Localized sensitivity of spiral waves in the complex Ginzburg-Landau equation

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Dynamics of spiral waves in perturbed (e.g., slightly inhomogeneous) two-dimensional autowave media can be described asymptotically in terms of Aristotelean dynamics, so that the velocities of the spiral wave drift in space and time are proportional to the forces caused by the perturbation. These forces are defined as convolutions of the perturbation with the so-called response functions. In this paper, we find the response functions numerically for the spiral waves in the complex Ginzburg-Landau equation, and show that they exponentially decrease with distance. [S1063-651X(98)06603-3]

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Problem formulation. Spiral waves are observed in two-dimensional nonlinear active systems of various natures, e.g., Belousov-Zhabotinsky reaction [1] cardiac tissue [2], social microorganisms [3], neural tissue [4], and catalytic oxidation of CO [5]. They attract attention as model self-organizing structures, and demonstrate remarkable stability. In this paper, we show that spiral waves have a very selective sensitivity to perturbations.

Spiral waves are often studied in terms of ''reaction-diffusion'' PDE systems,

$$\partial_t \mathbf{u} = \mathcal{D}\nabla^2 \mathbf{u} + \mathbf{f}(\mathbf{u}) + \varepsilon \mathbf{h}(\vec{R},t), \tag{1}$$

where $\vec{R}\in\mathbf{R}^2$, $\mathbf{u}(\vec{R},t)=(u_1,u_2,\ldots)^{\mathrm{T}}\in\mathbf{R}^{\ell}$ is a column-vector of reagent concentrations, $\mathbf{f}\in\mathbf{R}^{\ell}$ are nonlinear reaction rates, $\mathcal{D}\in\mathbf{R}^{\ell\times\ell}$ is matrix of diffusion coefficients, $\ell\geqslant 2$ and $\varepsilon\mathbf{h}\in\mathbf{R}^{\ell}$ is a perturbation. As shown in [6], if the last term in (1) is of a more general form of parametric perturbation $\varepsilon\mathbf{h}(u,\vec{R},t)$, this still reduces to (1) in the first order in ε, so without loss of generality here we consider the simpler form $\varepsilon\mathbf{h}(\vec{R},t)$. Physical origin of the perturbation may be various; the most frequent in applications is inhomogeneity of medium parameters, but the analysis can be also extended to external influence, anisotropy, etc.

The simplest case of spiral wave is that of the steadily rotating spiral,

$$\mathbf{u} = \overline{\mathbf{U}}(\vec{R},t) = \mathbf{U}(\mathrm{P},\Theta+\omega t), \tag{2}$$

where ω is its angular velocity and $\mathrm{P}=\mathrm{P}(\vec{R})$, $\Theta=\Theta(\vec{R})$ are polar coordinates. This may be observed in perfectly homogeneous unbounded stationary media, i.e., at $\varepsilon\mathbf{h}=0$. In the presence of perturbations, the spiral will drift in space and accelerate or decelerate its rotation, i.e., ''drift in time.'' This can be represented by

$$\mathbf{u}(\vec{R},t) = \overline{\mathbf{U}}\left(\vec{R}-\vec{R}_c(t), t-\frac{1}{\omega}\Phi(t)\right) + \varepsilon\mathbf{v}(\vec{R},t), \tag{3}$$

where $\vec{R}_c=(X_c,Y_c)$ is the vortex rotation center and Φ is its initial rotation phase.

The asymptotic theory of such drifts has been developed in [6]. It leads to Aristotelian motion equations, where the drift velocities are proportional to the forces caused by perturbation $\varepsilon\mathbf{h}$,

$$\partial_t\Phi = H^{(0)}, \quad \partial_t(X_c+iY_c)=H^{(1)}. \tag{4}$$

In the first approximation, the forces are linear convolution-type functionals of the perturbation,

$$H^{(n)}(\vec{R}_c,\Phi,t) = \varepsilon e^{-in(\omega t-\Phi)}\int \langle \mathbf{W}^{(\mathbf{n})}(\vec{r}),\mathbf{h}(\vec{R},t)\rangle\, \mathrm{d}^2\vec{R} + O(\varepsilon^2), \quad n=0,1, \quad \langle\mathbf{a},\mathbf{b}\rangle \equiv \sum_{i=1}^{\ell} a_i^* b_i, \tag{5}$$

where $\vec{r}\in\mathbf{R}^2$ is the radius vector in the frame of references attached to the spiral wave, where the polar coordinates are

$$\rho = \mathrm{P}(\vec{R}-\vec{R}_c), \quad \vartheta = \Theta(\vec{R}-\vec{R}_c)+\omega t-\Phi. \tag{6}$$

