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# Envelope Equations for Modulated Non-conservative Waves

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**ABSTRACT.**

In many applications people deal with waves that are locally plane and periodic, but at large distances and/or over long intervals of time change their characteristics, i.e. *modulated waves*. An efficient way to study such waves is the method of envelope equations, when the original wave equations are replaced by equations describing the slowly varying parameters of the waves. The practical approaches to this problem are numerous; however, many of them have limitations, either in achievable accuracy, or in the wave equations to which they could apply (e.g. only conservative systems), or both. The purpose of the present paper is to review results of a particular approach of this kind, which is free from these disadvantages. This approach is mostly illustrated for *autowaves*, which, in the author's opinion, should play the same role in the theory of waves, as auto-oscillations=limit cycles play in the theory of oscillations.

## Contents

# 1 Introduction

## 1.1 Modulated waves

When differential equations, describing a natural or technological process, are too complicated to be solved exactly, one needs to do it approximately. This can be done either numerically, or analytically, e.g. by using some asymptotic methods. It is known that approximate methods may not only serve for pure purposes of calculation, but also be an "instrument of understanding" of complex systems.

The simplest case is if the differential equations have the right-hand sides containing small parameters, and the systems becomes much simpler, e.g. can be treated exactly, if these parameters are equal to zero. Then for nonzero but small values of the parameters, solutions can be obtained by "perturbation techniques".

The perturbation technique may be also applied to systems without any small parameters. This may be the case if we are interested in solutions of a special form, containing such parameters. A well known example is small-amplitude oscillations in a nonlinear system. By scaling the dynamic variables to "normalise" the amplitude of oscillations, one can bring this problem to a problem explicitly depending on the small parameter, which becomes linear if this small parameter is equal to zero.

There is a less trivial example of the same kind. This is the famous "geometric optics" approximation. An important case of geometric optics is the "quasiclassical approximation" in quantum mechanics, which is so vital for revealing its interrelation with classical mechanics. Classical interpretation of geometric optics is that the consideration is restricted to solutions with wavelengths small in comparison with other characteristic sizes of the problem [4]. This works well with classical, linear wave equations, which admit wave solutions with arbitrarily short wavelengths. In general, this restriction may be impractical, e.g. if wave solution may not exist with wavelengths less than a certain minimum. Then the idea of the geometric optics is re-formulated as the idea of modulated waves[1], or slowly varying waves [3]: the characteristic sizes of the problem, in particular of the initial conditions for the equations, should be much larger than a typical wavelength. This means that in relatively small regions, the waves are close to plane and periodic, but the parameters of these waves, including direction of propagation and the period, slowly change in time and/or in space, becoming significant at large distances and/or after long time intervals.

The classical geometric optics approximation for linear wave equations heavily relies on the specific properties of these equation, notably, the superposition principle. This of course is no good for nonlinear waves. A well known method for nonlinear waves is the Whitham procedure, or the Whitham modulation theory [1, 2]. One form of the method uses knowledge of conservation laws, valid for many wave systems originating from physics, and derives the evolution equations for slowly varying parameters from these conservation laws. Another form of the method also uses the properties of physical origin, namely, the fact that the field equations can be written in the form of a Lagrange variational principle. The evolution equations are then derived from also from a Lagrangian principle, where the Lagrangian is rewritten in the form that depends upon the new independent variables describing the slowly-varying solutions. This method has been applied to many classical nonlinear equations, such as nonlinear Klein-Gordon equation, Korteweg-deVries equation and others.

