ENVELOPE EQUATIONS FOR MODULATED NON-CONSERVATIVE WAVES

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Abstract Modulated waves, i.e. waves that are locally plane and periodic, but at large distances and/or over long intervals of time change their characteristics, appear in many applications. An efficient way to study such waves is the method of envelope equations, when the original wave equations are replaced by equations describing the slowly varying parameters of the waves. The practical approaches to this problem are numerous; however, many of them have limitations, either in achievable accuracy, or in the wave equations to which they could apply (e.g. only conservative systems), or both. In this paper we discuss an approach of this kind, which appear to be free from these disadvantages. This approach is illustrated for *autowaves*, which, in the author's opinion, should play the same role in the theory of waves, as auto-oscillations=limit cycles play in the theory of oscillations: as the basic, least degenerate type of solutions.

1. INTRODUCTION.

When differential equations, describing a natural or technological process, are too complicated to be solved exactly, one needs to do it approximately. This can be done either numerically, or analytically, e.g. by using some asymptotic methods. Approximate methods may not only serve for pure purposes of calculation, but also be an "instrument of understanding" of complex systems.

The simplest case is if the right-hand sides of the differential equations contain small parameters, and the system becomes much simpler, e.g. can be treated exactly, if these parameters are equal to zero. Then for nonzero but small values of the parameters, solutions can be obtained by perturbation techniques.

The perturbation technique may also be applied to systems without any small parameters. This may be the case if we are interested in solutions from a special class, depending on such parameters. A well known example is small-amplitude oscillations in a nonlinear system. By scaling the dynamic variables to normalise the amplitude of oscillations, one can bring this problem to a problem explicitly depending on the small parameter, which becomes linear if this small parameter is equated to zero.

There is a less trivial example of the same kind. This is the famous geometric optics approximation. The classical interpretation of geometric optics is that consideration is restricted to solutions with wavelengths small in comparison with other characteristic scales of the problem [1]. This works well with classical, linear wave equations, which admit wave solutions with arbitrarily short wavelengths. In general, this restriction may be impractical, e.g. if wave solution may not exist with wavelengths less than a certain minimum. Then the idea of the geometric optics is reformulated as the idea of modulated waves[2], or slowly varying waves [3]: the characteristic sizes of the problem, in particular of the initial conditions for the equations, should be much larger than a typical wavelength. This means that in relatively small regions, the waves are close to plane and periodic, but the parameters of these waves, including direction of propagation and the period, slowly change in time and/or in space, becoming significantly different at large distances and/or after long time intervals.

The classical geometric optics approximation for linear wave equations heavily relies on the specific properties of these equation, e.g. the superposition principle. This is no good for nonlinear waves. A well known method for nonlinear waves is the Whitham procedure (modulation theory) [2, 4]. One form of the method uses knowledge of conservation laws, available for many wave systems originating from physics, and derives the evolution equations for slowly varying parameters from these conservation laws. Another form of the method also uses the properties of physical origin, namely, the fact that the field equations can be written in the form of a Lagrange variational principle. The evolution equations are then derived also from a Lagrangian principle, where the averaged Lagrangian is rewritten as a function of the new independent variables describing the slowly-varying solutions. This method has been applied to many classical nonlinear equations, such as nonlinear Klein-Gordon equation, Korteweg-deVries equation and others. Recent development in this direction can be found e.g. in [5].

2. THE PROBLEM.

There is an important class of nonlinear waves, for which Whitham's approach can not be applied — so called *autowaves*. Speaking physically, these are waves that propagate unchanged not because there is no dissipation, but because the dissipation is compensated by the constant supply of energy. Examples are electric pulses in nerve and heart tissues, and waves in some chemical reactions where the consumed reagents are either supplied, or are stored in substantial amounts so that their decrease during the wave period is negligible. Mathematically, such systems are most often described by systems of partial differential equation of reaction-diffusion type,

$$\partial u/\partial t = D\Delta u + f(u). \tag{1.1}$$

Here $u = u(x,t) \in \mathbb{R}^\ell$ is a column-vector of concentrations of the reagents, $f(u) \in \mathbb{R}^\ell$ is a column-vector of nonlinear reaction terms (interesting behaviour starts from $\ell \geq 2$), D is an $\ell \times \ell$ matrix of diffusion coefficients of the reagents, which we assume symmetric; the space coordinates $x \in \mathbb{R}^n$ where the dimension of physical space, n, can be equal to 1, 2 or 3, and Δ is the Laplacian in $\mathbb{R}^n$.

