Drift of a Reverberator in an Active Medium due to the Interaction with Boundaries

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The behaviour of a reverberator in a bounded medium of interacting nonlinear oscillators is considered. The expressions obtained for the reverberator drift velocity and the shift of its frequency may be interpreted in terms of "field-particle".

1. INTRODUCTION

One of the specific features of eigenwave conducting media (e.g. the heart tissue of the Belousov-Zhabotinsky reaction medium) is the ability of such media to contain eigenwave sources. For two-dimensional media, rotating spiral wave sources called also vortices or reverberators, are typical.

The existence of a reverberator may not be associated with medium inhomogeneities, but could be provided only by initial conditions /2,6,11, 12/. Such a source behaves as an outstanding, long-living object, thrusting its inherent rhythm upon environment, and possessing some stability features with respect to external forces. Small disturbances cause small changes in the autonomous evolution of the reverberator. The effect of the disturbances descends fastly with the increasing of the distance between an "application point" of external forces and some specific region near the spiral wave rotation center, called kernel /3,4,13/.

Let us consider the case of the simplest reverberator evolution, when the reverberator in a homogeneous unbounded medium radiates a spiral wave rotating with a constant frequency around a fixed center. Then the influence of medium boundaries, weak inhomogeneities or other eigenwave sources will cause a shif in frequency and a change of the reverberator location in the course of time, i.e. location and phase drifts occur.

Current concepts of such influence mechanisms are phenomenological and insufficient. It seems to be useful to consider simple cases, allowing analytical approach, in order to develop a more exact theory.

In this paper the drift of a reverberator in a bounded medium is studied in terms of "field-particle" formalism. A reverberator is re-

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garded to be a "point" particle, interacting with environment, which is governed by a "field" equation with fast oscillation being reduced.

2. MODEL. THE FREE REVERBERATOR

One of the simplest eigenwave models, belonging to lambda-omega systems /7/, which describe diffusion-conducted nonlinear oscillating media, has been chosen:

$$\dot{u} = (1-i\Omega)u - (1-i\alpha)u|u|^2 + \nabla^2 u; \qquad (1)$$

$u = u(r,t) \in C$, $\alpha, \Omega \in R$, $r \equiv (x,y)$, $\nabla \equiv (\partial/\partial x, \partial/\partial y)$, $\cdot \equiv \partial/\partial t$. (This is also a special case of the generalized Ginzburg-Landau equation /9/). Variable u corresponds to the complex amplitude of oscillations, its phase rate specifying the frequency of oscillations and its gradient - the local wave number. The replacement $\alpha \to -\alpha$ is equivalent to the replacement $\Omega \to -\Omega$, $u \to u^*$ (asterisk * means complex conjugate), for definiteness let $\alpha > 0$.

In the framework of this model Hagan /6/ has studied particularly the limit $\alpha \to 0$, that has proved to be simple enough to get an asymptotic representation for solutions like a stationary rotating reverberator of the form

$$u^0 = e^{im\phi - i\omega^0 t} U(p) \qquad (2)$$

and to investigate their stability. Here p, ϕ are the polar coordinates (rotation center being assumed to take place at the origin). U is a complex function, $m \neq 0$ is the "topological charge" (number of the arms of the spiral); for definiteness let $m > 0$. This representation satisfies the following condition:

$$\frac{d\,\arg(U)}{dp} \to k^0, \quad p \to \infty; \qquad \omega^0 = \Omega - \alpha + \alpha k^{0\,2} \qquad (3)$$

i.e. at large radii, lines of equal phase are close to Archimedian spirals, and

$$k^0 \approx \frac{2}{\alpha} \exp\left\{-\frac{\pi}{2m\alpha} + C'(m)\right\}$$

is uniquely defined for every α, i.e. it appears to be an eigenvalue of the problem; here C' are constants determined numerically. The solution will be referred to as the free reverberator.

3. FIELD EQUATION

Let us transform (1) in terms of amplitude and phase a, w, $u = a\exp(iw)$:

$$\dot{a} = (1-a^2-(\nabla w)^2)a + \nabla^2 a;$$

$$\dot{w} = -\Omega + \alpha a^2 + \frac{\nabla(a^2)\nabla w}{a^2} + \nabla^2 w.$$

If the derivatives ∇a, $\nabla^2 w$ and parameter α are small, then variable a is fast with respect to w, its adiabatical exclusion yields to the closed equation for w:

$$\dot{w} = -\Omega + \alpha - \alpha(\nabla w)^2 - \frac{\nabla w \nabla(\nabla w)^2}{1-(\nabla w)^2} + \nabla^2 w.$$

If $|\nabla w|$ is also small (which does take place in the model for a free reverberator at sufficiently large radii), then the fourth term at the right-hand side may be neglected. The remaining equation with the substitution

$$w = -1/\alpha \ln(W) - (\Omega-\alpha)t$$

results in the linear "field" equation:

$$\dot{W} = \nabla^2 W$$

(this development has been put forward in /10/). We are interested in some unusual (infinitely growing) solutions of the equation. For instance, plane wave solution of (1) with wave number k corresponds to $W = \exp(\alpha^2k^2t-\alpha kx)$. However, if there are close circuits, by-passing of which will cause change of w for $2\pi m$, $m \neq 0$, then function W is also ambiguous: when by-passing the circuits it will ascend (descend) by a factor of $\exp(2\pi m\alpha)$.