## 1.2 Autowaves

There is an important class of nonlinear waves, for which the Whitham approach can not be applied, as they are neither Hamiltonian, nor have any conserved quantities. These are so called "autowaves". From the physical viewpoint, these are waves that propagate not because there is no dissipation, but because the dissipation is compensated by the constant supply of energy. This can happen in open spatially distributed systems far from thermodynamic equilibrium. Examples are nerve and heart tissues, and waves in some chemical reactions where the consumed reagents are either supplied, or are stored in substantial amounts so that their decrease during the wave period is negligible. Mathematically, such systems are most often described by systems of partial differential equation of "reaction-diffusion" type,

$$u_t = D\Delta u + f(u). \tag{1}$$

Here $u = u(x,t) \in \mathbb{R}^\ell$ is a column-vector of concentrations of the reagents, $f(u) \in \mathbb{R}^\ell$ is a column-vector of nonlinear reaction terms (interesting behaviour starts from $\ell \geq 2$), $D$ is an $\ell \times \ell$ matrix of diffusion coefficients of the reagents, which we assume symmetric; the space coordinates $x \in \mathbb{R}^n$ where the dimension of physical space, $n$, can be equal to 1, 2 or 3, and $\Delta$ is the Laplacian in $\mathbb{R}^n$. Here and throughout the paper subscript by a dynamic variable denotes partial differentiation, so $u_t$ means $\partial u/\partial t$.

Systems of the form (1), describing real autowave systems, do not have any conserved quantities at all, and, in particular, are not Hamiltonian. The simplest form of autowaves in (1) are plane periodic waves,

$$u(x,t) = U\left((k,x) - \omega(k^2)t + \phi_0, k^2\right) = U(\xi,\eta), \tag{2}$$

where

$$\xi = (k,x) - \omega(k^2)t + \phi_0, \quad \eta = k^2, \quad k \in \mathbb{R}^n. \tag{3}$$

Here $k$ is the wavevector, $\omega$ is the frequency, and $\phi_0$ is an arbitrary initial phase. Brackets $(,)$ denote the scalar product in the physical space $\mathbb{R}^n$. The characteristic features of the family of solutions (2):

- Function $U(\xi,\eta)$ is $2\pi$-periodic in its first argument,

  $$U(\xi + 2\pi, \eta) \equiv U(\xi,\eta). \tag{4}$$

  This corresponds to waves periodic in space with the period $2\pi/k$, where $k$ is the wavenumber. An important class of systems that only admit non-periodic solutions, e.g. propagation of flame without subsequent replenishment of fuel, can be considered formally as a special limit case of (2) with $k \to 0$.

- Solutions (2), as a rule, can not be found analytically. As it can be seen by direct substitution, function $U(\xi,\eta)$ obeys a system of ordinary differential equations

  $$\eta D U_{\xi\xi}(\xi,\eta) + \omega(\eta) U_\xi(\xi,\eta) + f(U) = 0. \tag{5}$$

  depending on $\eta$ as a parameter.

- Typically, if the reaction-diffusion system has wave solutions of some period, then it will have wave solutions at close periods. That is, since dependence of the problem (5,4) on $\eta$ is continuous, it solution normally depends on $\eta$ continuously. We will assume that (2) exist at least for $k$ belonging to an interval,

  $$k \in (k_1, k_2); \qquad \eta \in (k_1^2, k_2^2). \tag{6}$$

- Functions $U$ (up to the arbitrary phase shift) and $\omega$ uniquely depend on $k^2$. That is, for every wavenumber $k$, the shape of the waves, and the temporal period (and therefore the phase velocity) are fixed. At most, for a given $k^2$ there could be a *discrete number* of different waves, possibly with different frequencies. From the physical viewpoint, this happens because the propagation of waves is the result of the unique balance between energy supply and its dissipation. Mathematically, this means that the periodic orbits in (5) are isolated; the proof of this fact is beyond the scope of this paper. The dependence

$$\omega = \omega(k^2) \tag{7}$$

is often called *dispersion relationship* or *dispersion curve* (especially if it is represented graphically). In conservative systems, dispersion relationship would involve also an amplitude or amplitudes of the wave; in autowaves, the wavevector is the only argument in (7).

- We show the dependence on $k^2$ rather than $k$ to ensure we don't forget that if waves in (1) can propagate in one direction, similar waves certainly can propagate in the opposite direction, thus $U$ should be an even function of $k$.

