Drift of large-core spiral waves in inhomogeneous excitable media

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Abstract. Spiral waves in excitable media may drift due to interaction with medium inhomogeneities. We describe this drift asymptotically, within the kinematic (eikonal) approximation.

Key words: kinematic approximation. eikonal, asymptotics, inhomogeneity, drift, spiral wave, excitable medium, autowave, reaction-diffusion system.

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1. Introduction

One of the most typical two-dimensional patterns in autowave media is a spiral wave, which has been observed *e.g.* in the Belousov-Zhabotinsky reaction [1, 2] cardiac tissue [3], social microorganisms [4], neural tissue [5] and heterogeneous catalytic chemical reactions [6]. Spiral waves of excitation in heart tissue underlie certain types of cardiac arrhythmias [7].

In a perfectly homogeneous medium, the spiral wave may rigidly rotate around a fixed center determined by initial conditions. Real media are often inhomogeneous. This usually leads to variation of rotation frequency and drift of the rotation center [8]. Understanding of this drift is important, for the prediction and control of spiral wave behaviour.

The mathematical description of autowave media is usually in terms of nonlinear partial differential equations. Description of the dynamics of spiral waves or there 3D analogues, scroll waves, can be done by perturbation techniques, assuming that the unperturbed spiral wave solution, the *free spiral*, is known [9, 10]. This theory has a heuristic value, *e.g.* it predicts that the drift of the spiral is governed by 'Aristotelean' motion equations: the velocity is proportional to the 'force' caused by inhomogeneity or other perturbation, and this force, in a first approximation, is a linear convolution-type functional of this perturbation. The kernels of convolution integrals, the *response functions*, can be found numerically [11].

There are limiting cases where some results can be achieved analytically. The 'kinematic approach' [12, 13] applies to an important class of *excitation waves*, which include the waves in heart tissue. If the excitation waves are rare and locally nearly planar, then the description of a wave is reduced to the description of a curve, the crest line of the wave, and motion of this curve is determined by its local geometry. For brevity, we call this line just the *wave*. Spiral wave is a broken wave, so we need also to describe the motion of its free end, the *tip*. Such motion equations have been proposed, in [12] from semi-phenomenological and in [13] from asymptotical considerations; these two versions differ in some details.

In this paper, we study the drift of spiral waves caused by inhomogeneity of the medium properties, in terms of the kinematic approach as formulated in [13]. Equations of motion of the wave and the tip in inhomogeneous media can be derived asymptotically, as for homogeneous media; this leads to spatial dependence of the coefficients, and also to additional terms, that are proportional to spatial derivatives of the medium properties. In this paper, to reveal the key features of interaction with inhomogeneity and to simplify formulas, we neglect the spatial derivatives and restrict consideration only to spatial dependence of the coefficients.

To describe the drift, we use the 'adiabatic' perturbation technique proposed in [14]. We consider the spiral wave in the comoving frame of reference (CMF), moving together with the center of the spiral wave. So, this center is always at the origin of the CMF. This leads to a problem with non-stationary coefficients, even for a stationary medium. As the drift is expected to be slow, the instant distribution of medium properties is considered, in the first approximation, as stationary. The solution in the CMF with frozen coefficients should be periodic in time. This requirement yields additional conditions, which allows the determination of the velocity of the CMF as a functional of the instantaneous distribution of the medium properties, *i.e.* of the current location of the spiral in the laboratory frame of reference.

Though obtained by a different technique, the present results are in agreement with the general theory of [10]. Namely, the drift velocity and frequency shift are linear convolution-type functionals of the perturbations of medium properties. This gives explicit analytical expressions for the response functions with respect to all varied medium parameters.

We conclude by considering examples of media with linear and with stepwise gradients of parameters.

2. Kinematic approach

The kinematic approach is based on the equation for the normal velocity of wave propagation, v_n, and curvature of the wave, K, [15, 16, 12]:

$$v_n(s,t) = V - DK(s,t), \tag{1}$$

where K is considered positive if the wave is convex in the direction of propagation, s is arclength along the wave measured from the tip, and V and D are parameters of the medium.

As mentioned in the Introduction, we also need motion equations for the wave tip. Let $G(t)$ be the tangential ("growth") component of tip velocity. Then quantities $G(t)$, $K_0(t) = K(0,t)$, $K_0'(t) = \partial_s K(0,t)$ and $\partial_t \alpha(t)$ are related by equations

$$\begin{aligned} &\nu_0 + \nu_1 K_0 - G = 0; \\ &-\nu_2 + \nu_3 K_0 + \nu_4 K_0' + \nu_5 K_0^2 + \partial_t \alpha = 0; \\ &-(\nu_1 + \nu_5) K_0^2 - (\nu_0 + \nu_3) K_0 + (\bar{D} - \nu_4) K_0' + \nu_2 = 0 \end{aligned} \tag{2}$$

where ν_{0-5} are medium parameters [13].

We suppose now that in an inhomogeneous medium, all the kinematic parameters depend on space,

$$
\begin{aligned}
&V=\bar{V}(1+\tilde{V}(\vec{r})), \quad D=\bar{D}(1+\tilde{D}(\vec{r})),\\
&\nu_0=\bar{V}\bar{\nu}_0(1+\tilde{\nu}_0(\vec{r})), \quad \nu_1=\bar{D}\bar{\nu}_1(1+\tilde{\nu}_1(\vec{r})), \quad \nu_2=\bar{V}^2\bar{\nu}_2(1+\tilde{\nu}_2(\vec{r}))/\bar{D},\\
&\nu_3=\bar{V}\bar{\nu}_3(1+\tilde{\nu}_3(\vec{r})), \quad \nu_4=\bar{D}\bar{\nu}_4(1+\tilde{\nu}_4(\vec{r})), \quad \nu_5=\bar{D}\bar{\nu}_5(1+\tilde{\nu}_5(\vec{r})).
\end{aligned}
\tag{3}
$$

where $\tilde{V}$, $\tilde{D}$ and ν_{0-5} are small and localised functions of spatial coordinates $\vec{r}$.

3. Adiabatic approach

Let $\vec{r}(s,t)$ be the radius-vector of a wave point s at time t, and $\tau(\vec{r})$ be the *eikonal*, a multivalued function, with the values being the time instants when the wave has visited point $\vec{r}$. So, $\forall t, s$, $\{\tau(\vec{r}(s,t))\} \ni t$. We suppose that in the CMF our solution is approximately periodic in time and close to the free spiral. In this frame of reference, the perturbations $\tilde{V}$ and $\tilde{D}$ depend on time. However, as the drift is slow, the instantaneous distribution of the medium properties can be considered as stationary. Denote the instantaneous velocity of the CMF by $\vec{c}$ and the period of the spiral by τ_0.