Interaction of eigenwaves with impenetrable boundaries could be described in the framework of (5). So, the wave falling from $x \to \infty$ under condition $\partial u/\partial x = 0$, $x \to 0$ is described by function

$$W = \cosh(\alpha kx)\exp(\alpha^2k^2t).$$

4. ADIABACITY

Now let us consider eigenwave medium in a bounded (for simplicity - finite) region with impenetrable boundaries:

$$(n\nabla)\ u|_{\Gamma} = 0, \tag{6}$$

where n is a normal to boundary Γ'. In order to deal with a drifting reverberator, let us consider a framework moving along some trajectory $r = r^0(t)$ with respect to the laboratory one. At this framework, instead of (1,6), we get

$$\dot{u} = (1-i\Omega)u - (1-i\alpha)\ u|u|^2 + \nabla^2 u + (\frac{dr^0}{dt}\nabla)u, \tag{7}$$

$$(n\nabla)\ u\ (r)|_{(r+r^0(t))\ \epsilon\ \Gamma} = 0, \tag{8}$$

or $$(n'(t)\nabla)\ u\ (r)|_{r\ \epsilon\ \Gamma'(t)} = 0. \tag{8'}$$

This is a boundary problem with moving boundaries.

Let the framework move together with the reverberator, i.e., let the solution of (7,8) be nearly periodical and close to the free reverberator within some neighbourhood of the origin at any time (near the boundaries the closeness is impossible since the reverberator cannot satisfy (8)).

From the numerical experiments /3/ one can see that the speed of the drift descends rapidly with the distance between the reverberator and the boundaries increasing. The estimation made in this paper agrees with this observation: the speed descends exponentially.

On the other hand, it is natural to expect the time of a stable stationary state regeneration, if any, to grow with the region dimension growing (e.g., for heat-conduction equation the time is known to be proportional to the square of medium dimensions, see /5/).

If so, then with the distance between the reverberator and the boundary being sufficiently large, the boundary will not be displaced essentially for the time of relaxation to the stationary state corresponding to the current boundary position.

For this stationary state we get a boundary problem:

$$(1-i\Omega)u - (1-i\alpha)u|u|^2 + \nabla^2 u + (c\nabla)u = -i\omega u \tag{9}$$

$$(n\nabla)\ u\ (r)|_{\Gamma'} = 0, \tag{10}$$

or in the trigonometrical form

$$\begin{aligned} \nabla^2 a + (1-a^2-(\nabla w)^2)a &= -(c\nabla)a \\ a\nabla^2 w + 2(\nabla a \nabla w) &= a(\Omega-\omega-\alpha a^2-(c\nabla)w), \end{aligned} \tag{9'}$$

$$(n\nabla)\, a\big|_{\Gamma'} = 0, \qquad (n\nabla)\, w\big|_{\Gamma'} = 0. \qquad (10')$$

Velocity vector c and frequency ω appear to be eigenvalues: when α and Γ' are given, they are uniquely defined. According to /6,8/, the free reverberator is stable for $m = 1$ and sufficiently small α. We shall assume the stationary state /9,10/ also to be stable. Vice versa, the free reverberators for $m > 1$ are unstable, therefore the case $m = 1$ seems to be enough; but the generalization for $m > 1$ does not meet any obstacles. Below we shall restrict ourselves with the analysis of the problem (9,10).

To a certain degree, such an approximation is analogous to the use of electrostatics equations for describing the motion of a system of electric charges, in case their velocities are small with respect to the speed of light. According to the analogy, applicability condition of such an approximation would consist in

$$|c| \ll \alpha k^2,$$

since the wave number k in the model corresponds to the information transmission speed of $2\alpha k$.

5. SOLUTION TECHNIQUE

The solution of problem (9,10) can be found with the help of asymptotic expansions matching method, if the boundaries are sufficiently remote. Let the solution at p not too large be close to the free reverberator, and at p not too small be described by "field" equation (5). Comparing the corresponding expressions within a region where they both are applicable, yields to the estimations of the reverberator frequency and the drift velocity. For the lack of space only the basic steps of calculations are presented, for more details see /1/.

a) *Internal region.* Under the assumptions made, it is natural to solve (8) at not large radii with the help of perturbation technique, taking into account that the free reverberator is a zero-order approximation, and c and ω^0 are small parameters. Let us use the fact that at not large p the free reverberator is, in turn, close to the solution of type (2) of equation (1) with $\alpha = 0$, and could be obtained from it by the perturbation technique with respect to the small parameter α.