Systems of the form (1.1), describing real autowave systems, do not have any conserved quantities at all, and, in particular, are not Hamiltonian. The simplest autowave solutions of (1.1) are the plane periodic waves,

$$u(x,t) = U\left((k,x) - \omega(k^2)t + \phi_0, k^2\right) = U(\xi,\eta), \tag{1.2}$$

$$\text{where} \qquad \xi = (k,x) - \omega(k^2)t + \phi_0, \quad \eta = k^2, \quad k \in \mathbb{R}^n \tag{1.3}$$

$$\text{and} \qquad \eta D U_{\xi\xi}(\xi,\eta) + \omega(\eta)U_\xi(\xi,\eta) + f(U) = 0. \tag{1.4}$$

Here k is the wavevector, $0 \leq \eta_1 \leq k^2 \leq \eta_2 < \infty$, ω is the frequency, and ϕ_0 is an arbitrary initial phase. Brackets $(,)$ denote the scalar product in the physical space $\mathbb{R}^n$.

Modulated autowaves (see fig. 1) are solutions of the form

$$u(x,t) = U\left(\phi, (\nabla\phi)^2\right) + v(x,t), \tag{1.5}$$

where $\nabla\phi$ is a local wavevector slowly varying in space and time, and v is a small correction, so that

$$\phi = \epsilon^{-1}\Phi(\epsilon x, \epsilon t, \epsilon), \qquad \epsilon \ll 1, v \ll 1. \tag{1.6}$$

The question is, what conditions should the phase ("eikonal") variable $\phi(x,t)$ satisfy. It happens, that the condition can be written in the form of a partial differential equation, the *evolution equation*, which will be the analogue of the eikonal equation in the geometric optics. Our purpose is to develop a method of derivation of this equation for every given reaction-diffusion system (1.1). One thing we should always bear in mind: as a rule, this *cannot be done entirely analytically*, there usually will be some bits to do numerically. One reason for that is quite obvious: the basic solutions (1.2) can not be found analytically.

3. HEURISTIC DERIVATION OF THE EVOLUTION EQUATION

By substituting (1.5) into (1.1), we get a ℓ-component vector equation for two unknown functions: vector function $v \in \mathbb{R}^\ell$ and scalar function ϕ. So, the equation is under-determined. In linear approximation in v it takes the form

$$v_t(x,t) = D\nabla^2 v(x,t) + F[\phi(x,t)]v(x,t) + h[\phi(x,t)], \tag{1.7}$$

where the Jacobian matrix F and the free term h depend on the unknown $\phi(x,t)$ so that

$$F[\phi] = \left.\frac{\partial f(u)}{\partial u}\right|_{u=U}, \qquad \text{and} \tag{1.8}$$

$$\begin{aligned} h[\phi] \;=\; & -\left(\omega((\nabla\phi)^2) + \phi_t\right) U_\xi - U_\eta \frac{\partial}{\partial t}(\nabla\phi)^2 \\ & + D\Big(U_\eta \nabla^2(\nabla\phi)^2 + 2U_{\xi\eta}\left(\nabla\phi, \nabla\left((\nabla\phi)^2\right)\right) \\ & + U_\xi \nabla^2\phi + U_{\eta\eta}\left(\nabla((\nabla\phi)^2)\right)^2\Big). \end{aligned} \tag{1.9}$$

Here the function U is assumed with arguments $U = U(\phi, (\nabla\phi)^2)$.

The requirement that v is small leads to certain restrictions on the function ϕ. These restrictions are just the desired evolution equation.

