega_{R/D}$. For this, it is enough to explain how each $(E, J) \in (R/D)_{q\text{-crys}}$ functorially yields an object $(E \rightarrow E/([p]_q) \leftarrow R^{(1)})$ of $(R^{(1)}/D)_\Delta$. As $E \rightarrow E/J \cong R$ is a q -PD thickening, we have $\phi_E(J) \subset [p]_q E$, so ϕ_E yields a map $R \cong$

$E/J \rightarrow E/([p]_q)$ that is linear over ϕ_D . By linearization, this can be viewed as a D -linear map $R^{(1)} \rightarrow E/([p]_q)$, which then gives the desired object $(E \rightarrow E/([p]_q) \leftarrow R^{(1)})$ of $(R^{(1)}/D)_\Delta$.

We have constructed the map $\alpha : \Delta_{R^{(1)}/D} \rightarrow q\Omega_{R/D}$. It remains to check that α is an isomorphism. We are now allowed to make choices. In particular, using Theorem 16.16, we may assume $I = (q-1)$; here we implicitly use that R lifts to a p -completely smooth algebra along $D/(q-1) \rightarrow D/I$. To check α is an isomorphism, it suffices to do so after $(p, [p]_q)$ -completed derived base change along $D \rightarrow D/(q-1)$ by Lemma 16.5 (2). Now $D \rightarrow D/(q-1)$ refines to a map $(D, ([p]_q)) \rightarrow (D/(q-1), (p))$ of prisms by our assumptions on D . So base change for prismatic cohomology identifies $\Delta_{R^{(1)}/D} \widehat{\otimes}_D^L D/(q-1)$ with $\Delta_{\overline{R}^{(1)}/(D/(q-1))}$, where $\overline{R}^{(1)}$ is the smooth $D/(p, q-1)$ -algebra defined by $R^{(1)}$ via base change. Using base change for prismatic cohomology again, this can also be written as $\widehat{\phi}_D^* \Delta_{\overline{R}/(D/(q-1))}$, where \overline{R} is the $D/(p, q-1)$ -algebra defined by R via base change (i.e., $\overline{R} = R/p$, so $\overline{R}^{(1)}$ is the Frobenius twist of \overline{R} relative to $D/(q-1, p)$, as the notation suggests). But Theorem 5.2 identifies this with $R\Gamma_{\text{crys}}(\overline{R}/(D/(q-1)))$, which coincides with $R\Gamma_{\text{crys}}(R/(D/(q-1)))$ and hence also $q\Omega_{R/D} \widehat{\otimes}_D^L D/(q-1)$ by Theorem 16.14. \square

16.3. q -de Rham cohomology. In this section, we fix a q -PD pair (D, I) with D being A -flat as well as a p -completely smooth D/I -algebra R .

Construction 16.18 (The q -de Rham complex of a framed D -algebra). A *framed D -algebra* is a pair (P, S) , where P is a p -completely ind-smooth D -algebra and $S \subset P$ is a set of p -completely ind-étale co-ordinates, i.e., the map $\square : D[S] := D[\{X_s\}_{s \in S}] \rightarrow P$ is p -completely ind-étale. There are unique (and compatible) δ - D -algebra structures on $D[S]$ and P determined by the requirement $\delta(X_s) = 0$ for all $s \in S$. For each $s \in S$, there is a unique δ - D -algebra automorphism γ_s of $D[S]$ given by scaling $X_s \mapsto qX_s$ and $X_t \mapsto X_t$ for $t \neq s$. As this automorphism is congruent to the identity modulo the topologically nilpotent element $qX_s - X_s$, it extends uniquely to a δ - D -algebra automorphism γ_s of P that is also congruent to the identity modulo $qX_s - X_s$. Since D was A -flat, the same holds true for P . In particular, $qX_s - X_s$ is a nonzerodivisor on P , so we obtain a q -derivation $\nabla_{q,s} : P \rightarrow P$ given by the formula

$$\nabla_{q,s} := \frac{\gamma_s(f) - f}{qX_s - X_s}.$$

Note that $\nabla_{q,s}$ lifts the derivative $\frac{\partial}{\partial X_s}(-)$ on both $D[S]/(q-1)$ and $P/(q-1)$: this is clear for $D[S]$ by explicit computation and follows for P by formal étaleness¹⁰. Varying s we can assemble these q -derivations into a map

$$\nabla_q : P \rightarrow \Omega_{P/S}^1 := \widehat{\bigoplus_{s \in S} PdX_s}$$