We call kernels $\mathbf{W}^{(0,1)}$ *response functions* (RF's). They determine the influence of particular perturbations at a particular site and instant onto the phase (*temporal* RF, $\mathbf{W}^{(0)}$) and location (*spatial* RF, $\mathbf{W}^{(1)}$) of the spiral wave. As seen in Eq. (5), graphs of these functions rotate together with their spiral wave.

The RF's are interesting characteristics of the spiral wave. Known experiments and numerics may be interpreted so that these functions decrease with distance. This decrease may provide convergence of integrals (5) for nonlocalized perturbations, e.g., caused by variation of properties of the whole medium. The viewpoint of [7] was that these functions are asymptotically periodic, similarly to the spiral wave itself. Our viewpoint [8,6] is that these functions should quickly decay. In other words, although spiral waves do not *look* like localized objects, they *behave* as such in their dynamics. We are unaware of any attempts to prove or disprove this property directly.

In this paper, we study this question for the complex Ginzburg-Landau equation. This equation is one of the most basic equations of nonlinear science; another reason for this choice is its internal symmetry, which simplifies the analysis,

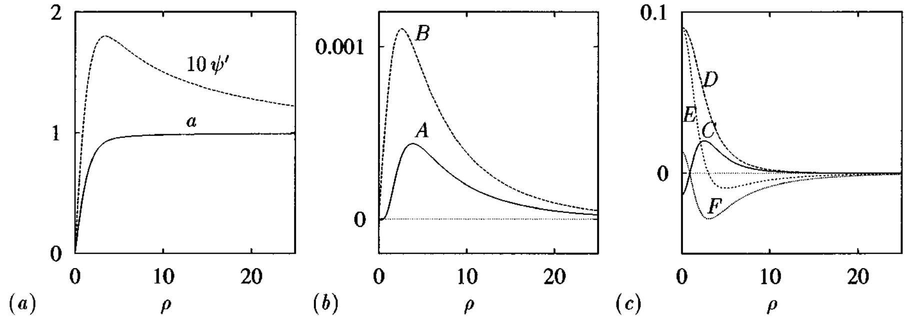

FIG. 1. The nonlinear problem solution, (b) temporal mode components, and (c) spatial mode components, as functions of ρ.

reducing the two-dimensional eigenvalue problem to a one-dimensional one. For this model, we find the RF's numerically and show that they have the expected localized form.

The linearized theory. Linearization of reaction-diffusion system (1) on (3) in the frame of reference (6) leads to an equation with a time-independent linear operator,

$$\mathcal{L}=\mathcal{D}\nabla^2-\omega\partial_\vartheta+\mathcal{F}(\vec{r}), \tag{7}$$

where $\mathcal{F}(\vec{r})=\partial_{\mathbf{u}}\mathbf{f}|_{\mathbf{u}=\mathbf{U}(\vec{r})}$. This operator has three neutral stability eigenvalues,

$$\mathcal{L}\mathbf{V}^{(n)}=i\omega n\mathbf{V}^{(n)}, \quad n=0,\pm 1, \tag{8}$$

corresponding to the translations in space and time, with the eigenfunctions, the translation modes being

$$\mathbf{V}^{(0)}=\frac{1}{\omega}\partial_t\overline{\mathbf{U}}(\vec{R},t)=\partial_\vartheta\mathbf{U}(\vec{r}),$$

$$\mathbf{V}^{(\pm 1)}=\tfrac{1}{2}e^{\mp i\omega t}(\partial_x\mp i\partial_y)\overline{\mathbf{U}}(\vec{R},t)=\tfrac{1}{2}e^{\mp i\vartheta}(\partial_\rho\mp i\rho^{-1}\partial_\vartheta)\mathbf{U}(\vec{r}). \tag{9}$$