So, at not large p we shall look for a solution of (9') in the form $a = a^0 + a'$, $w = w^0 + w'$, $a' \ll a^0$, $w' \ll w^0$, assuming α, c and $(\omega-\omega^0)$ to be small parameters and restricting ourselves to a linear approximation with respect to each of the variables.

A non-perturbed solution is described in /6/ by the functions $w^0 = m\Phi$, $a^0 = a^0(p) = P(p,m)$, where $P(p,m)$ is the solution of Greenberg /5/ equation:

$$\frac{d^2 a^0}{dp^2} + \frac{1}{p}\frac{da^0}{dp} + a^0\left(1 - a^{0\,2} - \frac{m^2}{p^2}\right) = 0$$

and

$$P(p,m) \approx 1 - \frac{m^2}{2p^2}, \quad p \to \infty; \quad P(p,m) \approx \epsilon(m)p^m, \quad p \to 0.$$

While studying equations for corrections a', w' under the assumption that they are growing not too rapidly with $p \to \infty$ (which is necessary for matching with external expression), the following problems arise. At

$$1 \ll p \ll \exp(\sqrt{2/\alpha m^2}) \tag{11}$$

holds

$$w(p,\phi) \approx m\phi + \alpha m^2 \ln(p)\left[\frac{1}{2}\ln(p) + C(M)\right] + cmp\left[\frac{1}{2}\ln(p) + B(m)\right]\sin(\phi - \theta) -$$

$$- \sum_{n=2}^{\infty} 2p^n \, \mathrm{Im}[C_n e^{in\phi}], \tag{12}$$

and similarly for $a(p,\phi)$, where additive terms meaning small arbitrary shifts of unperturbed solution in space and time. Here

$$c = (c\cos\theta, c\sin\theta);$$

C_n - arbitrary (not too large) coefficients, $B(m)$, $C(m)$ - constants to be found numerically. According to /6/

$$C(1) \approx -0.098; \quad C(2) \approx -0.998; \quad C(3) \approx -1.3; \quad \ldots \tag{13}$$

Constants $B(m)$ have also been estimated by the author:

$$B(1) \approx -0.31, \quad B(2) \approx -0.78, \quad B(3) \approx -1.01, \quad \ldots \tag{14}$$

b) *External region.* Problem (9,10) results in the following equations for the "field" variable

$$\nabla^2 W + (c\nabla)W - \alpha^2 k^2 W = 0. \tag{15}$$

Here k is the wave number according to the frequency $\omega = [illegible] - \alpha + \alpha k^2$. With allowance for ambiguity of W noted above, the general solution of (15) is

$$W = e^{-(cr)/2 - i\alpha m\phi} \sum_{n=-\infty}^{+\infty} e^{in\phi}[G^n K_{n+i\alpha m}(\alpha k'p) + F^n I_{n+i\alpha m}(\alpha k'p) \tag{16}$$

with arbitrary F^n, G^n. Here $(k')^2 = k^2 - c^2/4\alpha$; $K_{n+i m\alpha}$, $I_{n+im\alpha}$ are linearly independent solutions of the modified Bessel equation with complex index $n+im\alpha$, and

$$K_\mu(z) \approx \frac{1}{\sqrt{z}} e^{-z}, \quad I_\mu(z) \approx \frac{1}{\sqrt{z}} e^{z}, \quad z \to \infty, \quad I_\mu(z) = 0(1), \quad z \to 0;$$

$$K_\mu \equiv K_{-\mu}, \quad I_\mu \equiv I_{-\mu}, \quad (K_{\mu^*})^* \equiv K_\mu, \quad (I_{\mu^*})^* \equiv I_\mu.$$

The solution (16) is real provided

$$G^{-n} \equiv (G^n)^*; \quad F^{-n} \equiv (F^n)^*.$$

For the free reverberator all F^n, G^n vanish, besides of G, which may be put equal to 1. The moving reverberator differs slightly from the free one at small $|r'|$ if $G^n \ll 1$, $n \neq 0$. In other respects G^n, F^n are to be chosen to satisfy boundary conditions

$$(n\nabla)W|_{\Gamma'} = 0. \tag{17}$$

The final result is most easily achieved if the boundary Γ' may be described as $\rho = R(\theta)$, and $R(\theta)$ satisfies the inequalities

$$kR \gg 1, \quad \frac{1}{R}\frac{dR}{d\theta} \ll \alpha kR.$$

Then after some calculations we get

$$F^n = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta} e^{-2\alpha k' R(\theta)} d\theta. \tag{18}$$

Below we shall see that F^n play a part of "forces", having an effect on the reverberator and causing its drift. The forces being determined already as belonging to the order of zero with respect to c (the difference between k and k' appears to be not essential), make the assumption of adiabacity self-consistent.

c) *Matching*. . Compare (12,16) in the region (11) by transforming (16) for variable w. Matching can be accomplished separately for every angle mode.