¹⁰More precisely, as $(q-1)$ is a nonzerodivisor on P , it suffices to check that the endomorphism $f \mapsto \gamma_s(f) - f$ of P coincides with $X_s \cdot \frac{\partial}{\partial X_s}(-)$ modulo $(q-1)^2$, which follows from formal étaleness of $D[S] \rightarrow P$ and the interpretation of the endomorphism $f \mapsto \gamma_s(f) - f$ of $D[S]/(q-1)^2$ as the element of $H^0(T_{D[S]/D} \otimes_D (q-1)/(q-1)^2)$ classifying the infinitesimal automorphism γ_s of the square-zero extension $D[S]/(q-1)^2 \rightarrow D[S]/(q-1)$.

given by the rule $\nabla_q(f) := \sum \nabla_{q,s}(f) dX_s$. Taking the Koszul complex on the commuting endomorphisms $\nabla_{q,s}$, we obtain the q -de Rham complex

$$q\Omega_{P/D}^{*,\square} := \left(P \xrightarrow{\nabla_q} \Omega_{P/D}^1 \xrightarrow{\nabla_q} \Omega_{P/D}^2 \rightarrow \dots \right),$$

regarded as a chain complex of D -modules. This construction has the following functoriality property: given two framed D -algebras (P, S) and (P', S') as well as a map $P \rightarrow P'$ of D -algebras carrying S into S' , we obtain an induced morphism $q\Omega_{P/D}^{*,\square} \rightarrow q\Omega_{P'/D}^{*,\square}$ of chain complexes.

Construction 16.19 (The q -de Rham complex of a framed q -PD pair). A *framed q -PD datum* is a triple (P, S, J) , where (P, S) is a framed D -algebra as in Construction 16.18 and $J \subset P$ is the kernel of a D -algebra surjection $P \rightarrow R$. Let $D_{J,q}(P)$ be the q -PD envelope as in Construction 16.10. We have the following:

Lemma 16.20. *For each $s \in S$, the automorphism γ_s of P extends uniquely to an automorphism of $D_{J,q}(P)$ that is congruent to the identity modulo $qX_s - X_s$.*

Proof. We will check that γ_s extends to $D_{J,q}$ as an endomorphism that is congruent to the identity modulo $qX_s - X_s$; this will force the extension to be an automorphism, proving the lemma. Consider the composition $P \xrightarrow{\gamma_s} P \rightarrow D_{J,q}(P)$. As all rings in sight are A -flat, it suffices to show that this composition carries $\phi(f)$ into $[p]_q D_{J,q}(P)$ for all $f \in J$. As γ_s is congruent to the identity modulo $qX_s - X_s$, for any $f \in P$, we can write

$$\gamma_s(f) = f + (q - 1)X_s g$$

for suitable $g \in P$. Applying ϕ shows

$$\phi(\gamma_s(f)) = \phi(f) + [p]_q \cdot (q - 1) \cdot X_s^p \cdot \phi(g).$$

As $\phi(f) \in [p]_q D_{J,q}(P)$ for $f \in J$, the same must hold true for $\phi(\gamma_s(f))$ by the previous formula. This proves that γ_s extends to $D_{J,q}$ as an endomorphism. In fact, this formula also shows that the resulting endomorphism of $D_{J,q}$ is congruent to the identity modulo $qX_s - X_s$, as wanted. \square

We can thus define q -derivatives $\nabla_{q,s}$ of $D_{J,q}$ by the same formula as in Construction 16.18; note that these derivatives continue to lift the operator $\frac{\partial}{\partial X_s}$ on $D_{J,q}/(q - 1)$: this follows by identifying the latter as the pd-envelope of $P/(q - 1) \rightarrow R$ and reducing to the analogous statement for P using torsionfreeness and density. Taking the Koszul complex of the commuting endomorphisms $\nabla_{q,s}$ as in Construction 16.18, we obtain a q -dR complex

$$q\Omega_{D_{J,q}(P)/D}^{*,\square} := \left(D_{J,q}(P) \xrightarrow{\nabla_q} D_{J,q}(P) \hat{\otimes}_P \Omega_{P/D}^1 \xrightarrow{\nabla_q} D_{J,q}(P) \hat{\otimes}_P \Omega_{P/D}^2 \rightarrow \dots \right),$$

regarded as a chain complex of D -modules. The construction $(P, S, J) \mapsto q\Omega_{D_{J,q}(P)/D}^{*,\square}$ has the following functoriality property: given two framed q -PD data (P, S, J) and (P', S', J') and a map $P \rightarrow P'$ of D -algebras carrying S into S' and J into J' , we obtain an induced morphism $q\Omega_{D_{J,q}(P)/D}^{*,\square} \rightarrow q\Omega_{D_{J',q}(P')/D}^{*,\square}$.