The adjoint linear operator is

$$\mathcal{L}^{+}=\mathcal{D}^{\mathrm{T}}\nabla^2+\omega\partial_\vartheta+\mathcal{F}^{\mathrm{T}}(\vec{r}), \tag{10}$$

and its eigenfunctions

$$\mathcal{L}_n^{+}\mathbf{W}^{(n)}=-i\omega n\mathbf{W}^{(n)}, \qquad \int\langle\mathbf{W}^{(n)},\mathbf{V}^{(m)}\rangle d^2\vec{r}=\delta_{nm},$$

$$n,m=0,\pm 1 \tag{11}$$

serve as projectors onto these modes, and are the RF's. The requirement that $\mathbf{v}$ in Eq. (3) is orthogonal to $\mathbf{W}^{(n)}$ leads to the motion equations (4) [6].

Application to the complex Ginzburg-Landau equation. This equation can be written in the form

$$\partial_t u=u-(1-i\alpha)u|u|^2+(1+i\beta)\nabla^2 u \tag{12}$$

for $u\in\mathbf{C}$ with real parameters α and β. In this paper, we restrict ourselves to the case of $\alpha=0.5$ and $\beta=0$ (and omit β). To apply the general theory of [6] we first rewrite Eq. (12) in real vector form [9]. Let us denote

$$\mathbf{u}(\vec{R},t)=\begin{pmatrix}\mathrm{Re}\ u\\ \mathrm{Im}\ u,\end{pmatrix}, \quad \mathcal{I}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \quad \mathcal{C}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},$$

$$\mathbf{1}=\begin{pmatrix}1\\ 0\end{pmatrix}. \tag{13}$$

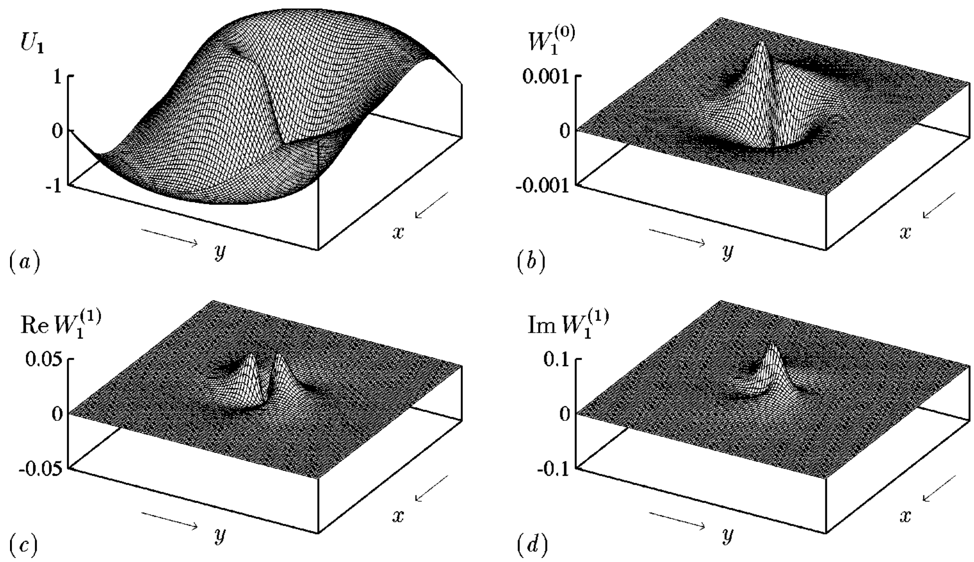

FIG. 2. (a) Spiral wave U_1, (b) temporal RF, $W_1^{(0)}$, (c) real part of spatial RF, Re $W_1^{(1)}$, and (d) imaginary part of spatial RF, Im $W_1^{(1)}$. Spiral wavelength is about 67; $(x,y)\in[-30,30]\times[-30,30]$.