In the CMF, the scaling of variables

$$
\vec{R}=(\bar{V}/\bar{D})\vec{r}; \quad T(\vec{R})=(\bar{V}^2/\bar{D})\tau(\vec{r}); \quad \vec{C}=\bar{V}\vec{c} \tag{4}
$$

bring equation (1) to

$$
(1+\tilde{V})|\nabla T|^3-(1+\tilde{D})\left[(\nabla T)^2\nabla^2 T-\frac{1}{2}(\nabla T\nabla)(\nabla T)^2\right]=(\nabla T)^2\left[1+\langle\vec{C},\nabla T\rangle\right], \tag{5}
$$

where ∇ is now the gradient over R, and values of the scaled eikonal T are equidistant in $T_0=\bar{V}^2\tau_0/\bar{D}$.

Scaling (4) brings equations at the tip to

$$
\begin{aligned}
0 &= -[\bar{\nu}_0(1+\tilde{\nu}_0)+\bar{\nu}_3(1+\tilde{\nu}_3)]p\kappa-[\bar{\nu}_1(1+\tilde{\nu}_1)+\bar{\nu}_5(1+\tilde{\nu}_5)](p\kappa)^2\\
&\quad +\bar{\nu}_2(1+\tilde{\nu}_2)+[1+\tilde{D}-\bar{\nu}_4(1+\tilde{\nu}_4)]\sigma\kappa';\\
\dot{\alpha} &= \bar{\nu}_1(1+\tilde{\nu}_1)(p\kappa)^2+\bar{\nu}_0(1+\tilde{\nu}_0)p\kappa-(1+\tilde{D})\sigma\kappa';\\
\dot{X}_0 &= -[1+\tilde{V}-(1+\tilde{D})p\kappa]\sin\alpha\\
&\quad -[\bar{\nu}_0(1+\tilde{\nu}_0)+\bar{\nu}_1(1+\tilde{\nu}_1)p\kappa]\cos\alpha-C_x;\\
\dot{Y}_0 &= [1+\tilde{V}-(1+\tilde{D})p\kappa]\cos\alpha\\
&\quad -[\bar{\nu}_0(1+\tilde{\nu}_0)+\bar{\nu}_1(1+\tilde{\nu}_1)p\kappa]\sin\alpha-C_y;
\end{aligned}
\tag{6}
$$

$$\langle \nabla T, \vec{t} \rangle = 0;$$
$$\langle \nabla T, \vec{n} \rangle = \left[1 + \tilde{V} - p(1+\tilde{D})\kappa\right]^{-1}$$

where X_0 and Y_0 are scaled tip coordinates, the dot stands for the derivative with respect to the scaled time $\bar{V}^2 t/\bar{D}$, p and σ are dimensionless parameters of the unperturbed solution:

$$p = \bar{D}\bar{K}_0/\bar{V}, \quad \sigma = p^2 \bar{K}_0'/\bar{K}_0^2, \tag{7}$$
$$\kappa = K_0/\bar{K}_0, \quad \kappa' = K_0'/\bar{K}_0', \tag{8}$$

$\bar{K}_0$, $\bar{K}_0'$ are the curvature and its derivative at the tip for the free spiral, $\vec{t}$ is the tangent and $\vec{n}$ is the normal unit vectors. We consider $\vec{t}$ and $\vec{n}$ and the shape of the core quasi-stationary in the CMF, similarly to $\tilde{V}$ and $\tilde{D}$.

Imposing requirements of boundedness of $|\nabla T|$ at large ρ and periodicity of X_0, Y_0 and α onto (5), (6), we obtain problem for three unknowns: the function $T(\vec{R})$, describing shape of the spiral wave, the scalar T_0, the rotation period of spiral wave, and the vector $\vec{C}$, corresponding to the drift velocity.

We present a solution to this problem below.

4. The free spiral

We begin with the description of the free spiral, *i.e.* the spiral wave solution in the homogeneous medium. We look for solution of Eq. (5) at $\tilde{V} = \tilde{D} = 0$ and $\vec{C} = 0$ in the form of a stationary, counterclockwise rotating wave with angular velocity ω_0, which in polar coordinates (ρ, θ) is

$$T(\rho, \theta) = \bar{T} \equiv \frac{1}{\omega_0}(\theta + \psi_0(\rho)); \qquad \psi_0' \geq 0. \tag{9}$$

Substitution

$$\chi = \rho \psi_0'(\rho) \tag{10}$$

with (9) brings (5) to the form

$$\chi' - (\chi^2 + 1)^{3/2} + (\chi/\rho + \omega_0 \rho)(\chi^2 + 1) = 0, \tag{11}$$

and boundary conditions (6) become

$$R_0\omega_0 = \left[(1-p)^2 + (\bar{\nu}_0 + \bar{\nu}_1 p)^2\right]^{1/2},$$
$$\alpha = \omega_0 t - \beta,$$
$$\tan\beta = (\bar{\nu}_0 + \bar{\nu}_1 p)/(1-p), \tag{12}$$

$$\omega_0 = \bar{\nu}_2 - \bar{\nu}_3 p - \bar{\nu}_4\sigma - \bar{\nu}_5 p^2,$$
$$-(\bar{\nu}_1 + \bar{\nu}_5)p^2 - (\bar{\nu}_0 + \bar{\nu}_3)p + (1-\bar{\nu}_4)\sigma + \bar{\nu}_2 = 0,$$
$$\chi(R_0) = \tan\beta,$$

where R_0 is the scaled core radius ($R_0\bar{D}/\bar{V}$ is the real core radius) and β is the angle between the tip tangent and the tip radius-vector, counted from the wave tangent.

Equation (11) coincides with the far asymptotics of the small-core spiral waves considered in [17, 15, 18, 19], but in our case it is valid for all radii under consideration. The exact spiral wave solution of (11) can be found analytically [13, 20]. However, it is rather complicated and we do not need it in this paper. To obtain closed-form results in the examples, we use its uniformly valid asymptotics.