Theorem 16.21 (*q*-de Rham and *q*-crystalline comparison). *Let (P, S, J) be a framed *q*-PD datum in the sense of Construction 16.19. Then there is a canonical quasi-isomorphism $q\Omega_{R/D} \simeq q\Omega_{D_{J,q}(P)/D}^{*,\square}$.*

Proof. Consider the Čech nerve P^\bullet of $D \rightarrow P$. The étale co-ordinates S on $P = P^0$ naturally define étale co-ordinates S^\bullet on P^\bullet by taking coproducts, yielding a cosimplicial framed D -algebra (P^\bullet, S^\bullet) . Likewise, letting $J^\bullet \subset P^\bullet$ be the cosimplicial ideal defined as the kernel of the augmentation $P^n \rightarrow P \rightarrow R$, we obtain a cosimplicial framed *q*-PD datum $(P^\bullet, S^\bullet, J^\bullet)$. Applying Construction 16.19, we obtain a cosimplicial complex $M^{\bullet,*} := q\Omega_{D_{J^\bullet,q}(P^\bullet)}^{*,\square}$. The cosimplicial D -module $M^{\bullet,i}$ given by the “*i*-th row” is acyclic (even homotopy equivalent to 0 as a cosimplicial module) for $i > 0$ by a standard argument. On the other hand, for any face map $P^i \rightarrow P^j$ in the cosimplicial ring P^\bullet , the induced map $M^{i,*} \rightarrow M^{j,*}$ of chain complexes is a quasi-isomorphism: this holds true modulo $(q - 1)$ by the Poincaré lemma, and thus holds true by $(q - 1)$ -completeness. It formally follows that the chain complex attached to the cosimplicial D -module $M^{\bullet,0}$ is quasi-isomorphic to the 0-th column $M^{0,*}$. The former computes $q\Omega_{R/D}$, while the latter is $q\Omega_{D_{J,q}(P)/D}^{*,\square}$, so we obtain the desired quasi-isomorphism. \square

17. COMPARISON WITH $A\Omega$

In this section, we fix a perfectoid field C of characteristic 0 containing μ_{p^∞} as well as a p -completely smooth \mathcal{O}_C -algebra R . The goal of this section is to prove the comparison between the prismatic cohomology of R and the $A\Omega$ -complexes defined in [BMS18] via the intermediary of q -de Rham cohomology. Let us introduce the relevant notation first.

Notation 17.1. Choosing a compatible system $\{\zeta_{p^n} \in \mu_{p^n}(C)\}$ of primitive p -power roots of 1, we obtain the rank 1 element $q = [\epsilon] \in A := A_{\text{inf}}(\mathcal{O}_C)$, where $\epsilon = (1, \zeta_p, \dots) \in \mathcal{O}_C^b$. This allows us to view A as a perfect $\delta\text{-}\mathbf{Z}_p[[q-1]]$ that is also flat. Let $\xi = \phi^{-1}([p]_q)$, so (A, ξ) is a q -PD pair corresponding to the q -PD thickening $A \rightarrow A/(\xi) \cong \mathcal{O}_C$.

With this notation, our goal is to show the following:

Theorem 17.2. *There is a canonical isomorphism $q\Omega_{R/A} \simeq A\Omega_R$, and thus*

$$A\Omega_R \simeq q\Omega_{R/A} \simeq \Delta_{R(1)/A} = \phi_A^* \Delta_{R/A}.$$

All these maps are isomorphisms of E_∞ - A -algebras compatible with the Frobenius.

Remark 17.3. The proof will a priori give an isomorphism of E_1 - A -algebras functorially in R and compatibly with the Frobenius. However, this can be upgraded to an isomorphism of E_∞ - A -algebras a posteriori: By left Kan extension, one can extend both sides to quasiregular semiperfectoid R -algebras, where it induces an isomorphism (as associative rings) of discrete commutative rings, which then tautologically upgrades to an isomorphism as E_∞ -rings. Now by quasisyntomic descent, one gets the desired isomorphism of E_∞ - A -algebras.