Since the local wavevector $k = \nabla\phi$ varies slowly, the coefficients F and the free term h are approximately periodic functions of only one variable $\xi = \phi(x,t) \approx (k,x) - \omega(k^2)t$ (locally). Then the boundedness of solutions v of (1.7) at large times requires, via the Fredholm alternative,

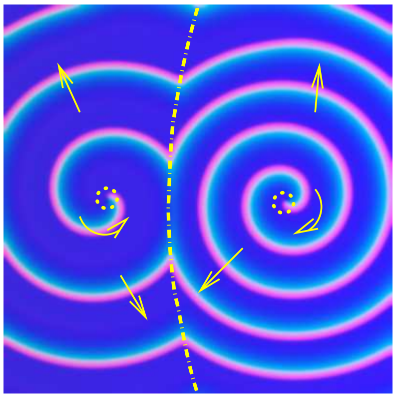

Figure 1: Modulated waves in a reaction+diffusion system. Notice the cores of the spiral waves, and the shock structure between the spiral wave domains. Everything else is modulated waves.

that

$$\oint \langle W_0(\xi,\eta), h[\phi]\rangle \, \mathrm{d}\xi = 0, \tag{1.10}$$

where $W_0(\xi,\eta) \in \mathbb{R}^\ell$ is the 0-eigenfunction of the adjoint $\mathcal{L}^+$ of the linearised operator $\mathcal{L}$,

$$\begin{aligned} \mathcal{L}v &= \eta D v_{\xi\xi} + D\Delta_y v + \omega v_\xi + F(\xi)v, \\ \mathcal{L}^+ w &= \eta D w_{\xi\xi} + D\Delta_y w - \omega w_\xi + F^T(\xi)w. \end{aligned} \tag{1.11}$$

Now we substitute (1.9) into condition (1.10). The resulting equality does not depend x and t explicitly, but only on time and space derivatives of ϕ, and so can be viewed as the required evolution equation. It makes sense if the derivatives of ϕ are no longer considered constant, but allowed to slowly vary in time and space, i.e. "unfrozen". Generically, this equation will contain a term with $\frac{\partial}{\partial t}(\nabla\phi)^2$. However, the coefficient at that term can be made to disappear, by the following transformation. Notice that if $U(\xi,\eta)$ is a base wave solution of (1.4), then $U(\xi+\chi(\eta),\eta)$ is also a solution, for arbitrary χ. This function $\chi(\eta)$ can be chosen in such a way that $\oint \langle W_0(\xi,\eta), DU_{\xi\eta}(\xi,\eta)\rangle \, \mathrm{d}\xi = 0$, see [6, 7] for details. Ultimately, we have

$$\phi_t + \omega((\nabla\phi)^2) = P\left((\nabla\phi)^2\right)\nabla^2\phi + Q\left((\nabla\phi)^2\right)(\nabla\phi\nabla)(\nabla\phi)^2 \tag{1.12}$$

where

$$\begin{aligned} P(\eta) &= \oint \langle W_0(\xi,\eta), DU_\xi(\xi,\eta)\rangle \, \mathrm{d}\xi, \\ Q(\eta) &= 2\oint \langle W_0(\xi,\eta), DU_{\xi\eta}(\xi,\eta)\rangle \, \mathrm{d}\xi. \end{aligned} \tag{1.13}$$

The key points of this heuristic derivation are elimination of the secular growth (Fredholm alternative), and "freezing" and "unfreezing" the derivatives of ϕ. The latter procedure is potentially flawed. E.g., the ignored non-periodicity of F and h leads to an error of the order of ϵ even at a distance of one period, though the final equation (1.12) keeps the terms $\mathcal{O}(\epsilon)$ preserved. Thus, more sophisticated tools are required.

4. PERTURBATION OF A MANIFOLD OF STABLE EQUILIBRIA

The ideas of the more accurate asymptotic approaches can be introduced using a finite dimensional example. Consider a perturbed system of ODEs

$$u_t = f(u) + \epsilon h(u), \quad u \in \mathbb{R}^n \tag{1.14}$$

and assume that at $\epsilon = 0$ it has an m-dimensional attracting manifold of equilibria $U(a)$ with coordinates $a \in A \subset \mathbb{R}^m$, $m < n$, so that $f(U(a)) = 0$, $\forall a \in A$.