This family of solutions is the basis of the whole problem, and we call them *basic waves* or *basic solutions.*

### 1.3 Modulated autowaves

Now we can formulate our goal. We want to describe a certain class of solutions to (1), based on the plane periodic waves (2). We now want to consider not plane periodic waves, but waves that are in some sense close to them. The difference from plane and periodic waves should be obvious on time and space scales much larger than the period and length of the waves.

Mathematically, this can be expressed in the form

$$u(x,t) = U\left(\phi, (\nabla\phi)\right) + v(x,t), \tag{8}$$

where $\nabla\phi$ is a local wavevector slowly varying in space and time, and $v$ is a small correction, so that

$$\phi = \epsilon^{-1}\Phi(\epsilon x, \epsilon t, \epsilon), \qquad \epsilon \ll 1, v \ll 1. \tag{9}$$

and function $\Phi$ is a smooth function of its three arguments. Indeed, one can see that if $\epsilon$ is very small, then $\phi$ in any bounded domain will be close to a linear function of $x$ and $t$, and thus (8) will be close, up to a phase shift, to a periodic wave of the family (2). The difference from that periodic wave will be due to the departure of $\phi$ from a linear function in that domain, and also due to the correction $v$.

The necessity to have the small correction $v$ in the Ansatz (8) may not be obvious from the beginning, but is a result of calculations. For now it suffices to note that such a correction is admissible, and as long as it is small, solution (8) in every bounded domain is close to one or another plane periodic wave.

Formulas (8) and (9) formally define the class of solutions we are interested in. We will call them modulated waves, or slowly varying waves (SVW). Our expectation is that if the initial values of (1) are SVW, then the solution will remain SVW, at least in large regions of space and for a long time.

The question is, what conditions should the phase ("eikonal") variable $\phi(x,t)$ satisfy. It happens, that the solution can be written in the form of a partial differential equation, the *evolution equation*, which will be the analog of the eikonal equation in the geometric optics. Our purpose is to develop a method of derivation of this equation for every given reaction-diffusion system (1). One thing we should always bear in mind: as a rule, this *cannot be done entirely analytically*, there usually will be some bits to do numerically. One reason for that is quite obvious: the basic solutions (2) can not be found analytically.

### 1.4 Structure of the paper

In Section 2 we will approach the problem on a heuristic level, in order to "guess" the correct answer without worrying too much about the correctness of the procedures used. Properties and consequences of the obtained evolution equation are analysed in Section 3. By analysing the limitations of the methods used in Section 2, we show the necessity of more formal procedures. The main ideas of that procedure are introduced in section Section 4, on the example of a "model" problem for ordinary differential equations. You will see however, that this model actually covers many of the well known asymptotic methods as special cases. This more formal approach is upgraded to be applied to the problem of SVW in the next two sections, where we will introduce the method of the detecting operator (Section 5) and of the sub-centre manifold (Section 6); both of them can be used to derive the evolution equations of arbitrary asymptotic precision.

Thus the overall progress of the paper is from intuitive understanding of the behaviour of SVW by informal and inefficient ways, towards technically efficient but formal methods.

## 2 Heuristic derivation of the SVW evolution equation

### 2.1 Preliminary comments

First of all, we immediately try to obtain the evolution equation, on the "physical" level of strictness and in the lowest nontrivial order of asymptotic precision. We assume that we know, analytically or numerically, the plane wave solutions (2) to the PDE system (1), and we want to obtain an evolution equation for the phase $\phi$ of the slowly varying wave solutions (8).

To proceed with this programme, we need to do one more comment. Note that the choice of $U(\xi,\eta)$ in (2) is not unique. For different $\eta$ the arbitrary initial phase $\phi_0$ may be chosen independently. Mathematically, if a function $U(\xi,\eta)$ satisfies (5) and (4), so does any function $U^K(\xi,\eta)$ given by the "gauge transformation"

$$U^K(\xi,\eta) = U(\xi + K(\eta), \eta). \tag{10}$$

for arbitrary $K(\eta)$. Later we will see that no all choices of $K$ are alike, and some will give simpler evolution equations than others.