Proof. We shall build a functorial map $\mu_R : q\Omega_{R/A} \rightarrow A\Omega_R$ by constructing one between functorial representatives of both sides; to check this is a quasi-isomorphism, we give an abstract argument (ultimately reducing to the Hodge-Tate comparison theorem for both sides). In fact, it suffices to build a strictly functorial map under the additional assumption that R is *very small*, i.e., when R/p is generated over \mathcal{O}_C/p by units; the comparison map for general R is then obtained by glueing as any smooth formal \mathcal{O}_C -scheme has a basis of opens of the form $\text{Spf}(R)$ with R very small. For the rest of the proof, we assume R is very small.

Computing $q\Omega_{R/A}$ by an explicit and functorial complex. Since R is very small, we have a canonical surjection $P \rightarrow R$ where $P = A[\{x_s^{\pm 1}\}_{s \in S}]^\wedge$ is a p -completely ind-smooth infinite dimensional torus over A with (invertible) co-ordinates $S := R^*$. We may regard P as a δ - A -algebra in a unique way by letting $\delta(x_s) = 0$ for each $s \in S$. Writing $D_{J,q}(P)$ for the q -PD envelope of $P \rightarrow R$, we can compute $q\Omega_{R/A}$ by the complex

$$q\Omega_{D_{J,q}(P)/A}^{*, \square} := \text{Kos}_c(D_{J,q}(P); \{\nabla_{q,s}\}_{s \in S}),$$

where the right side is defined as the $(p, [p]_q)$ -completed filtered colimit over all finite subsets $\Sigma \subset S$ of the corresponding Koszul complexes for $\{\nabla_{q,s}\}_{s \in \Sigma}$.

Computing $A\Omega_R$ by an explicit and functorial complex. Let $P \rightarrow R$ be as in the previous paragraph. This gives a surjection $\overline{P} := P/(\xi) \rightarrow R$, which may be regarded as an embedding of $\text{Spf}(R)$

into a canonical infinite dimensional formal torus over \mathcal{O}_C . Let \overline{P}_∞ be the standard perfectoid cover of \overline{P} obtained by passing to the inverse limit along the p -power maps of the torus $\mathrm{Spf}(\overline{P})$, so $\overline{P} \rightarrow \overline{P}_\infty$ is a pro-étale Δ -torsor on generic fibres, where $\Delta := \prod_{s \in S} \mathbf{Z}_p(1)$; note that \overline{P}_∞ can also be described as the perfectoid ring corresponding to the perfection of the prism $(P, (\xi))$. Let R_∞ denote the p -completed integral closure of R in the corresponding torsor over $R[1/p]$. By the almost purity theorem, R_∞ is perfectoid and comes equipped with a Δ -action; for $s \in S$, write σ_s for the automorphism of $A_{\mathrm{inf}}(R_\infty)$ induced by functoriality from the Galois automorphism of R_∞ corresponding to the basis vector of Δ determined by $s \in S$. Then $A\Omega_R$ is computed by the complex

$$\eta_{q-1} \mathrm{Kos}_c(A_{\mathrm{inf}}(R_\infty); \{\sigma_s - 1\}_{s \in S}),$$

defined again as a $(p, [p]_q)$ -completed filtered colimit of the analogous construction indexed by all finite subsets of S .

Constructing the comparison map $\mu_R : q\Omega_{R/A} \rightarrow A\Omega_R$. To construct a functorial comparison map, thanks to the complexes described above, it suffices to construct a canonical map

$$D_{J,q}(P) \rightarrow A_{\mathrm{inf}}(R_\infty)$$

of A_{inf} -algebras that intertwines σ_s with γ_s . Indeed, we will then get a map

$$\mathrm{Kos}_c(D_{J,q}(P); \{\nabla_{q,s}\}_{s \in S}) \cong \eta_{q-1} \mathrm{Kos}_c(D_{J,q}(P); \{\gamma_s - 1\}_{s \in S}) \rightarrow \eta_{q-1} \mathrm{Kos}_c(A_{\mathrm{inf}}(R_\infty); \{\sigma_s - 1\}_{s \in S}).$$