5. Method of solution

We expect that the drifting spiral in CMF is close to the free spiral, and so we linearise equation (5) at the solution (9), taking into account the smallness of the drift velocity $\vec{c}$ and the perturbations $\tilde{T}$ and $\tilde{T}_0$ of the eikonal $\bar{T}$ and rotation period $\bar{T}_0$,

$$T = \bar{T} + \tilde{T}, \qquad T_0 = \bar{T}_0 + \tilde{T}_0, \tag{13}$$

and will look for the perturbations in the form of Fourier series,

$$\begin{aligned} \tilde{T} &= \frac{\tilde{T}_0}{2\pi}\theta + \sum T_n(\rho)\exp(in\theta), & \alpha &= \omega t - \beta + \sum \alpha_n \exp(in\omega t), \\ \kappa &= 1 + \sum \kappa_n \exp(in\omega t), & \kappa' &= 1 + \sum \kappa'_n \exp(in\omega t), \\ \rho_0(\omega t) &= R_0\left(1 + \sum r_n \exp(in\omega t)\right), & \theta(\omega t) &= \omega t + \sum \theta_n \exp(in\omega t), \end{aligned} \tag{14}$$

where $T_{-n} = T_n{}^*$, $\alpha_{-n} = \alpha_n{}^*$ etc, all the perturbations (except perhaps T_n) are assumed small compared to unity, $\rho_0(\theta)$ is the equation of the tip trajectory in polar coordinates and ω is the perturbed frequency,

$$\omega = 2\pi/(\bar{T}_0 + \tilde{T}_0) \approx \omega_0 + \delta\omega. \tag{15}$$

This leads to linear second-order ODEs for the eikonal modes $T_n(\rho)$ of the form

$$\begin{aligned} &T''_n + p_n(\rho)T'_n + q_n(\rho)T_n = \qquad (16) \\ &= H_n(\rho) + h_0(\rho)\delta\omega\delta_{n,0} + h_1(\rho)(C_x - iC_y)\delta_{n,1} + h_{-1}(\rho)(C_x + iC_y)\delta_{n,-1}. \end{aligned}$$

Equations for C_x and C_y in (6) give the Fourier series for the complex tip velocity $\dot{X}_0 + i\dot{Y}_0$. Its zeroth harmonics should vanish to provide periodicity of the tip trajectory. This requirement leads to a linear relationship between $C_x - iC_y$ and κ_1'. Consideration of the zero-order and first harmonics of $\dot{X}_0 + i\dot{Y}_0$ yields expressions for the parameters of the core shape $r_{0,1}, \theta_{0,1}$ through $\kappa_{0,1}'$.

Linearisation of the tip equations (6) gives boundary conditions for $T_{0,1}(\rho)$ at the tip trajectory. Extrapolating or interpolating functions $T_{0,1}$ and taking into account relations between the core shape and $\kappa_{0,1}'$, we write these boundary conditions in the form

$$T_0'(R_0) = g_0\kappa_0' + G_0, \quad T_1(R_0) = f_1\kappa_1' + F_1, \quad T_1'(R_0) = g_1\kappa_1' + G_1. \tag{17}$$

Explicit formulae for the coefficients in (16) and (17) are tedious and we omit them. Boundary condition at infinity is obtained from the requirement of boundedness of the gradient of the solution and has the form

$$T_n'(\infty) = 0 \tag{18}$$

For $n = 0, \pm 1$, solutions of (17) are easily found analytically, using the fact that $\partial_\theta \bar{T}$ and $\partial_x \bar{T}$ satisfy the homogeneous linear equations. These solutions satisfy the boundary conditions only for a unique choice of C_x, C_y and $\delta\omega$. And this choice is the solution to our problem.

6. The general results

The frequency deviation and velocity drift come out as linear functionals of parameter perturbations, as it predicted by the general theory [10]. We classify the kernels of these functionals, the *response functions* (RF's), as *rotational* RF's, determining shift in rotation frequency, and *translational* RF's, determining the drift of the rotation center (in [10, 11], the RF's were classified as *temporal* and *spatial*, as for steadily rotating waves rotation in space is equivalent to translation in time). We denote this by subscripts, 0 for rotational and 1 for translational RF's. Another classification of the RF's is by the parameter the influence of which they describe. We denote this by superscripts in parentheses.

Namely, the frequency deviation is

$$\delta\omega \;=\; \int_{R_0}^{\infty}\int_0^{2\pi} \left[\tilde{W}_0^{(V)}(\rho)\tilde{V}(\rho,\theta) + \tilde{W}_0^{(D)}(\rho)\tilde{D}(\rho,\theta)\right] \mathrm{d}\theta\rho\mathrm{d}\rho \tag{19}$$

$$+\int_0^{2\pi}\left[\hat{W}_0^{(V)}\tilde{V}(R_0,\theta)+\hat{W}_0^{(D)}\tilde{D}(R_0,\theta)+\sum_{j=0}^{5}\hat{W}_0^{(j)}\bar{\nu}_j\tilde{\nu}_j(R_0,\theta)\right]\mathrm{d}\theta,$$

where the rotational RF's are

$$\begin{aligned}
&\tilde{W}_0^{(V)}(\rho)=-\mathcal{W}_0^0\frac{\omega_0}{1-p}\exp\left(-\omega_0\int_{R_0}^{\rho}\rho_1\chi(\rho_1)\mathrm{d}\rho_1\right),\\
&\tilde{W}_0^{(D)}(\rho)=\tilde{W}_0^{(V)}(\rho)\left[\omega_0\rho(\chi^2+1)^{-1/2}-1\right],\\
&\hat{W}_0^{(V)}=\mathcal{W}_0^0 p/\omega_0,\quad \hat{W}_0^{(D)}=-\frac{\sigma\Delta}{2\pi}\left(2\bar{\nu}_5 p+\bar{\nu}_3\right)-\mathcal{W}_0^0\left(\sigma\Delta\mathcal{W}_0^1-\frac{p^2}{\omega_0}\right), \qquad (20)\\
&\hat{W}_0^{(0)}=\frac{p\Delta}{2\pi}\left(2\bar{\nu}_5 p+\bar{\nu}_3\right)+\mathcal{W}_0^0\left(p\Delta\mathcal{W}_0^1-\frac{1-pR_0\sin\beta}{1-p}\right),\quad \hat{W}_0^{(1)}=\hat{W}_0^{(0)}p,\\
&\hat{W}_0^{(2)}=\frac{\Delta}{2\pi}\left(2\bar{\nu}_1 p+\bar{\nu}_0\right)+\mathcal{W}_0^0\Delta\mathcal{W}_0^2,\\
&\hat{W}_0^{(3)}=-\hat{W}_0^{(2)}p,\quad \hat{W}_0^{(4)}=-\hat{W}_0^{(2)}\sigma,\quad \hat{W}_0^{(5)}=-\hat{W}_0^{(2)}p^2
\end{aligned}$$