For $\epsilon \neq 0$, there is an invariant manifold in the vicinity of U, with a slow dynamics on it:

$$u = U(a(t)) + \epsilon v(t) \tag{1.15}$$

Here $v = \mathcal{O}(1)$, and a, v are ϵ-slowly varying functions, i.e. they depend upon t only via the combination ϵt, and $v(t)$ can be found as a functional of a in all orders of ϵ. To make (1.15) unambiguous, we require that $v(t)$ is always orthogonal to U at the point $a(t)$, say

$$\langle W_j(a), v\rangle = 0, \quad j = 1..m, \tag{1.16}$$

where W_j are eigenvectors of the transposed Jacobian matrix, $F^T(a)W_j(a) = \overline{\lambda_j}(a)W_j(a)$, biorthogonal $(\big\langle W_j(a), V_{j'}(a)\big\rangle = \delta_{j,j'})$ to the eigenvectors of the Jacobian $F = \partial f(U(a))/\partial u$ itself, including the tangent vectors to the stationary manifold $V_j(a) = \partial U(a)/\partial a_j$, corresponding to $\lambda_j(a) = 0, j = 1\ldots m$.

Substitution of the Ansatz (1.15) into (1.14), introducing notation for the flow on the slow manifold, $\dot{a}_j = \epsilon\mathcal{G}_j$, $\mathcal{G} = \mathcal{O}(1)$ and expanding $v(t)$ in the eigenvector basis, $v(t) = \sum_j^n V_j(a(t))v_j(t)$, $v_j \in \mathbb{R}$, leads to

$$\begin{aligned}\dot{v}_j &= \lambda_j(a)v_j + (h_j(a) - \mathcal{G}_j(t)) \\ &\quad + \epsilon\left\{\sum_k^n h_{jk}v_k + \sum_{k,l}^n (K_{jkl}\mathcal{G}_k v_l + f_{jkl}v_k v_l)\right\} + \mathcal{O}(\epsilon^2)\end{aligned} \tag{1.17}$$

where $\mathcal{G}_j \equiv 0$ for $j > m$, h_{jk}, f_{jkl} etc. are Taylor coefficients of the functions h and f, and $K_{jkl} = \langle \partial W_j(a)/\partial a_k, V_l(a)\rangle = -\langle W_j, \partial V_l(a)/\partial a_k\rangle$. Imposing now conditions of orthogonality (1.16), $v_j(t) \equiv 0, \quad j = 1\ldots m$, we determine iteratively $a_j(t)$ and $v_j(t)$ with higher and higher precision, by considering alternately (1.17) for $j = m+1\ldots n$ as equations for v_j, and $j = 1\ldots m$ as equations for $\mathcal{G}_j$. E.g., in the second order we get

$$\begin{aligned}\dot{a}_j &= \epsilon h_j + \epsilon^2\left\{-\sum_k h_{jk}h_k/\lambda_k + \sum_{k,l}\Big(K_{jkl}h_k h_l/\lambda_l + f_{jkl}h_k h_l/(\lambda_k\lambda_l)\Big)\right\} \\ &\quad + \mathcal{O}(\epsilon^3).\end{aligned} \tag{1.18}$$

We see, that this iterative procedure yields successively more and more precise evolution equations. The significant feature of the procedure is that their solutions approximate exact solutions not only with successively decreasing (in asymptotic sense) error, but also become valid at successively growing time scales. This becomes possible only by *keeping terms of different order in the same equation.*

5. SUBCENTER MANIFOLD EXPANSION

The described method is associated with the method of the (sub)centre manifold, which is both a fundamentally important theoretical concept and an efficient practical tool.

The procedure of alternating increase of asymptotic precision of the $\mathcal{G}(a)$ and $v(a)$ leads to building asymptotic series in ϵ for these functions. If these series converge for some ϵ, this means that for each of those ϵ we will have an invariant manifold

$$\mathcal{U}(a, \epsilon) = U(a) + \epsilon v(a, \epsilon) \tag{1.19}$$

and the flow on that manifold defined by

$$\dot{a} = \mathcal{G}(a, \epsilon). \tag{1.20}$$

The unique dependence of v on a arises when we select a unique solution to the differential equation, say the one that remains finite for $t \to -\infty$.