Note that we have a commutative diagram

$$\begin{array}{ccccc} P & \longrightarrow & \widehat{P_{\mathrm{perf}}} \simeq A_{\mathrm{inf}}(\overline{P}_\infty) & \longrightarrow & A_{\mathrm{inf}}(R_\infty) \\ \downarrow & & \downarrow & & \downarrow \\ \overline{P} & \longrightarrow & \overline{P}_\infty & & \\ \downarrow & & \downarrow & & \downarrow \\ R & \longrightarrow & R \widehat{\otimes}_{\overline{P}} \overline{P}_\infty & \longrightarrow & R_\infty, \end{array}$$

where all maps in the top horizontal row are δ -maps, all vertical maps are surjective, and the two smaller squares on the left are pushout squares. It is a standard calculation that the γ_s automorphism of P is compatible with the σ_s automorphism of $A_{\mathrm{inf}}(\overline{P}_\infty)$ and thus also with the σ_s -automorphism $A_{\mathrm{inf}}(R_\infty)$. As $A_{\mathrm{inf}}(R_\infty) \rightarrow R_\infty$ is a q -PD thickening, the above commutative diagram shows that the δ -map $P \rightarrow A_{\mathrm{inf}}(R_\infty)$ extends uniquely to a δ -map $D_{J,q}(P) \rightarrow A_{\mathrm{inf}}(R_\infty)$. It also follows by uniqueness that the γ_s -action on $D_{J,q}(P)$ is compatible with the σ_s -action on $A_{\mathrm{inf}}(R_\infty)$: the maps

$$D_{J,q}(P) \xrightarrow{\gamma_s} D_{J,q}(P) \rightarrow A_{\mathrm{inf}}(R_\infty) \quad \text{and} \quad D_{J,q}(P) \rightarrow A_{\mathrm{inf}}(R_\infty) \xrightarrow{\sigma_s} A_{\mathrm{inf}}(R_\infty)$$

of δ - A -algebras are induced by the universal property of q -PD envelopes from the maps

$$P \xrightarrow{\gamma_s} P \rightarrow A_{\mathrm{inf}}(R_\infty) \quad \text{and} \quad P \rightarrow A_{\mathrm{inf}}(R_\infty) \xrightarrow{\sigma_s} A_{\mathrm{inf}}(R_\infty),$$

so the former pair must coincide as the latter pair does. This gives the promised map $D_{J,q}(P) \rightarrow A_{\text{inf}}(R_\infty)$ of δ -rings that intertwines the γ_s automorphism of the source with the σ_s automorphism of the target, and thus also a comparison map

$$\mu_R : q\Omega_{R/A} \rightarrow A\Omega_R$$

by the explicit complexes constructed above.

Proving that μ_R is an isomorphism. To check that μ_R is an isomorphism, we use the criterion in Lemma 17.4 with $d = [p]_q$. By Theorem 16.17, we can identify $q\Omega_{R/A} \simeq \Delta_{R^{(1)}/A}$, where $R^{(1)} := R \otimes_{A_{\text{inf}}, \phi} A$. By the Hodge-Tate comparison for $\Delta_{R^{(1)}/A}$, we then obtain a canonical map

$$\eta_{q\Omega_{R/A}} : R^{(1)} \rightarrow H^0(q\Omega_{R/A}/[p]_q)$$

that induces an isomorphism

$$(\Omega_{R^{(1)}/(A/[p]_q)}^*, d_{dR}) \simeq (H^*(q\Omega_{R/A}/[p]_q), \beta_{[p]_q})$$

of commutative differential graded $A/[p]_q$ -algebras. On the other hand, by [BMS18, Theorem 9.2] for $A\Omega_R$, we have a canonical identification¹¹

$$\tilde{\Omega}_{R^{(1)}} \simeq A\Omega_R/[p]_q$$

of commutative $A/[p]_q$ -algebras. This gives a comparison map

$$\eta_{A\Omega_R} : R^{(1)} \rightarrow H^0(A\Omega_R/[p]_q).$$

By the Hodge-Tate comparison [BMS18, Theorem 8.3], the map $\eta_{A\Omega_R}$ induces an isomorphism

$$(\Omega_{R^{(1)}/(A/[p]_q)}^*, d_{dR}) \simeq (H^*(A\Omega_R/[p]_q), \beta_{[p]_q})$$

of commutative differential graded $A/[p]_q$ -algebras. We leave it to the reader to check that the comparison map μ_R intertwines $\eta_{q\Omega_{R/A}}$ with $\eta_{A\Omega_R}$ by reduction to the case where $R = \mathcal{O}_C[x^{\pm 1}]^\wedge$. Lemma 17.4 then implies that μ_R is an isomorphism. \square