and notations introduced for brevity

$$\begin{aligned}
\mathcal{W}_0^0 &= -\frac{1}{2\pi}\left[\frac{\mathcal{Z}_0}{1-p}+\frac{pR_0-\sin\beta(1+\cos^2\beta)}{\omega_0\cos\beta}-\frac{pa}{b}\left(\frac{p}{\omega_0}+\bar{\nu}_1\frac{1-pR_0\sin\beta}{1-p}\right)\right]^{-1},\\
\mathcal{W}_0^1 &= \left[\frac{\mathcal{Z}_0}{1-p}-\frac{pR_0-\sin\beta(1+\cos^2\beta)}{\omega_0\cos\beta}\right](2\bar{\nu}_5 p+\bar{\nu}_3)+\frac{p}{\omega_0}+\bar{\nu}_1\frac{1-pR_0\sin\beta}{1-p}, \qquad (21)\\
\mathcal{W}_0^2 &= \left[\frac{\mathcal{Z}_0}{1-p}-\frac{pR_0-\sin\beta(1+\cos^2\beta)}{\omega_0\cos\beta}\right](2\bar{\nu}_1 p+\bar{\nu}_0)-\frac{p}{\omega_0}-\bar{\nu}_1\frac{1-pR_0\sin\beta}{1-p},\\
\mathcal{Z}_0 &= \int_{R_0}^{\infty}\frac{\rho}{(\chi^2+1)^{3/2}}\left[(\chi^2+1)^{1/2}(1-2\chi^2)+2\chi^2(\omega_0\rho+\frac{\chi}{\rho})\right]\exp\left(-\omega_0\int_{R_0}^{\rho}\rho_1\chi(\rho_1)\mathrm{d}\rho_1\right)\mathrm{d}\rho
\end{aligned}$$

The drift velocity is

$$\begin{aligned}
C_x-iC_y &= \int_{R_0}^{\infty}\int_0^{2\pi}\left[\tilde{W}_1^{(V)}(\rho,\theta)\tilde{V}(\rho,\theta)+\tilde{W}_1^{(D)}(\rho,\theta)\tilde{D}(\rho,\theta)\right]\mathrm{d}\theta\rho\mathrm{d}\rho \qquad (22)\\
&+\int_0^{2\pi}\left[\hat{W}_1^{(V)}(\theta)\tilde{V}(R_0,\theta)+\hat{W}_1^{(D)}(\theta)\tilde{D}(R_0,\theta)+\sum_{j=0}^{5}\hat{W}_1^{(j)}(\theta)\bar{\nu}_j\tilde{\nu}_j(R_0,\theta)\right]\mathrm{d}\theta
\end{aligned}$$

with the translational RF's

$$\tilde{W}_1^{(V,D)}(\rho,\theta)=\tilde{W}_1^{(V,D)}(\rho)\exp(-i\theta),\quad \hat{W}_1^{(j)}(\theta)=\hat{W}_1^{(j)}\exp(-i\theta),\quad j=V,D,0...5,$$

$$
\begin{aligned}
&\tilde{W}_1^{(V)}(\rho) = -2i\mathcal{W}_1\left[bR_0 + pa(1+i\bar{\nu}_1)\exp(i\beta)\right]\frac{\chi+i}{\rho}\exp\left[-\int\limits_{R_0}^{\rho}\left(\omega_0\rho_1+\frac{2i}{\rho_1}\right)\chi(\rho_1)\mathrm{d}\rho_1\right]\\
&\tilde{W}_1^{(D)}(\rho) = \tilde{W}_1^{(V)}(\rho)\left[\frac{\omega_0\rho}{(\chi^2+1)^{1/2}}-1\right],\\
&\hat{W}_1^{(V)} = \mathcal{W}_1\mathcal{W}_1^1,\quad \hat{W}_1^{(D)} = \mathcal{W}_1\left[-(\sigma\Delta+p)\mathcal{W}_1^1+\bar{\nu}_1\sigma\Delta\mathcal{W}_1^2-(2\bar{\nu}_5 p+\bar{\nu}_3)\sigma\Delta\mathcal{W}_1^3\right],\\
&\hat{W}_1^{(0)} = \mathcal{W}_1\left[p\Delta\mathcal{W}_1^1+(1-\bar{\nu}_1 p\Delta)\mathcal{W}_1^2+(2\bar{\nu}_5 p+\bar{\nu}_3)p\Delta\mathcal{W}_1^3\right],\quad \hat{W}_1^{(1)} = \hat{W}_1^{(0)}p, \qquad (23)\\
&\hat{W}_1^{(2)} = \mathcal{W}_1\left[-\Delta\mathcal{W}_1^1+\bar{\nu}_1\Delta\mathcal{W}_1^2+(2\bar{\nu}_1 p+\bar{\nu}_0)\Delta\mathcal{W}_1^3\right],\\
&\hat{W}_1^{(3)} = -\hat{W}_1^{(2)}p,\quad \hat{W}_1^{(4)} = -\hat{W}_1^{(2)}\sigma,\quad \hat{W}_1^{(5)} = -\hat{W}_1^{(2)}p^2
\end{aligned}
$$

and notations introduced for brevity

$$
\begin{aligned}
\mathcal{W}_1 &= \frac{1}{2\pi}\left[\frac{pR_0\sin\beta-1}{1-p}\left(bR_0-pa\exp(-i\beta)(1+i\bar{\nu}_1)\right)-i\left(bR_0+pa(1+i\bar{\nu}_1)\exp(i\beta)\right)\cdot\right.\\
&\quad\left.\cdot\int\limits_{R_0}^{\infty}\frac{(\chi+i)^2}{(\chi^2+1)^{1/2}}\exp\left(-\int\limits_{R_0}^{\rho}\left(\omega_0\rho_1+\frac{2i}{\rho_1}\right)\chi(\rho_1)\mathrm{d}\rho_1\right)\mathrm{d}\rho\right]^{-1},\\
\mathcal{W}_1^1 &= 2\frac{b}{\omega_0}(i-pR_0\cos\beta)-2\frac{p^2a}{\omega_0}(1+i\bar{\nu}_1), \qquad (24)\\
\mathcal{W}_1^2 &= 2\frac{b}{\omega_0}(1-pR_0\sin\beta),\quad \mathcal{W}_1^3 = 2i\frac{pa}{\omega_0}(1-pR_0\sin\beta)(1+i\bar{\nu}_1)
\end{aligned}
$$

As it is seen from (19) and (22), the RF's are further classified on their spatial character. RF's $\tilde{W}_{0,1}^{(*)}$ determine the contributions from parameter perturbations in the whole medium outside the core, whereas $\hat{W}_{0,1}^{(*)}$ only from the perturbations at the core boundary. So, if expressed as functions of space, $\hat{W}_{0,1}^{(*)}$ will be singular functions, proportional to $\delta(\rho-R_0)$. Respectively, we call $\tilde{W}_{0,1}^{(*)}$ and $\hat{W}_{0,1}^{(*)}$ the *regular* and *singular* components of the RF's. Parameters that influence only the motion of the tip have only singular RF components. The rotational RF's $\tilde{W}_0^{(j)}$, $\hat{W}_0^{(j)}$ are real, while translational RF's $\tilde{W}_1^{(j)}$, $\hat{W}_1^{(j)}$ are complex as they determine not only the speed but also the direction of the drift.