In the extended phase space $\mathbb{R}^n \times \mathbb{R} = \{u, \epsilon\}$, manifold U is an intersection of the manifold $\mathcal{U}$ by the hyperplane $\epsilon = 0$. It be easily seen that the difference between $\mathcal{U}$ and U at small ϵ is along V_0, which means that $\mathcal{U}$ is tangent to the centre subspace of U. Thus it is a centre manifold or a sub-centre manifold; the technical difference is not important for us here, as all we use is the formalism.

This motivates an alternative approach to building the asymptotic evolution equation: from the very beginning, to look for representations of (1.19) and (1.20) in the form of power series in ϵ straightaway, instead of coming to (1.19) via the complicated procedure described above. This certainly is a very efficient method from a practical viewpoint.

And as far as actual calculations rather than their motivation in this finite-dimensional example are concerned, the subcenter manifold approach and the method described in Section 4 are strikingly similar. Method of Section 4, modified for a PDE problem in the form of the "method of the detecting operator", has been applied for the problem of the modulated strongly nonlinear waves, including autowaves in reaction-diffusion system (1.1) and conservative waves in the nonlinear Klein-Gordon equation, in [7]. Here we treat the modulated autowaves using the formalism of the subcenter manifold. This is based on the idea of [8] originally applied to another class of problems (nearly linear waves).

6. FUNCTIONAL SUBCENTER MANIFOLD OF MODULATED WAVES

We consider modulated waves defined by (1.5) as a "functional manifold". Thus, the whole phase distribution $\phi(x)$, in its entirety, is a coordinate on this manifold.

We consider the starting PDE system (1.1) as an ordinary differential equation in a Banach space $\mathcal{Y}$,

$$\frac{\mathrm{d}\hat{u}}{\mathrm{d}t} = \hat{f}(\hat{u})$$

where $\hat{u} : \mathbb{R} \to \mathcal{Y}; \quad t \mapsto u(x,t)$. The Ansatz is

$$u(x,t) = \mathcal{U}[\phi(x,t)], \tag{1.21}$$

where square brackets [] denote functional dependence, i.e. $[\phi]$ denotes dependence on ϕ and *all its spatial derivatives*, $\mathcal{U}(\phi, \partial\phi/\partial x_i, \partial^2\phi/\partial x_i \partial x_j, \dots)$. In Banach-vector form,

$$\hat{u}(t) = \hat{\mathcal{U}}(\hat{\phi}(t)), \tag{1.22}$$

where $\hat{\phi}$ is a vector of a "smaller" functional space $\mathcal{X}$, representing the spatial distribution of the phase $\phi(x,t)$ at a particular time instant t: $\hat{\phi} : \mathbb{R} \to \mathcal{X}; \quad t \mapsto \phi(x,t)$, and $\hat{\mathcal{U}} : \mathcal{X} \to \mathcal{Y}$.

Space $\mathcal{X}$ is "smaller" than $\mathcal{Y}$, e.g. in the sense that it consists of scalar functions rather than $\mathbb{R}^\ell$-valued functions as $\mathcal{Y}$. Another (non-formalised) difference is that $\hat{\phi}$ represents functions $\phi(x,t)$ with slowly varying spatial derivatives.

Evolution of the phase $\phi(x,t)$ is sought in the form

$$\frac{\partial \phi}{\partial t} = \mathcal{G}[\phi(x,t)] \quad \text{or} \quad \frac{\mathrm{d}\hat{\phi}}{\mathrm{d}t} = \hat{\mathcal{G}}(\hat{\phi}(t)) \tag{1.23}$$

where $\hat{\mathcal{G}} : \mathcal{X} \to \mathcal{X}$ (assuming $\mathcal{X}$ is linear).