Lemma 17.4. *Let A be a commutative ring equipped with a nonzerodivisor d . Let R be a smooth A/d -algebra. Let E and F be derived d -complete commutative algebras in $D(A)$ with E/d and F/d being coconnective and $H^*(E/d)$ and $H^*(F/d)$ being strictly graded commutative. Assume we are given A/d -algebra maps $\eta_E : R \rightarrow H^0(E/d)$ and $\eta_F : R \rightarrow H^0(F/d)$ such that the map $(\Omega_{R/(A/d)}^*, d_{dR}) \rightarrow (H^*(E/d), \beta_d)$ induced by the universal property of the de Rham complex is an isomorphism, and similarly for F . Then any commutative A -algebra map $\alpha : E \rightarrow F$ that intertwines η_E with η_F must be an isomorphism.*

A similar statement holds true if (A, d) is a bounded prism, R is assumed to be p -completely smooth over A/d , and E and F are assumed to be derived (p, d) -complete.

¹¹The Frobenius twist in $\tilde{\Omega}_{R^{(1)}}$ is implicit but not mentioned in [BMS18, Theorem 9.2]: the p -completely smooth \mathcal{O} -algebra R (corresponding to the formal scheme \mathfrak{X}) is regarded in *loc. cit.* as an A_{inf} -algebra via the map $A_{\text{inf}} \xrightarrow{\tilde{\theta}} \mathcal{O} \rightarrow R$, which is identified in the present context with the map $A_{\text{inf}} \xrightarrow{\theta} \mathcal{O} \rightarrow R \xrightarrow{\phi} R^{(1)}$.

Proof. To show α is an isomorphism, it is enough to show that $\bar{\alpha} : E/d \rightarrow F/d$ induces an isomorphism on $H^*(-)$. Now $H^*(\bar{\alpha})$ is a map of graded A/d -algebras that lifts to a map of commutative differential graded A/d -algebras $(H^*(E/d), \beta_d) \rightarrow (H^*(F/d), \beta_d)$. This latter map intertwines η_E with η_F by assumption, so we obtain a commutative diagram

$$\begin{array}{ccc} (\Omega_{R/(A/d)}^*, d_{dR}) & \xlongequal{\quad} & (\Omega_{R/(A/d)}^*, d_{dR}) \\ \downarrow & & \downarrow \\ (H^*(E/d), \beta_d) & \xrightarrow{H^*(\bar{\alpha})} & (H^*(F/d), \beta_d) \end{array}$$

of maps of commutative differential graded A/d -algebras. The left and right vertical maps are induced by η_E and η_F , and are isomorphisms by assumption. It follows that the bottom horizontal map is also an isomorphism, as wanted. \square

18. A UNIQUENESS CRITERION FOR COMPARISON ISOMORPHISMS

Let (A, I) be a perfect prism corresponding a perfectoid ring R . In this section, we show that the prismatic cohomology functor $S \mapsto \Delta_{S/R}$ has no non-trivial automorphisms that are compatible with the Hodge-Tate structure map $\eta_S : S \rightarrow \overline{\Delta}_{S/R}$. In particular, the comparison isomorphisms relating prismatic cohomology to q -crystalline cohomology (Theorem 16.17), the $A\Omega$ -theory (Theorem 17.2), or the p -adic Nygaard complexes arising from topological Hochschild homology (Theorem 13.1) are essentially unique. To formulate our result precisely, we introduce some ∞ -categorical notation.

Notation 18.1. Let Sm_R denote the category of p -completely smooth R -algebras. Let $\mathcal{C}_{\mathrm{Sm}_R}$ be the ∞ -category of pairs (G, η) , where $G : \mathrm{Sm}_R \rightarrow \mathcal{D}_{(p,I)\text{-comp}}(A)$ is a symmetric monoidal functor and $\eta : \mathrm{id} \rightarrow G \otimes_A^L R$ is a natural transformation of symmetric monoidal functors $\mathrm{Sm}_R \rightarrow \mathcal{D}_{p\text{-comp}}(R)$; here $G \otimes_A^L R$ is the $\mathcal{D}_{p\text{-comp}}(R)$ -valued functor determined by $S \mapsto G(S) \otimes_A^L R$. Prismatic cohomology together with the Hodge-Tate comparison naturally gives an object $\Delta_{-/A} \in \mathcal{C}_{\mathrm{Sm}_R}$.