Note that all the regular RF's decay quickly (superexponentially) at large ρ.

7. Examples

7.1. Approximate solution for the typical case

In this section, we consider the case $\bar\nu_2 \ll 1$ and $\bar\nu_0$ nonzero and of the order of unity, all other $\bar\nu_j$ being arbitrary and of the order of unity. This is "the most typical" case, as $\bar\nu_2$ being small or zero is a necessary condition for applicability of the kinematic approach. The main parameters of the free spiral, in the principal orders in $p = \bar\nu_2/\left(\bar\nu_3+\bar\nu_0\right)$, are [13]

$$\sigma = -p^2\bar\nu_0 \ll 1, \quad \beta = \arctan\bar\nu_0 \propto 1, \quad \omega_0 = p\bar\nu_0 \ll 1, \quad R_0 = \frac{1-p}{p\sin\beta} \gg 1. \tag{25}$$

The shape of the spiral is asymptotically described by

$$\chi \approx (\omega_0^2\rho^2-1)^{1/2} + \frac{\omega_0^3\rho^2}{\omega_0^2\rho^2-1} + O\left(\omega_0^2\right), \tag{26}$$

uniformly on $\rho \in [R_0, +\infty)$. The first term here corresponds to the involute of a circle. To leading orders in p, the rotational regular RF's simplify to

$$\begin{aligned}
\tilde W_0^{(V)}(\rho) &\approx \frac{\omega_0}{4\pi}\cdot p\bar\nu_0^2\left(1+\bar\nu_0^2\right)\exp\left[\frac{\bar\nu_0^2}{3p} - \frac{(\omega_0^2\rho^2-1)^{3/2}}{3p\bar\nu_0}\right], \\
\tilde W_0^{(D)}(\rho) &\approx -\frac{\omega_0}{(\omega_0^2\rho^2-1)^{1/2}}\tilde W_0^{(V)}(\rho),
\end{aligned} \tag{27}$$

rotational singular RF's to

$$\begin{aligned}
&\hat W_0^{(V)} \approx -\frac{\omega_0}{4\pi}\cdot\left(1+\bar\nu_0^2\right), \quad \hat W_0^{(D)} \approx \frac{\omega_0}{4\pi}\cdot p\frac{\bar\nu_0\left(\bar\nu_0^2-1\right)+\bar\nu_3\left(\bar\nu_0^2+1\right)}{\bar\nu_0+\bar\nu_3}, \\
&\hat W_0^{(0)} \approx \frac{\omega_0}{2\pi}\cdot\frac{\bar\nu_3}{\bar\nu_0\left(\bar\nu_0+\bar\nu_3\right)}, \quad \hat W_0^{(1)} \approx \frac{\omega_0}{2\pi}\cdot\frac{p\bar\nu_3}{\bar\nu_0\left(\bar\nu_0+\bar\nu_3\right)}, \\
&\hat W_0^{(2)} \approx \frac{\omega_0}{2\pi\bar\nu_2}, \quad \hat W_0^{(3)} \approx -\frac{\omega_0}{2\pi}\cdot\frac{1}{\bar\nu_0+\bar\nu_3}, \quad \hat W_0^{(4)} \approx \frac{\omega_0}{2\pi}\cdot\frac{p\bar\nu_0}{\bar\nu_0+\bar\nu_3}, \quad \hat W_0^{(5)} \approx -\frac{\omega_0}{2\pi}\cdot\frac{p}{\bar\nu_0+\bar\nu_3},
\end{aligned} \tag{28}$$

translational regular RF's to

$$\begin{aligned}
\tilde W_1^{(V)}(\rho) &\approx \frac{p\bar\nu_0^2}{\pi(1+\bar\nu_0^2)^{1/2}}\exp\left(\frac{\bar\nu_0^2}{3p}+2i\bar\nu_0\right)\left[1+i\left(\omega_0^2\rho^2-1\right)^{1/2}\right]\cdot \\
&\quad\cdot\exp\left[-\frac{(\omega_0^2\rho^2-1)^{3/2}}{3\omega_0} - 2i\left(\omega_0^2\rho^2-1\right)^{1/2}\right], \\
\tilde W_1^{(D)}(\rho) &\approx -\frac{\omega_0}{(\omega_0^2\rho^2-1)^{1/2}}\tilde W_1^{(V)}(\rho),
\end{aligned} \tag{29}$$

and translational singular RF's to

$$
\begin{aligned}
\hat{W}_1^{(V)} &\approx -\frac{1+i\bar{\nu}_0}{\pi\left(1+\bar{\nu}_0^2\right)}, \quad \hat{W}_1^{(D)} \approx \frac{p(1+i\bar{\nu}_0)}{\pi\left(1+\bar{\nu}_0^2\right)}, \\
\hat{W}_1^{(0)} &\approx \frac{p\bar{\nu}_0(1+i\bar{\nu}_0)}{\pi\left(1+\bar{\nu}_0^2\right)}\left[\frac{1+i\bar{\nu}_0}{1+\bar{\nu}_0^2}-\frac{1}{\bar{\nu}_0(\bar{\nu}_0+\bar{\nu}_3)}\right], \\
\hat{W}_1^{(1)} &\approx \frac{p^2\bar{\nu}_0(1+i\bar{\nu}_0)}{\pi\left(1+\bar{\nu}_0^2\right)}\left[\frac{1+i\bar{\nu}_0}{1+\bar{\nu}_0^2}-\frac{1}{\bar{\nu}_0(\bar{\nu}_0+\bar{\nu}_3)}\right], \\
\hat{W}_1^{(2)} &\approx \frac{1+i\bar{\nu}_0}{\pi\left(1+\bar{\nu}_0^2\right)(\bar{\nu}_0+\bar{\nu}_3)}, \quad \hat{W}_1^{(3)} \approx -\frac{p(1+i\bar{\nu}_0)}{\pi\left(1+\bar{\nu}_0^2\right)(\bar{\nu}_0+\bar{\nu}_3)}, \\
\hat{W}_1^{(4)} &\approx \frac{p^2\bar{\nu}_0(1+i\bar{\nu}_0)}{\pi\left(1+\bar{\nu}_0^2\right)(\bar{\nu}_0+\bar{\nu}_3)}, \quad \hat{W}_1^{(5)} \approx -\frac{p^2(1+i\bar{\nu}_0)}{\pi\left(1+\bar{\nu}_0^2\right)(\bar{\nu}_0+\bar{\nu}_3)}.
\end{aligned}
\tag{30}
$$