### 2.2 Linear approximation

By substituting (8) into (1), we get a $\ell$-component vector equation for two unknown functions: vector function $v \in \mathbb{R}^\ell$ and scalar function $\phi$. So, the equation is under-determined. In linear approximation in $v$ it takes the form

$$v_t(x,t) = D\nabla^2 v(x,t) + F[\phi(x,t)]v(x,t) + h[\phi(x,t)], \tag{11}$$

where the Jacobian matrix $F$ and the free term h depend on the unknown $\phi(x,t)$ so that

$$F[\phi] = \left.\frac{\partial f(u)}{\partial u}\right|_{u=U}, \tag{12}$$

$$\begin{aligned} h[\phi] \quad = \quad & -\left(\omega((\nabla\phi)^2) + \phi_t\right) U_\xi - U_\eta \frac{\partial}{\partial t}(\nabla\phi)^2 \\ & + D\left(U_\eta \nabla^2 (\nabla\phi)^2 + 2U_{\xi\eta}\left(\nabla\phi, \nabla\left((\nabla\phi)^2\right)\right)\right. \\ & \left. + U_\xi \nabla^2\phi + U_{\eta\eta}\left(\nabla((\nabla\phi)^2)\right)^2\right). \end{aligned} \tag{13}$$

Here the function $U$ is assumed with arguments $U = U(\phi, (\nabla\phi)^2)$. In (13), the function $f$ and term with $U_{\xi\xi}$ are absent due to (5).

### 2.3 Eliminating the secular growth

We make sense of the under-determined system (11) by the following arguments. Formally, the function $\phi(x,t)$ can be chosen arbitrary, and then the system (11) solved with respect to $v$. However, not every such choice would guarantee that the resulting $v$ will be small. And this smallness is necessary for the linear approximation and the definition of SVW to be valid.

The requirement that $v$ be small leads to certain limitations the function $\phi$ should obey. These limitations are just the desired evolution equation.

If $\phi$ is fixed, equation (11) is a linear inhomogeneous equation for the unknown $v$.

Since the local wavevector $k = \nabla\phi$ varies slowly, the coefficient matrix and the free term in (11) are in a large region close to periodic, with the space periods $2\pi/|k|$ and time periods $2\pi/\omega$.

On this stage, we approach the condition that $v$ be small, neglecting the difference and assuming that $F$ and $h$ are exactly periodic functions with these periods. More precisely, we will consider the function $\phi$ included in (12) and (13) as an argument of $U$ as linear, and its derivatives, including higher order derivatives, as constants. In other words, every time $\phi$ occurs in (12) and (13), we retain only the principal terms. The derivatives are used as parameters until we get a closed equation for $\phi$. For $(\nabla\phi)^2$ we use the notation $(\nabla\phi)^2 = \eta$.

Next, we pass over to a moving frame of reference, by changing independent variables $(t, x)$ to $(\tau, \xi, y)$, where $\tau = t$, $\xi = \phi(x,t) \approx (k, x) - \omega(k^2)t$, and $y$ stands for all spatial coordinates perpendicular to $k$.

Thereafter, the coefficients $F$ and the free term $h$ become periodic functions of one independent variable $\xi$ only. Instead of (11) we then get

$$v_\tau = \mathcal{L}v + h(\xi) \tag{14}$$

where $\mathcal{L}$ is the linear differential operator with periodic coefficients defined as

$$\mathcal{L}v = \eta D v_{\xi\xi} + D\Delta_y v + \omega v_\xi + F(\xi)v \tag{15}$$

For brevity, we omit the dependence on the derivatives of $\phi$.