Theorem 18.2. *The object $\Delta_{-/A} \in \mathcal{C}_{\mathrm{Sm}_R}$ has no nontrivial endomorphisms, i.e., $\mathrm{End}(\Delta_{-/A}) = \{1\}$.*

Somewhat surprisingly, we do not need to *a priori* impose compatibility with the Frobenius endomorphism of prismatic cohomology in formulating the above uniqueness assertion.

Proof. Let $\mathrm{rsPerfd}_R$ denote the category of R -algebras S that are quotients of p -completely flat perfectoid R -algebras by p -completely regular sequences relative to R . In particular, $\mathrm{Kos}(S; p)$ is flat over $\mathrm{Kos}(R; p)$ for any such S . Let $\mathcal{C}_{\mathrm{rsPerfd}_R}$ be the ∞ -category defined the same way as $\mathcal{C}_{\mathrm{Sm}_R}$ with $\mathrm{rsPerfd}_R$ replacing Sm_R . Derived prismatic cohomology and its Hodge-Tate comparison then similarly defines an object $\Delta'_{-/A} \in \mathcal{C}_{\mathrm{rsPerfd}_R}$. As any p -completely smooth \mathcal{O} -algebra S admits a quasisyntomic hypercover (or even a Čech cover) $S \rightarrow S^\bullet$ with each $S^i \in \mathrm{rsPerfd}_R$, it suffices by descent to show that $\Delta'_{-/A}$ has no nontrivial endomorphisms. As the prismatic cohomology of any $S \in \mathrm{rsPerfd}_R$ takes on discrete values, this reduces to the following concrete statement:

Lemma 18.3. *Assume that for each $S \in \mathrm{rsPerfd}_R$, we are given an A -algebra endomorphism ϵ_S of $\Delta_{S/A}$ with the following properties:*

- (1) *The endomorphism ϵ_S is functorial in S .*
- (2) *The endomorphism ϵ_S is compatible with the identity map on S under the natural map $\eta_S : S \rightarrow \overline{\Delta}_{S/A}$.*

Then $\epsilon_S = \mathrm{id}$ for all S .

Proof. Let us first check that $\epsilon_S = \mathrm{id}$ when S is perfectoid. In this case, we have $\Delta_{S/A} \cong A_{\mathrm{inf}}(S)$, so ϵ_S can be regarded as an A -algebra endomorphism of $A_{\mathrm{inf}}(S)$. By A -linearity, this endomorphism carries $IA_{\mathrm{inf}}(R)$ to itself. As any endomorphism of a perfect p -complete δ -ring is automatically compatible with the δ -structure (Corollary 2.31), we may regard ϵ_S as an endomorphism of the perfect prism $(A_{\mathrm{inf}}(S), IA_{\mathrm{inf}}(S))$. By assumption (2), this endomorphism gives the trivial endomorphism of $S \cong A_{\mathrm{inf}}(S)/IA_{\mathrm{inf}}(S)$. By Theorem 3.10, it follows that $\epsilon_S = \mathrm{id}$ for perfectoid S .

Now fix some $S \in \mathrm{rsPerfd}_R$. By assumption, we can write $S = R'/(f_1, \dots, f_r)$, where R' is a p -completely flat perfectoid R -algebra and f_1, \dots, f_r is a sequence in S that is p -completely regular relative to R . Then $\Delta_{R'/A} \cong A_{\mathrm{inf}}(R')$, so $\Delta_{S/A}$ is naturally a δ - $A_{\mathrm{inf}}(R')$ -algebra. Moreover, if we fix

a distinguished element $d \in I$ as well as $g_i \in A_{\inf}(R')$ lifting $f_i \in S$ for $i = 1, \dots, r$, then the presentation $S = R'/(f_1, \dots, f_r)$ gives a presentation of $\Delta_{S/A}$ as the δ - $A_{\inf}(R')$ -algebra $A_{\inf}(S)\{\frac{g_1}{d}, \dots, \frac{g_r}{d}\}^\wedge$ obtained by freely adjoining $\frac{g_i}{d}$ for $i = 1, \dots, r$ to $A_{\inf}(S)$ in the world of (p, d) -complete simplicial δ -rings. Applying assumption (1) to the map $R' \rightarrow S$, we obtain the commutative diagram

$$\begin{array}{ccc} A_{\inf}(R') & \xrightarrow{\text{can}} & A_{\inf}(R')\{\frac{g_1}{d}, \dots, \frac{g_r}{d}\}^\wedge \\ \downarrow \epsilon_{R'} & & \downarrow \epsilon_S \\ A_{\inf}(R') & \xrightarrow{\text{can}} & A_{\inf}(R')\{\frac{g_1}{d}, \dots, \frac{g_r}{d}\}^\wedge. \end{array}$$