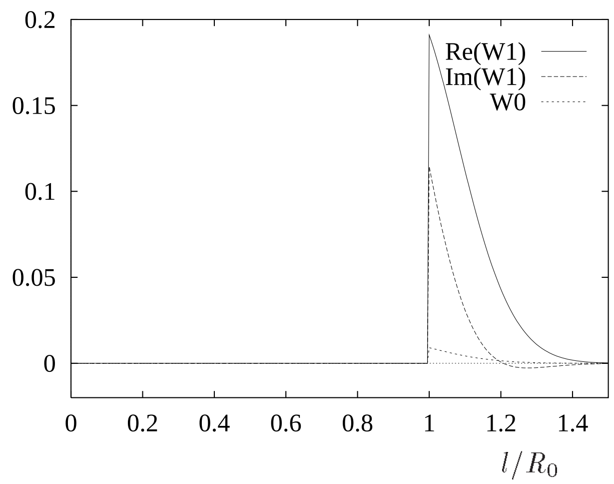

Figure 1. The response functions $\tilde{W}_{0,1}^{(V)}$, representing the sensitivity to perturbation of the wave propagation velocity, as functions of the distance from the center normalised by the core radius, l/R_0, for $\bar{\nu}_0 = 0.6$ and $p = 0.2$.

Typical shapes of the most essential regular RF components, $\tilde{W}_{0,1}^{(V)}$, are illustrated on Figure 1.

$\tilde{W}_1^{(V)}$ and $\hat{W}_1^{(V)}$ are the largest amongst the spatial RFs. So if perturbation of the propagation velocity $\tilde{V}$ is present, it is the main factor determining the drift. If all other perturbations are neglected, then

$$\delta\omega = \frac{\omega_0}{4\pi\cos^2\beta}\oint\left\{-\tilde{V}(R_0,\theta)+\frac{\sin\beta}{\cos^2\beta}\int\limits_0^\infty \exp\left(-\frac{\sin\beta}{\cos^2\beta}\zeta\right)\tilde{V}(R_0+\zeta,\theta)\mathrm{d}\zeta\right\}\mathrm{d}\theta,$$

$$C_x - iC_y = \frac{\mathrm{e}^{i\beta}}{\pi}\oint\left\{-\tilde{V}(R_0,\theta)\cos\beta+\frac{\sin\beta}{\cos^2\beta}\int\limits_0^\infty \exp\left(-\frac{\sin\beta}{\cos^2\beta}\zeta\right)\tilde{V}(R_0+\zeta,\theta)\mathrm{d}\zeta\right\}\mathrm{d}\theta. \tag{31}$$

7.2. Linear gradient inhomogeneity

Let us consider a perturbation in the form of a linear gradient of all parameters in the x-direction:

$$\tilde{D} = k_D\rho\cos\theta, \quad \tilde{V} = k_V\rho\cos\theta, \quad \tilde{\nu}_j = k_j\rho\cos\theta. \tag{32}$$

These perturbations are large at large x. However, as the RF's decrease rapidly with distance, only perturbations in a small neighbourhood of the core are of interest. In other words, if we consider perturbations, that are described by (32) in a neighbourhood of the core, and vanish outside that neighbourhood, the result would be almost the same as that for (32).

Substitution of (32) into (19) and (22) gives

$$\delta\omega = 0,$$

$$C_x - iC_y = \frac{1+i\bar{\nu}_0}{(1+\bar{\nu}_0^2)^{1/2}}\left[\left(1-(1+\bar{\nu}_0^2)^{-1/2}\right)R_0k_V + k_D\right. \tag{33}$$

$$\left.+\left(\bar{\nu}_0\frac{1+i\bar{\nu}_0}{1+\bar{\nu}_0^2}-\frac{1}{\bar{\nu}_0+\bar{\nu}_3}\right)(\bar{\nu}_0k_0+p\bar{\nu}_1k_1)+k_2+\frac{1}{\bar{\nu}_0+\bar{\nu}_3}(-\bar{\nu}_3k_3+p\bar{\nu}_0\bar{\nu}_4k_4-p\bar{\nu}_5k_5)\right].$$

The zero perturbation of the spiral rotation frequency simply means that the influences of the faster and the slower neighbourhoods of the spiral cancel each other in the first approximation. The drift is mainly determined by variations of the conduction velocity, as only k_V is multiplied by the large coefficient R_0. Apart from this coefficient, the drift velocity in this main order is a simple function of a single parameter $\bar{\nu}_0$. In particular, the angle between the drift velocity and the gradient of the conduction velocity is always approximately equal to the angle between the tip tangent and radius-vector of the core.

7.3. Stepwise inhomogeneity

Suppose the medium consists of two domains with different properties, separated by straight line $x = l$ (in the CMF), and the parameters of the medium are uniform within each of the domains. In polar coordinates,

$$\tilde{D} = \delta D \mathrm{sgn}(l-\rho\cos\theta), \quad \tilde{V} = \delta V \mathrm{sgn}(l-\rho\cos\theta), \quad \tilde{\nu}_j = \delta\nu_j \mathrm{sgn}(l-\rho\cos\theta) \tag{34}$$

Again, the seeming contradiction with the assumption on the localisation of the perturbation is resolved by noting that it's only the nearest neighbourhood of the core that matters, and it suffices if the parameters are described by (34) in this neighbourhood and vanish outside it.