The stability of plane periodic waves is determined by the spectrum of $\mathcal{L}$,

$$\mathcal{L}V_j(\xi,\eta) = \lambda_j(\eta)V_j(\xi,\eta). \tag{16}$$

Differentiation of (5) by $\xi$ and comparison with (15,16) proves that

$$V_0(\xi,\eta) \equiv U_\xi(\xi,\eta), \tag{17}$$

the so called *shift mode*, or *Goldstone mode*, is an eigenfunction of $\mathcal{L}$ corresponding to the eigenvalue zero,

$$\mathcal{L}V_0 = 0 \tag{18}$$

We will assume that all other $\lambda_j$ have negative real parts, which is the strongest stability we could expect for this system. The existence of zero eigenvalue and the corresponding Goldstone mode is due to the translational invariance of the reaction-diffusion equations: even if the wave is stable, there is always indifferent stability with respect to its phase shift: if we perturb the wave, the perturbation will decay, but the wave may get slightly shifted after that. In terms of the equation (14), this means that for $h = 0$ and generic initial conditions, all the components of the solution $v$ in the basis of eigenfunctions $V_j$ die out, excepted for the shift mode (17).

**Remark** 2.3.1 Actually, 0 is not an isolated eigenvalue of (14) considered in the entire space, as there are other modes which decay arbitrarily slowly. This not just an *a priori* possibility, but a necessary consequence of the very asymptotic theory described here, and will become evident later. The slowly decaying modes are the modes that correspond to very slow modulations of the periodic wave. These modes, when properly normalised, will be arbitrarily close to (17) in every prescribed finite region. Therefore, in every finite region the solution of the equation (14) with $h = 0$ approaches (17), with a factor depending on the region. This we may say that the slow modulations can be included into $\phi$, while not-so-slow modulations remain in $v$ and they do decay quickly.

Let us now consider the inhomogeneous form of (14). As the free term $h$ and the coefficients $F$ are periodic functions of $\xi$ only, there exists a partial solution periodic in $\xi$ and independent of $y$. For this solution to be bounded in time, it is necessary and sufficient that the free term is orthogonal to the shift mode

$$\oint \langle W_0(\xi,\eta), h[\phi] \rangle \, \mathrm{d}\xi = 0, \tag{19}$$

where $W_0(\xi,\eta) \in \mathbb{R}^\ell$ is the eigenfunction of the adjoint operator $\mathcal{L}^+$,

$$\mathcal{L}^+ w = \eta D w_{\xi\xi} + D\Delta_y w - \omega w_\xi + F^T(\xi) w \tag{20}$$

corresponding to the eigenvalue 0. Brackets $\langle , \rangle$ denote the scalar product in the concentration space $\mathbb{R}^\ell$, and the integral is taken over a period.

Provided (19) is satisfied and the initial values of (14) do not exceed $\epsilon$, we can be sure that, after a large enough span of time, the general solution of (14) becomes a nearly periodic function of $\xi$, slowly depending on $\tau$ and $y$ and having the order $\epsilon$, which is the order of the free term $h$.

Condition (19) with "unfrozen" derivatives of $\phi$ becomes the desired closed equation for $\phi$, which does not depend on $x$ and $t$ explicitly and can therefore be used as a macroscopic evolution equation.

## 2.4 The final equation

Let us normalise the adjoint eigenfunction $W_0$ so that

$$\oint \langle W_0(\xi,\eta), V_0(\xi,\eta) \rangle \, \mathrm{d}\xi = 1. \tag{21}$$

The evolution equation is simpler if

$$\oint \langle W_0(\xi,\eta), U_\eta(\xi,\eta) \rangle \, \mathrm{d}\xi = 0. \tag{22}$$

It proves always possible, thanks to the possibility of the gauge transformation (10), with appropriately chosen $K(\eta)$. This can be seen by direct substitution, with account of (21).