We must show that the A -module endomorphism $\epsilon_S - \text{id}$ of $A_{\inf}(R')\{\frac{g_1}{d}, \dots, \frac{g_r}{d}\}^\wedge$ is identically 0. Equivalently, if K denotes the kernel of this map, we must show $K = A_{\inf}(R')\{\frac{g_1}{d}, \dots, \frac{g_r}{d}\}^\wedge$. Note that K is an A -submodule of $A_{\inf}(R')\{\frac{g_1}{d}, \dots, \frac{g_r}{d}\}^\wedge$ that has the following stability properties:

- (1) K is stable under taking powers, i.e., if $f \in K$ then $f^n \in K$ for all $n \geq 0$: the kernel of a difference of ring homomorphisms always has this property.
- (2) K is naturally linear over $A_{\inf}(R')$, not merely A : we have $\epsilon_{R'} = \text{id}$ by the previously settled perfectoid case, so both ϵ_S and id are $A_{\inf}(R')$ -algebra maps. In particular, the image of $A_{\inf}(R')$ in $A_{\inf}(R')\{\frac{g_1}{d}, \dots, \frac{g_r}{d}\}^\wedge$ lies in K .
- (3) K is saturated in $A_{\inf}(R')\{\frac{g_1}{d}, \dots, \frac{g_r}{d}\}^\wedge$ with respect to multiplication by both p as well as $\phi^n(d)$ for all $n \geq 0$. Indeed, $A_{\inf}(R')\{\frac{g_1}{d}, \dots, \frac{g_r}{d}\}^\wedge$ is (p, d) -completely flat over A (Proposition 3.13). In particular, by Lemma 3.7 (2), the element p as well as all the Frobenius powers $\phi^n(d)$ for $n \geq 0$ are nonzerodivisors in this A -module, and hence the kernel of any endomorphism of this $A_{\inf}(R')$ -module has the stated saturation property.

We now show that $K = A_{\inf}(R')\{\frac{g_1}{d}, \dots, \frac{g_r}{d}\}^\wedge$. As K is derived $(p, [p]_q)$ -complete and an $A_{\inf}(R')$ -module by property (2), it suffices to check that a topological generating set for the $A_{\inf}(R')$ -module $A_{\inf}(R')\{\frac{g_1}{d}, \dots, \frac{g_r}{d}\}^\wedge$ lies in K . Using the operations $\{\delta_n\}_{n \geq 0}$ of Joyal (Remark 2.13 and Remark 2.14), it suffices to check that $\delta_n(\frac{g_i}{d}) \in K$ for all $n \geq 0$ and all i . We check this by induction on n (and fixed i). For $n = 0$, we must check that $\frac{g_i}{d} \in K$. But $g_i \in K$ by property (2), and hence $\frac{g_i}{d} \in K$ by the d -saturatedness of K from property (3). Assume now that $\delta_j(\frac{g_i}{d}) \in K$ for $j \leq n$. To show $\delta_{n+1}(\frac{g_i}{d}) \in K$, we use the formula

$$\phi^{n+1}(\frac{g_i}{d}) = (\frac{g_i}{d})^{p^n} + p\delta_1(\frac{g_i}{d})^{p^{n-1}} + \dots + p^n\delta_n(\frac{g_i}{d})^p + p^{n+1}\delta_{n+1}(\frac{g_i}{d}).$$

The term on the left lies in K : this follows after multiplication by $\phi^{n+1}(d)$ using property (2), and hence holds true on the nose as K is $\phi^{n+1}(d)$ -saturated by property (3). Moreover, all terms on the right except the last one also lie in K : this follows by induction and property (1). Thus, the last term $p^{n+1}\delta_{n+1}(\frac{g_i}{d})$ also lies in K . As K is p -saturated by property (3), it follows that $\delta_{n+1}(\frac{g_i}{d}) \in K$ as well, which finishes the induction. \square

\square

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