Then equations (19)–(22) give the frequency deviation

$$\frac{\delta\omega}{\omega_0} \approx \frac{2}{\pi(\bar{\nu}_0+\bar{\nu}_3)} \arcsin\frac{l}{l_0}\left[\frac{p}{2}(\bar{\nu}_0(\bar{\nu}_0^2-1)+\bar{\nu}_3(\bar{\nu}_0^2+1))\delta D\right. \tag{35}$$
$$\left.+\bar{\nu}_3\delta\nu_0 + p\frac{\bar{\nu}_3\bar{\nu}_1}{\bar{\nu}_0}\delta\nu_1 + (\bar{\nu}_0+\bar{\nu}_3)\delta\nu_2 - \bar{\nu}_3\delta\nu_3 + p\bar{\nu}_0\bar{\nu}_4\delta\nu_4 - p\bar{\nu}_5\delta\nu_5\right]$$

and the drift velocity

$$C_x - iC_y \approx -4\frac{2^{1/2}p\bar{\nu}_0^2\left[1+i(\omega_0^2 l_0^2-1)^{1/2}\right]}{\pi\omega_0^{3/2} l_0 (1+\bar{\nu}_0^2)^{1/2}(\omega_0^2 l_0^2-1)^{1/2}}\cdot$$
$$\cdot\exp\left[\frac{\bar{\nu}_0^3-(\omega_0^2 l_0^2-1)^{3/2}}{3\omega_0} + 2i\left(\bar{\nu}_0-(\omega_0^2 l_0^2-1)^{1/2}\right)\right]\cdot$$
$$\cdot\exp\left(\frac{\omega_0(l_0^2-l^2)(\omega_0^2 l_0^2-1)^{1/2}}{2}\right)\,\Gamma\!\left(\frac{3}{2},\frac{\omega_0(l_0^2-l^2)(\omega_0^2 l_0^2-1)^{1/2}}{2}\right)\delta V$$
$$+4\left(1-\frac{l^2}{l_0^2}\right)^{1/2}\frac{1+i\bar{\nu}_0}{\pi(1+\bar{\nu}_0^2)}\delta V - \frac{4}{\pi}\left(1-\frac{l^2}{l_0^2}\right)^{1/2}\frac{1+i\bar{\nu}_0}{1+\bar{\nu}_0^2}p\cdot \tag{36}$$
$$\left[p\delta D + \left(\bar{\nu}_0\frac{1+i\bar{\nu}_0}{1+\bar{\nu}_0^2} - \frac{1}{\bar{\nu}_0+\bar{\nu}_3}\right)(\bar{\nu}_0\delta\nu_0 + p\bar{\nu}_1\delta\nu_1) + \delta\nu_2\right.$$
$$\left.+\frac{1}{\bar{\nu}_0+\bar{\nu}_3}\left(-\bar{\nu}_3\delta\nu_3 + p\bar{\nu}_0\bar{\nu}_4\delta\nu_4 - p\bar{\nu}_5\delta\nu_5\right)\right]$$

where, for brevity, we introduced

$$l_0 = \max(l, R_0). \tag{37}$$

An example of the dependencies of C_x and C_y on l due to the perturbation of δV is shown on Figure 2. It can be seen, both on the graphs and from (36), that in the interval $l < R_0$, where the both

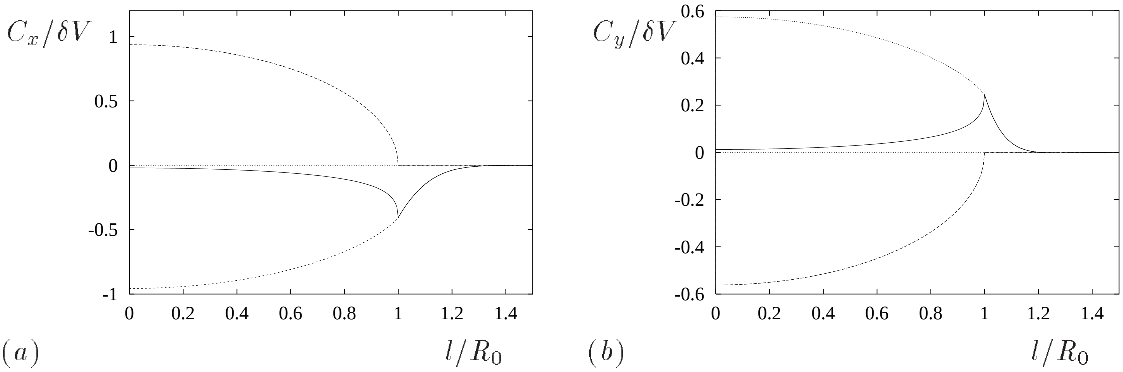

Figure 2. Velocities of the drift of the spiral wave, $C_x/\delta V$ and $C_y/\delta V$, near a stepwise inhomogeneity in the conduction velocity δV, normalised by the magnitude of this inhomogeneity, as functions of the distance from the spiral center to the step, normalised by the core radius, l/R_0, for $\bar{\nu}_0 = 0.6$ and $p = 0.2$. (a) Velocity across the inhomogeneity line, $C_x/\delta V$, (b) velocity along the inhomogeneity $C_y/\delta V$. Solid lines: the total velocity, dashed line: contribution of the "singular" component (tip motion equations), dotted line: contribution of the "regular" component (wavefront motion equation).

velocity components are considerable, they are, again, approximately proportional to each other with the coefficient $-\bar{\nu}_0$.

7.4. Comparison with numerical results

. We compared predictions of the previous subsection with numerical simulations of the piecewise-linear FitzHugh-Nagumo system [21]

$$\begin{aligned} u_t &= a_i u - v + b_i + \nabla^2 u, \\ v_t &= \epsilon_i (u - v), \\ i &= \begin{cases} 0, & u < u_1, \\ 1, & u_1 < u < u_2, \\ 2, & u_2 < u, \end{cases} \end{aligned} \tag{38}$$

with parameters $a_0 = -4.0$, $b_0 = 0$, $a_1 = 0.98$, $b_1 = -0.08964$, $a_2 = -15$, $b_2 = 15$, $u_1 = 0.018$, $u_2 = 0.944283$ and a stepwise inhomogeneity of parameters ϵ_1 from 3.0 (in the left) to 1.68 (in the right), ϵ_2 from 0.1 to 0.06 and ϵ_3 from 3.0 to 1.5. Results of simulations are presented on Figure 3. In agreement with the asymptotic theory, (i) the detectable drift happens only when the core crosses the line of the inhomogeneity, and throughout this processes (ii) the absolute value of the velocity changes, but (iii) direction of the drift is preserved.