Keeping in (19) only terms of the orders of $\mathcal{O}(1)$ and $\mathcal{O}(\epsilon)$, we obtain the desired evolution equation

$$\phi_t + \omega((\nabla\phi)^2) = P\left((\nabla\phi)^2\right)\nabla^2\phi + Q\left((\nabla\phi)^2\right)(\nabla\phi\nabla)(\nabla\phi)^2. \tag{23}$$

This form is equivalent to the following equation for the local wavevector $k(x,t) \equiv \nabla\phi$:

$$k_t = \mathrm{grad}\left(-\omega(k^2) + P(k^2)\mathrm{div}k + Q(k^2)(k, \mathrm{grad}(k^2))\right). \tag{24}$$

The coefficients $P$ and $Q$ are functions of $\eta$ and are defined as

$$P(\eta) = \oint \langle W_0(\xi,\eta), DU_\xi(\xi,\eta)\rangle \,\mathrm{d}\xi, \qquad Q(\eta) = 2\oint \langle W_0 xi,\eta), DU_{\xi\eta}(\xi,\eta)\rangle \,\mathrm{d}\xi. \tag{25}$$

Note that if the diffusion is scalar, that is $D_{ij} = d\delta_{ij}$, then $P \equiv d$.

### 2.5 On the accuracy of the equation

The heuristic derivation we have just performed leaves some question open. Why does the final equation (23) contain the terms of the orders of 1 and $\epsilon$ only? Would it be correct if we keep also the terms of the order of $\epsilon^2$? At last, a rhetorical question: could we get an absolutely precise equation if we do not omit any terms at all?

The key points of the heuristic derivation are elimination of the secular growth, and "freezing" and "unfreezing" the derivatives of $\phi$. The elimination of the secular growth seems quite reasonable, but the procedure of freezing and unfreezing looks somewhat artificial, especially if we allow for the fact that the ignored non-periodicity of $F$ and $h$ leads to an error of the order of $\epsilon$ even at a distance of one period, though the final equation (23) keeps the terms $\mathcal{O}(\epsilon)$ preserved. It is therefore obvious that even when this equation is correct, the method cannot provide more accurate approximations. A more sophisticated tools is required. Such a tool is developed in next sections.

But before that, we enjoy the results of our effort, and have a look at the obtained evolution equation and its properties.

## 3 Making sense of the SVW evolution equation

### 3.1 The physical meaning of the terms in the equation

The term $\omega(k^2)$ of the order of 1 means that the rate at which the phase changes is normally close to the frequency typical of the plane wave with the given wavevector, and can be determined by the dispersion relationship (7). This law controls the dispersion of the waves, or transport of the phase, and thus we may call it dispersional, or transport, term.

The two terms in the right-hand side of (23) are of the order of $\epsilon$ and vanish for plane periodic waves. They make the equation parabolic, and we may call them "diffusional" terms. The fact that there are two such terms points to that here are two distinct processes here: "longitudinal" and "lateral" diffusions. They can easily be illustrated with a case of nearly constant local wavevectors. Assume that

$$
\begin{aligned}
&\phi = (k_0, x) - \omega(k_0^2)t + \phi_1(x,y,t), \qquad |\nabla\phi_1| \ll k_0; \\
\omega(\eta) &= \omega_0 + (\eta - k_0^2)\omega_1 + \mathcal{O}((\eta - k_0^2)^2), \\
P(\eta) &= P_0 + \mathcal{O}(\eta - k_0^2), \\
Q(\eta) &= Q_0 + \mathcal{O}(\eta - k_0^2),
\end{aligned}
\tag{26}
$$

when $\eta \to k_0^2$.