Then Eq. (12), with added perturbation, takes the form

$$\partial_t \mathbf{u} = \mathbf{u} - (1-\alpha\mathcal{I})\cdot\mathbf{u}(\mathbf{u}^{\mathrm{T}}\cdot\mathbf{u}) + \nabla^2\mathbf{u} + \varepsilon\mathbf{h}(\vec{R},t), \tag{14}$$

The unperturbed spiral wave solution (2) to Eq. (12) has the form

$$\mathbf{U}(\vec{r}) = \exp(\mathcal{I}\vartheta)\cdot\mathbf{P}(\rho). \tag{15}$$

Here $\mathbf{P}(\rho)$ is a solution to the following boundary-value problem,

$$\mathbf{P}'' + \mathbf{P}'/\rho + [1 - \mathcal{I}\omega - (1-\mathcal{I}\alpha)(\mathbf{P}^{\mathrm{T}}\cdot\mathbf{P}) - 1/\rho^2]\cdot\mathbf{P} = \mathbf{0}, \tag{16a}$$

$$\mathbf{P}(0) = \mathbf{0}, \quad \mathbf{P}(\rho) \approx \sqrt{1-k^2}e^{k\rho\mathcal{I}+o(\rho)}\cdot\mathbf{1}, \quad \rho\to\infty, \ [10] \tag{16b}$$

where k is a nonlinear eigenvalue, and $\omega = \alpha(1-k^2)$. This problem was brought to a scalar form by substitution $\mathbf{P}(\rho) = a(\rho)\exp[\mathcal{I}\psi(\rho)]\cdot\mathbf{1}$ with real a and ψ. Solutions to this problem were studied, e.g., by Hagan [11]; they are illustrated below by Figs. 1(a) [for $a(\rho)$ and $\psi'(\rho)$] and 2(a) [for $U_1(x,y)$].

It can be seen that, due to the symmetry of Eqs. (14) and (15), the $\mathbf{C}^2$-valued RF's defined by Eq. (11) have the form

$$\mathbf{W}^{(n)}(\rho,\vartheta) = \exp[(\mathcal{I} - in)\vartheta]\cdot\mathbf{Q}^{(n)}(\rho), \quad n = 0, \pm 1. \tag{17}$$

This reduces the two-dimensional problems for $\mathbf{W}^{(n)}$ to one-dimensional problems for functions $\mathbf{Q}^{(n)}(\rho)$:

$$\mathbf{Q}^{(n)\prime\prime} + \frac{1}{\rho}\mathbf{Q}^{(n)\prime} + \left\{ 1 + \mathcal{I}\omega + \frac{(\mathcal{I}-in)^2}{\rho^2} - a^2[2(1+\mathcal{I}\alpha) + (1-\mathcal{I}\alpha)e^{2\mathcal{I}\psi}\mathcal{C}] \right\}\cdot\mathbf{Q}^{(n)} = \mathbf{0}, \tag{18a}$$

$$|\mathbf{Q}^{(n)}| < \infty, \quad \rho\to 0; \quad \mathbf{Q}^{(n)}\to\mathbf{0}, \quad \rho\to\infty. \tag{18b}$$

Method of solution and results. It can be seen that if $\mathbf{Q}^{(n)}$ tend to zero as $\rho\to\infty$, they do so exponentially, with decrement $\Lambda = \Lambda(\alpha,k)$ being the smallest positive root of the cubic equation

$$\Lambda^3 - 2(1-3k^2)\Lambda + 4k\alpha(1-k^2) = 0. \tag{19}$$

This requirement makes problems (18) formally overdetermined, as in fact they are EVP's, and that the eigenvalues are $i\omega n$ is only our expectation. To make them numerically treatable, they were reformulated as EVP's with eigenvalues $\lambda_0 \in \mathbf{R}$ for $n=0$ (temporal mode) and $i\omega + \lambda_1^r + i\lambda_1^i \in \mathbf{C}$ for $n=1$ (spatial mode), and the smallness of λ_0, λ_1^i, and λ_1^r was considered an estimation of the accuracy of the numerical procedure. The problems were brought to real scalar form by substitutions $\mathbf{Q}^{(0)} = (A + \mathcal{I}B)\cdot\exp(\mathcal{I}\psi)\cdot\mathbf{1}$ and $\mathbf{Q}^{(1)} = (C + \mathcal{I}D + iE + i\mathcal{I}F)\cdot\exp(\mathcal{I}\psi)\cdot\mathbf{1}$. The half-infinite interval