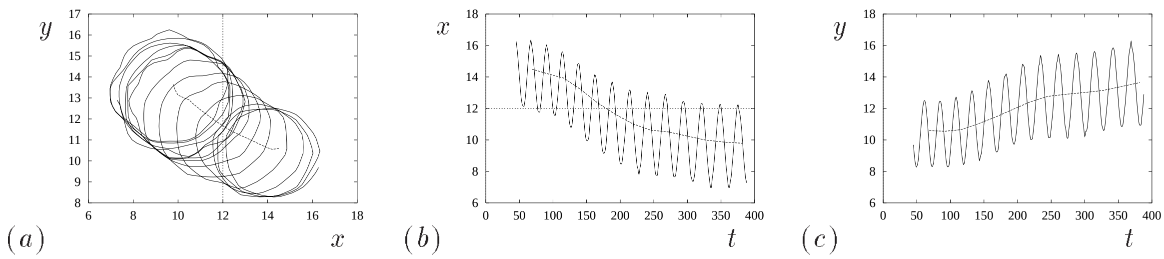

Figure 3. (a) Trajectory of the tip (solid line) and the rotation center (dashed line) of the spiral wave, in a numerical experiment with a stepwise inhomogeneous FitzHugh-Nagumo system (38). The vertical dotted line is the boundary between the domains. Coordinates of the rotation center were defined as the coordinates of the tip averaged over one rotation. (b), (c) x- and y-coordinates of the tip and of the center, as functions of time.

8. Conclusion

In this paper, we have studied the behaviour of large-core spiral waves in slightly inhomogeneous excitable media. We used the kinematic approach, in which an excitable medium is characterised by several parameters, two of which determine the motion of the wave (the *wave parameters*) and the motion of the wave tip depending also on the other six (*tip parameters*). These parameters can be found by perturbations of the half-plane-wave solutions of underlying reaction-diffusion equations [13] or phenomenologically by fitting experimental or numerical data.

We have obtained explicit analytical expressions of the frequency shift and drift velocity, as functionals of the perturbations of the medium parameters.

Some features of our results are:

1. The linear superposition principle is valid: influence of medium inhomogeneity onto the spiral wave rotation frequency and drift velocity is a sum of contributions of all infinitesimal regions with perturbed parameters. The contribution of each region depends on parameters' perturbations in this region, and (for perturbations of the wave parameters) on its distance to the spiral center.

2. At large distances, the decrease of the rotation frequency perturbation is superexponential, $\sim \exp(-(\rho/\Lambda)^3)$, and the decrease of the drift velocity is superexponential with oscillations, $\sim \exp(-(\rho/\Lambda)^3 - 2i\rho/\lambda_0)$. The period of these oscillations is $\lambda_0/2$, half the asymptotic wavelength of the free spiral.

3. Of all the parameters, the spiral wave is most sensitive to the inhomogeneity of V. Neglecting other inhomogeneities, the influence of a perturbation of V is described by simple equations (31), involving only parameters which are easily found phenomenologically. In particular, influence of V is significant even at some (relatively small) distance from the tip.

4. Influence of perturbation of D on the spiral drift and frequency deviation is always much smaller than the influence of a perturbation of V, and is significant only if it takes place at the tip (regular components of the RF's vanish in the linear approximation in p).

5. Only eight of the singular RF components, $\hat{W}_{0,1}^{V,D,0,2}$ are essentially different, while other eight may be expressed through these and parameters p and σ.

Some of these features may be explained heuristically and compared with similar features of problems of small-amplitude perturbations in CGLE [11] and boundary-induced drift in CGLE [14] and "coreless" spirals in Fife limit excitable media [17]. Naturally, features 1–5 are not identical to those of boundary-induced drift and when we comparing these features, we speak about similar but not identical properties.

Feature 1, is valid for the solution of [14, 11] and as we think, of [17] (those authors did not present solutions for general form of boundary and restricted only to some partial cases). The fact that small perturbations act additively is not surprising. However, neither it is obvious *a priori* for a nonlinear problem: *e.g.* in the dynamics of the domain wall in the Landau-Lifshits equation, it is the squares of the perturbations that are additive [22].

Feature 2 is valid for [17]. However, in [14] and [11], both frequency deviation and drift velocity show simple exponential dependence on distance. This difference can be understood intuitively in terms of diffusion of the autowave phase [23, 24]. The basic idea is that the perturbations influence the events around the core through the spiral wave radiated outwards. The spiral wave outside the core is locally close to plane periodic waves, and can be considered as a slowly varying wave (SVW)[23]. So, the transition of the perturbation influence to the core could be thought of in terms of diffusion of the phase of SVW on the background of its transport outwards, which may be only due to the phase diffusion[24]. Coefficients of longitudinal and lateral diffusions of phase may differ significantly. The model considered in [14] had the two diffusion coefficients equal, while in the model considered here and in [17], the longitudinal diffusion is absent. It is easy to see that Eq. (5) corresponds to the phase evolution equation of [24] with zero longitudinal diffusion. The reason is that in present approximation, the next excitation wave receives no information from the previous one, and the perturbation can propagate only along the wave. So, in CGLE, perturbations can propagate straightforwardly "upstream" the waves emitted by the spiral, so they have to overcome just the distance ρ. On the contrary, in the present case the perturbations have to propagate the much longer way along the wave, thus the faster dependence on the distance.

The oscillatory dependence of drift velocity on the distance can be partly understood in terms of the so called resonant drift. Note that perturbation significantly influences the spiral wave only when the wave moves through the perturbed region. Thus, this is a periodic influence synchronised with the spiral wave rotation. Now we can apply the same arguments as in the case of resonant parametric drift [25, 21, 26]. Namely, small effects of the perturbations are applied to the same phases of the spiral rotation and therefore add up. The essential difference here is that the influence strongly depends on the distance to the perturbed region. In particular, the direction of the drift is determined by the phase difference between the spiral rotation and the influence. This difference is determined by the distance to the boundary, measured by 2π times number of wavelengths. This explains the periodic dependence of the direction of the drift on the distance to the inhomogeneity, and relationship of this period with the wavelength of the spiral. The factor of $1/2$, however, remains unexplained and appears so far as a nontrivial result of the theory. On the contrary, in CGLE all the oscillation phases are exactly equivalent, and so the influence of inhomogeneity, in this sense, is not oscillating at all.

These heuristic interpretations predict that (i) the superexponential asymptotics are not common and are limited to the cases with zero longitudinal diffusion of autowave phase, while otherwise the exponential asymptotic are more probable, and (ii) the oscillatory character of drift direction is more common, and may be absent only in systems with special internal symmetries.

Features 3–5 are specific to the considered approximation. Of these, the most prominent seems the prediction 3, as it can be most easily tested in experiments and numerical simulations. In this paper, we presented only simplest qualitative comparison; this subject deserves further study.

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