Assume also that the physical space is a plane with coordinates $x, y$, and the vector $k_0$ is directed along the $x$ axis. Then the linear approximation in $\nabla\phi_1$ yields the equation of anisotropic diffusion with transport

$$(\phi_1)_t = -c_g(\phi_1)_x + R_0(\phi_1)_{xx} + P_0(\phi_1)_{yy}, \tag{27}$$

where the velocity of transport is none other than the group velocity defined by the standard expression

$$c_g = 2\omega_1 k_0 = \mathrm{d}\omega(k^2)/\mathrm{d}k, \tag{28}$$

the lateral diffusion coefficient is $P_0$ and the longitudinal diffusion coefficient is

$$R_0 = P_0 + 2k_0^2 Q_0. \tag{29}$$

## 3.2 Some special cases

### 3.2.1 Geometric optics limit

If we neglect the terms of the order of $\mathcal{O}(\epsilon)$ in (23), we get the simplest "eikonal" $\equiv$ Hamilton-Jacobi equation for the phase, obtained in the context of autowaves by Howard and Koppel [3]

$$\phi_t = -\omega\left((\nabla\phi)^2\right). \tag{30}$$

This into account the dispersion processes only.

### 3.2.2 "Deformed stripes" dissipative structures

The special case without dispersion, $\omega \equiv 0$, corresponds to basic waves being stationary dissipative structures in the form of stripes; then the SVW are deformed stripes. The evolution equation is

$$\phi_t = P(\nabla\phi^2) + Q(\nabla\phi^2)(\nabla\phi\nabla)(\nabla\phi)^2 \tag{31}$$

This equation has been obtained by Cross and Newell [20].

### 3.2.3 Long waves in self-oscillatory media

Assume that the local kinetics sustains a stable limit cycle so that every point of the medium is a self-oscillator. In terms of plane periodic waves this means that

$$\omega \approx \omega_0 - \omega_1 (\nabla\phi)^2, \qquad P \approx P_0, \qquad Q = \mathcal{O}(1) \quad \text{as } (\nabla\phi)^2 \to 0. \tag{32}$$

For small $\nabla\phi$ the evolution equation takes the form

$$\phi_t \approx -\omega_0 + \omega_1 (\nabla\phi)^2 + P_0 \nabla^2 \phi \tag{33}$$

For a special case of self-oscillatory systems, $\lambda - \omega$ systems, this equation has been suggested by Kuramoto and Tsuzuki [17].

### 3.2.4 An analytically solvable example

Consider a simple special case which easily yields the explicit forms of coefficients (25). This is the Complex Ginzburg-Landau equation with zero linear dispersion:

$$u_t = u - (1 - i\alpha) u |u|^2 + \nabla^2 u, \qquad u \in \mathbb{C}, \ \alpha \in \mathbb{R}. \tag{34}$$

The basic family is

$$U(\xi, \eta) = (1 - \eta)^{1/2} e^{i\xi}. \tag{35}$$

with the dispersion relationship

$$\omega(k) = \alpha(1 - k^2) \tag{36}$$

The adjoint zero eigenfunction is

$$W_0(\xi, \eta) = \frac{i(1-\eta)}{2\pi} e^{i\xi}. \tag{37}$$

Here we assume that the scalar product in the concentration space is defined as

$$\langle w, v \rangle = \mathrm{Re} \left( \overline{w}(\xi), v(\xi) \right). \tag{38}$$

So, the coefficients of the SVW equation are

$$P(\eta) = 1, \qquad Q(\eta) = -1/(1-\eta), \qquad R(\eta) = (1 - 3\eta)/(1 - \eta), \tag{39}$$

and the evolution equation

$$\phi_t = -\alpha + \alpha (\nabla\phi)^2 + \nabla^2 \phi - \frac{(\nabla\phi\nabla)(\nabla\phi)^2}{1 - (\nabla\phi)^2}. \tag{40}$$

The one-dimensional version of the equation has been derived by Malomed [18] and Bernoff [19].

### 3.3 Dissipative properties of the one-dimensional SVW equation

We have already seen in Section 3 that the diffusion terms make the evolution of autowaves with nearly constant local wave numbers irreversible. Here we describe two dissipative properties of the one-dimensional SVW equation which keep preserved even for the regimes with wavenumbers varying generally in a wide range. Both the properties consist in aligning the wavenumber.

