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Prismatic Cohomology

March 13th – 16th, 2020

Brief description of the seminar

In this seminar we will study the first part of the recent preprint "Prisms and Prismatic Cohomology" by Bhatt–Scholze [BS19]. Let p be a fixed prime number. Prismatic cohomology is a new p-adic cohomology theory which specialises to old theories such as cristalline cohomology, de Rham cohomology, or étale cohomology. To any so-called bounded prism (A, I) (a ring with a Frobenius lift ϕ and a locally invertible ideal subject to certain conditions) and any smooth p-adic formal scheme X over A/I there is assigned a so-called prismatic site $(X/A)_{\Delta}$ together with a structure sheaf $\mathcal{O}_{\Delta}$. The derived global sections

$$R\Gamma_{\Delta}(X/A) := R\Gamma\big((X/A)_{\Delta}, \mathcal{O}_{\Delta}\big)$$

fulfil several comparison isomorphisms whereof we will see the following ones in the seminar:

Crystalline comparison If $I = (p)$, then there is a canonical ϕ-equivariant isomorphism $R\Gamma_{\text{cris}}(X/A) \cong \phi^* R\Gamma_{\Delta}(X/A)$.

Hodge–Tate comparison If $X = \text{Spf}(R)$ is affine, then there is a canonical R-linear isomorphism

$$\Omega^i_{R/(A/I)}\{i\} \cong \text{H}^i\big(R\Gamma_{\Delta}(X/A)\big) \otimes^{\mathbf{L}}_A A/I$$

where $M\{i\} = M \otimes_{A/I} (I/I2)^{\otimes i}$ for an A/I-module M.

de Rham comparison There is a canonical isomorphism

$$R\Gamma_{\text{dR}}\big(X/(A/I)\big) \cong R\Gamma_{\Delta}(X/A) \otimes^{\mathbf{L}}_{A,\varphi} A/I$$

of commutative algebras in $\mathbf{D}(A)$.

Remark. This programme has been plagiarised by a seminar done at the University of Zürich, borrowing some ideas from a seminar by Moritz Kerz and Georg Tamme at Universität Regensburg.

Talk 1: Overview [Andreas]
A general introduction to Bhatt–Scholze's work.

Talk 2: Cristalline Cohomology [Valentina]
Background from cristalline cohomology, in particular the comparison isomorphism with de Rham cohomology, use [Bha18, §VI.1] and additionally [BdJ11].

Talk 3: Perfectoid rings [Netan]
This is a summary talk. Introduce the notion of (integral) perfectoid rings and discuss the tilting equivalence for those following [Mor17, §1]. In the definition of [Mor17, §1], in a perfectoid ring p is either 0 or a non-zero-divisor. There is a more general notion of perfectoid rings, but for simplicity we will always assume that p is 0 or a non-zero-divisor. Explain the tilting equivalence for perfectoid Tate rings following [Mor17, §2] and state the almost purity theorem [Sch12, Theorem 1.12].

Talk 4: δ-Rings [Filippo]
Introduce δ-rings, discuss basic characterizations, examples, permanence properties. Prove the equivalence between perfect δ-rings and perfect rings. References: [Bha18, Lecture II] and [BS19, §2.1–4].

Talk 5: Distinguished elements and derived completions [Alex]
Discuss distinguished elements in δ-rings and examples and prove some characterisations of these. Give a brief account of derived completions as needed for the next talk. References: [Bha18, Lecture III, §§1-2] and [BS19, §2.3].

Talk 6: Prisms [Oli/Valentina]
Discuss prisms [Bha18, §III.3] and perfect prisms [Bha18, §IV.1]. A complete reference is [BS19, §2.3 and §3]. Prove that the category of perfect prisms (A, I) where A/I is p-torsion free or of characteristic p is equivalent to the category of perfectoid rings (in the sense of [Mor17]—note that this definition differs from [Bha18, Definition IV.2.1]). For the last part use arguments from the proof of the tilting equivalence [Mor17, §1] and

Talk 4.

Talk 7: Prismatic cohomology and the Hodge-Tate comparison map [Chris]
Define the prismatic site and construct the Hodge–Tate map from differential forms to prismatic cohomology. References: either [BS19, §4.1-2] or[Bha18, Lecture V].

Talk 8: Crystalline comparison [Andreas]
Explain the connection between divided powers and δ-structures and use this to prove the crystalline comparison for prismatic cohomology. References: [BS19, §2.5, §5] and [Bha18, §VI.2-3].

Talk 9: Hodge–Tate comparison isomorphism [Oli]
Proof of the isomorphy of the Hodge–Tate comparison map. The main reference for this is [BS19, §4.3, §6]. A more incomplete reference is [Bha18, §VI.4]. Prove the de Rham comparison isomorphism [BS19, Theorem 6.4].

Talk 10: Free slot

References

[BdJ11] Bhargav Bhatt and Aise Johan de Jong, *Crystalline cohomology and de rham cohomology*, Lecture notes for the Eilenberg Lectures, 2011.

[Bha18] Bhargav Bhatt, *Prismatic cohomology*, `arxiv:1110.5001`, 2018.

[BS19] Bhargav Bhatt and Peter Scholze, *Prisms and prismatic cohomology*, `arXiv:1905.08229`, 2019.

[Mor17] Matthew Morrow, *Foundations of perfectoid spaces*, `PDF file`, 2017.

[Sch12] Peter Scholze, *Perfectoid spaces*, Publ. Math. Inst. Hautes Études Sci. **116** (2012), 245–313. MR 3090258