Then, the operators $\hat{\mathcal{U}}$ and $\hat{\mathcal{G}}$ are sought in the form of formal power series in the small parameter ϵ,

$$\mathcal{U} = \sum_{n=0}^{\infty} \mathcal{U}^{(n)} \epsilon^n, \ \mathcal{G} = \sum_{n=0}^{\infty} \mathcal{G}^{(n)} \epsilon^n, \qquad \text{or} \quad \hat{\mathcal{U}} = \sum_{n=0}^{\infty} \hat{\mathcal{U}}^{(n)} \epsilon^n, \ \hat{\mathcal{G}} = \sum_{n=0}^{\infty} \hat{\mathcal{G}}^{(n)} \epsilon^n. \tag{1.24}$$

Then we use the finite-dimensional procedure discussed above, expressed in terms of $\hat{\mathcal{U}}$ and $\hat{\mathcal{G}}$, as a guidance, but immediately translate obtained expressions to the straightforward form, without hats. The orbit derivative of (1.21) by the system (1.23) yields

$$\partial u/\partial t = \mathcal{U}_\phi \mathcal{G} + \mathcal{U}_{\nabla\phi} \nabla \mathcal{G} + \mathcal{O}(\epsilon^2) \tag{1.25}$$

since, as we mentioned, $\mathcal{U}$ is function of ϕ and all its derivatives, and derivatives of higher orders are, according to (1.6), of higher asymptotic orders in ϵ.

In turn, differentiation of $\mathcal{U}$ by spatial variables, using the chain rule, yields

$$\partial\mathcal{U}/\partial x_i = \mathcal{U}_\phi \phi_i + \mathcal{U}_{\phi_j}\phi_{ij} + \mathcal{O}(\epsilon^2). \tag{1.26}$$

Here and below, we denote spatial derivatives of ϕ by subscripts corresponding to the spatial coordinates, so ϕ_i means $\partial\phi/\partial x_i$ etc.; and assume summation by repeated indices.

Now, substitution of spatial and (1.25) into (1.1), with account of expansions (1.24), yields

$$\begin{aligned} &\mathcal{U}_\phi^{(0)}\mathcal{G}^{(0)} + \mathcal{U}_\phi^{(1)}\mathcal{G}^{(0)} + \mathcal{U}_\phi^{(0)}\mathcal{G}^{(1)} + \mathcal{U}_{\phi_i}^{(0)}\mathcal{G}_{\phi_j}^{(0)}\phi_{ij} + O(\epsilon^2) \\ &= f(\mathcal{U}^{(0)}) + F(\mathcal{U}^{(0)})\mathcal{U}^{(1)} \\ &+ D\left(\mathcal{U}_{\phi\phi}^{(0)}\phi_i\phi_i + 2\mathcal{U}_{\phi\phi_j}^{(0)}\phi_i\phi_{ij} + \mathcal{U}_\phi^{(0)}\phi_{ii} + \mathcal{U}_{\phi\phi}^{(1)}\phi_i\phi_i\right) + \mathcal{O}(\epsilon^2). \end{aligned} \tag{1.27}$$

Then we consider sequentially different orders in ϵ of this equation.

Order $\mathcal{O}(1)$. Equating terms of (1.27) of the order $\mathcal{O}(1)$ we have

$$-\mathcal{U}_{\phi^{(0)}}\mathcal{G}^{(0)} = f(\mathcal{U}^{(0)}) + D\mathcal{U}_{\phi\phi}^{(0)}\phi_i\phi_i. \tag{1.28}$$

This coincides with the equation (1.4) for the basic waves if $\mathcal{U}^{(0)} = U$, $\mathcal{G}^{(0)} = -\omega$, and $\phi_i\phi_i$ is identified with η. Together with the requirement of periodicity in ϕ, this order provides a nonlinear eigenvalue problem determining the basic solution $\mathcal{U}^{(0)} = U$ and the main frequency $-\mathcal{G}^{(0)} = \omega$ as functions of the local value of the slowly varying phase gradient $\eta = \phi_i\phi_i = (\nabla\phi)^2$.