The comparison for the 0-th mode demonstrates the following condition of compatibility of internal and external expression:

$$k = k'(1+\pi F^0),$$

or for the frequency

$$\omega \approx \omega^0 + 2\pi\alpha k^2 F^0, \tag{19}$$

where k^0 is the wave number of the free reverberator /6/ (compare with /4/):

$$k^0 \approx \frac{2}{\alpha}\exp\{-\frac{\pi}{2m\alpha} + C(m) - \tau\},$$

$\tau \approx 0.5771...$ is the Euler-Maskeroni constant.

The comparison for the first mode, in turn, yields the expression for the drift velocity:

$$ce^{-i\theta} \approx -2\pi i\, m\alpha^2 k\, [1+im\alpha(C(m)-1-2B(m)]\, F^1. \tag{20}$$

The comparison for other modes results in determining coefficients C_n (arbitrary for internal viewpoint) via boundary conditions.

The obtained equation of motion (19,20) solves the problem under the following consideration: specifying phase and location drift of the reverberator via its disposition with respect to boundaries. The dependence on the disposition appears in (19,20) via "generalized forces" F^0, F' that can be found by solving boundary problem (15,17) for the field equation, in most simple cases - by formulae (18).

d) *Example*. Let boundary Γ' be a circle with a radius $S \gg (\alpha k)^{-1}$, and the reverberator's kernel be shifted from the circle center for a distance s with and angle ψ with respect to x-axes, αks being either less or of the order of 1. Since $s \ll S$, we get approximately

$$R(\varphi) \approx S - s\cos(\varphi-\psi).$$

By substituting this into (18,19,20), we obtain that the frequency of the reverberator in the circular region is

$$\omega \approx \omega^0 + 2\pi\alpha k^2\, e^{-2\alpha kS}\, I_0(2\alpha ks),$$

and its spatial drift could be described by the equations of motion

$$s\frac{d\psi}{dt} \approx 2\pi\, m\alpha^2 k\, e^{-2\alpha kS}\, I_1(2\alpha ks),$$

$$\frac{ds}{dt} \approx -2\pi\, m^2\alpha^3 k\, B'(m)\, e^{-2\alpha kS}\, I_1(2\alpha ks).$$

Here I_0, I_1 are the modified Bessel functions of indices 0, 1; ω^0 is the free reverberator frequency, and coefficients $B'(m)$ are expressed via $B(m)$, $C(m)$ (13,14) by the formula

$$B'(m) = 1 + 2B(m) - C(m)$$

and are positive at least for small m:

$B'(1) \simeq 0.48, \quad B'(2) \simeq 0.44, \quad B'(3) \simeq 0.3.$

So, in the model under consideration the reverberator is repelled from the boundaries, the repulsion speed being much less than the speed of drift along the boundary (ratio is of the order of α).

6. DISCUSSION

The estimations of the interaction parameters between the reverberator and the boundaries in a simple model are obtained. The difference between the characters of the processes proceeding near the reverberator kernel and in the environment enables the application of the asymptotic technique.

In the chosen model, the difference is particularly evident. Function W may be treated as a potential of some short-distance field, which is "emanated" from the reverberator kernel. It influences back on the reverberator when "reflected" from the boundaries and causes it to drift.

Such a treatment results in the description of the evolution of a reverberator in terms of "field-particle" equations. The field satisfies linear equation (5), and coefficients F^0, F' describing the deformation of field W in the neighbourhood of the kernel, play a part of the forces effecting on the reverberator. In spite of the nonlinearity of the field equation, the superposition principle is applicable there with certain restrictions caused by the multivalency of "potential". That is also why the "forces" cannot be described as derivatives of the "external field", but should be determined indirectly.

Unfortunately, simple estimations demonstrate that direct numerous checking of the equations of motion cannot be easy because of exponential dependencies on small parameter α. So, for the validity of the assumptions made, the dimensions of the model medium must not be less than a specific length of the boundary influence descending. However, already with $\alpha = 0.2$ the length, according to the estimations made, should be about 10^3-10^4 times greater than the dimensions of the reverberator kernel. Apparently, some real way to avoid the difficulty may appear in the generalization of the theory presented here by including distances not too large or a finite α.

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