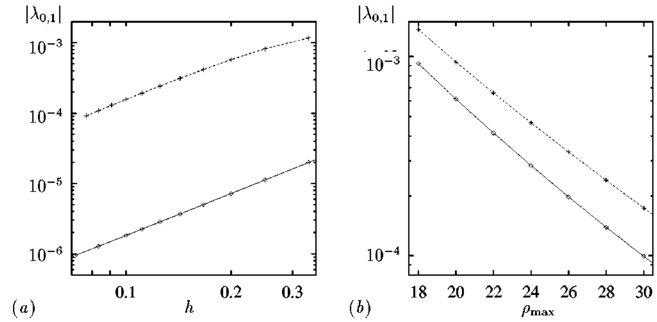

FIG. 3. The absolute values of the eigenvalues $|\lambda_0|$ (◇) and $|\lambda_1^r + i\lambda_1^i|$ (+) as functions (a) of discretization step h, at $\rho_{\max} = 100$, and (b) of cutoff radius $\rho_{\max}$, at $h = 0.05$.

$\rho\in[0,+\infty)$ was replaced by a finite interval $\rho\in[0,\rho_{\max}]$. Boundary conditions $A(0)=B(0)=C'(0)=D'(0)=0$, $E(0)=D(0)$, $F(0)=-C(0)$, and $\mathbf{Q}^{(n)\prime}(\rho_{\max}) = -\Lambda\mathbf{Q}^{(n)}(\rho_{\max})$ were posed based on conditions (18b) *via* asymptotics of acceptable solutions to the ODE system (18a). To select unique solutions of these homogeneous systems, we added conditions $B'(0)=C(0)=D(0)=1$, and normalized the solutions according to Eq. (11) afterwards. Thus posed boundary-value–eigenvalue problems have been studied in the double limit in the two numeric parameters, the cutoff radius $\rho_{\max}\to\infty$, and the discretization step $h\to 0$. The discretization was second order in h, and the solutions looked for should decrease exponentially at large ρ. Therefore, the expected behavior of the small eigenvalues is

$$\lambda_0, \lambda_1^r, \lambda_1^i = O[h^2 + \exp(-\Lambda\rho_{\max})], \quad h\to 0, \quad \rho_{\max}\to\infty. \tag{20}$$

This agrees well with the numerical results shown on Fig. 3, where the dependence on h is shown in logarithmic, and on $\rho_{\max}$ in semilogarithmic coordinates, so that the linear form of the graphs corresponds to the asymptotics (20). We consider this as a numerical proof of existence of solutions to the overdetermined problem (18). The solutions are shown in Figs. 1(b,c). Both temporal and spatial RF's do decay quickly, being essentially nonzero only in the core. The reconstructed shape of RF in the (x,y) plane is shown on Figs. 2(b)–(d). Only the first components are shown; the second components are the first ones rotated in the (x,y) plane by $\pi/2$. The behavior of the RF's at other tested values of α and β was analogous; at small $(\alpha-\beta)$, the spatial scale of all the functions grows rapidly, which is consistent with Hagan's asymptotics [11].

Conclusion. We have obtained numerically the response functions of spiral waves in the complex Ginzburg-Landau equation. As expected, these functions are localized around the core of the spiral, and decay exponentially outside it. The spatial scale of localization, Λ^{-1}, can be found analytically from Eq. (19). Unlike solitons, spiral waves look like essentially nonlocalized objects. On the other hand, their dynamic properties, determined by the RF's, are those of localized objects. This opposition between the nonlocal appearance and the infinity region of influence, on one side, and local sensitivity and independence on distant events, on the other side, makes spiral waves a very interesting example of a self-organization pattern. We believe that this physical property of localization is mathematically expressed as the existence of eigenvalues 0 and $\pm i\omega$ of the adjoint linearized operator in the space of functions integrable over the plane, and is common for all proper spiral waves in generic reaction-diffusion systems. The detailed conditions for this property is a subject for further study, and here we have shown only the first, to our knowledge, direct evidence of this viewpoint.

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[9] For calculations, it was convenient to keep treating Eq. (12) as a complex equation. The complexification of the linearized theory then leads to an algebra with two imaginary units, $\mathcal{I}$ from Eq. (12), and i of the linearized theory, with $i^2=\mathcal{I}^2=-1$ and $i\mathcal{I}=\mathcal{I}i$. It has divisors of zero, e.g., $(i+\mathcal{I})(i-\mathcal{I})=0$. We did not find in literature if it has any special name.

[10] In this model, the direction of group velocity is opposite to that of phase velocity, and so proper spiral waves, with group velocity directed outwards, look converging.

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