Order $\mathcal{O}(\epsilon)$. This order gives a linear equation for $\mathcal{U}^{(1)}$:

$$\mathcal{L}\mathcal{U}^{(1)} = h, \tag{1.29}$$

where the linear operator $\mathcal{L}$ is

$$\mathcal{L} = \eta D\partial_{\phi\phi} + \omega(\eta)\partial_\phi + F \tag{1.30}$$

and the free term h depends on $\mathcal{G}^{(1)}$,

$$h = \mathcal{G}^{(1)}\mathcal{U}_\phi^{(0)} + \mathcal{U}_{\phi_i}^{(0)}\mathcal{G}_{\phi_j}^{(0)} - D\mathcal{U}_{\phi\phi}^{(0)}\phi_{ii} - 2D\mathcal{U}_{\phi\phi_j}^{(0)}\phi_i\phi_{ij}. \tag{1.31}$$

This is a differential problem in ϕ and as such is exactly 2π-periodic. Operator $\mathcal{L}$ is singular, its zero eigenfunction is the Goldstone mode $V_0 =$

$\partial U/\partial\phi$. By Fredholm, (1.29) is solvable iff the free term is orthogonal to the zero-eigenfunction W_0, of the adjoint operator $\mathcal{L}^+$,

$$\mathcal{L}^+ = \eta D\partial_\phi^2 - \omega(\eta)\partial_\phi + F^T. \tag{1.32}$$

This leads to the following result:

$$\mathcal{G}^{(1)} = \left(\oint \left\langle W_0, D\mathcal{U}_\phi^{(0)}\right\rangle \mathrm{d}\phi\right)\phi_{ii} + 2\left(\oint \left\langle W_0, D\mathcal{U}_{\phi\phi_j}^{(0)}\right\rangle \mathrm{d}\phi\right)\phi_i\phi_{ij}. \tag{1.33}$$

Noting that $\mathcal{U}^{(0)}$ depends on ϕ only via $\eta = \phi_i^2$, we arrive ultimately to the evolution equation precisely coinciding, up to the notation difference, with (1.12), with identical definitions of P and Q.

7. DISCUSSION

In this paper, we describe a new method of derivation of the envelope equations for strongly nonlinear, non-conservative waves. The method is illustrated for the generic reaction-diffusion system (1.1), without any assumptions on the exact form of the reaction term $f(u)$ or diffusion matrix D. It has been demonstrated that the detecting operator technique, closely related to the sub-centre manifold technique, is equally applicable classical conservative nonlinear wave equations, such as the nonlinear Klein-Gordon equation, and produces the same results[7], so we may expect the present method to be similarly flexible. The detecting operator technique described in [7] is rather involved, and construction of the detecting operator itself is a non-algorithmic part of the procedure. The method described here is free from that disadvantage and provides a much easier, algorithmic way, producing the same results.

References

[1] L. D. Landau and E. M. Lifshitz. *The Classical Theory of Fields*, volume 2 of *Course of Theoretical Physics*. Pergamon Press, Oxford, 197.

[2] G. B. Whitham. *Linear and Nonlinear Waves*. Wiley, New York, 1974.

[3] L. N. Howard and N. Kopell. Slowly varying waves and shock structures in reaction-diffusion equations. *Stud. Appl. Math.*, 56(2):95–146, 1977.

[4] P. L. Bhatnagar. *Nonlinear Waves in One-Dimensional Dispersive Systems*. Clarendon Press, Oxford, 1979.

[5] A. M. Kamchatnov "New approach to periodic solutions of integrable equations and nonlinear theory of modulational instability", *Physics Reports*, 286:199-270, 1997.

[6] V. N. Biktashev. Diffusion of autowaves. *Physica D*, 40:83–90, 1989.

[7] V. N. Biktashev. On the geometrical optics for nonlinear waves. In L. Debnath, editor, *Nonlinear Dispersive Wave Systems*, pages 481–506. World Scientific, Singapore, 1992.

[8] A. J. Roberts. A sub-centre manifold description of the evolution and interaction of nonlinear dispersive waves. In L. Debnath, editor, *Nonlinear Dispersive Wave Systems*, pages 127–155. World Scientific, Singapore, 1992.