#### 3.3.1 Lyapunov functional

Consider the functional

$$E(t) = \frac{1}{2} \int\limits_{x_1}^{x_2} \phi_x^2(x,t) \mathrm{d}x, \tag{41}$$

where $(x_1, x_2)$ is an interval. By (23), its time derivative is

$$E_t = \left[\Omega(\phi_x) + \phi_x(R(\phi_x^2)\phi_{xx} - \omega(\phi_x^2))\right]_{x_1}^{x_2} - \int\limits_{x_1}^{x_2} R(\phi_x^2)\phi_{xx}^2 \mathrm{d}x \tag{42}$$

where

$$\Omega(k) = \int \omega(k^2) \mathrm{d}k \tag{43}$$

is a primitive of the dispersion curve. The integral in (42) sums up the internal factors, which cause the functional $E$ to decrease. The outside action (the non-integral terms) vanishes, for instance, when

1. there is a periodicity condition on the boundaries $x_1$, $x_2$, for example, the problem is put on a circle;

2. the problem is put on the real axis $(-\infty, +\infty)$, with the identical asymptotic values, so that $\phi_x \to k_0$, $\phi_{xx} \to 0$ as $x \to \pm\infty$;

3. the boundaries are impermeable, this makes sense, e.g., for self-oscillatory systems.

In these cases the (positively definite) functional $E$ monotonically decreases, and the wavenumber evolves to a constant, which is $2\pi N$ for some integer $N$ in the first case, $k_0$ in the second, and 0 in the third.

### 3.3.2 Maximum principle

Assume that local wavenumber has a local maximum at an interior point $x_0$ at some instant of time, $t_0$, so that

$$k_x(x_0, t_0) = 0, \qquad k_{xx}(x_0, t_0) < 0. \tag{44}$$

Then it decreases with time at this point. Indeed,

$$k_t = \frac{\mathrm{d}\omega}{\mathrm{d}k} k_x + \frac{\mathrm{d}P}{\mathrm{d}k} k_x^2 + R k_{xx} \Big|_{k_x=0} = R k_{xx} < 0 \tag{45}$$

if only $R(k(x_0, t_0))$ is positive. Consequently, if the local wavenumber is limited to a range, this range does not increase with time, and typically shrinks.

## 4 Perturbation of a manifold of stationary stationary points

As we noted in the end of Section 2, the heuristic derivation, although producing some reasonably looking result, is flawed, and in any case, can not produce higher-order results if required. So the rest of the paper is devoted to a more accurate treatment of the problem. In this section, we will consider a model problem, for a system of ordinary differential equations. This will be used to introduce the main ideas of the method. The method is well known in several different formulations; the purpose of discussing it here is to introduce the principal ideas in that very form that will be used for the main problem. Therefore, some notations in this section will coincide with the rest of the paper, while some will be different, and the reader is advised to watch out.

### 4.1 Statement of the problem

Let us consider a perturbed system of ordinary differential equations (ODEs)

$$u_t = f(u) + \epsilon h(u), \quad u \in \mathbb{R}^n \tag{46}$$

and assume that at $\epsilon = 0$ it has an $m$-dimensional manifold of stationary points:

$$f(U(a)) = 0, \quad a \in A \subset \mathbb{R}^m, \quad m < n. \tag{47}$$

Here $a$ are coordinates on the manifold. We assume that the stationary manifold is stable (attractive).

If $\epsilon$ is not zero, sufficiently small, the stationary manifold is generally destroyed. Since the manifold was stable, the perturbed system will have another invariant manifold in the vicinity of $U(A)$, but it will no longer be a manifold of stationary points; rather, we will observe a slow dynamics on it. That is, we expect existence of trajectories of the form

$$u = U(a(t)) + \epsilon v(t) \tag{48}$$

Here $v = \mathcal{O}(1)$, and $a$, $v$ are $\epsilon$-slowly varying functions, i.e. they depend upon $t$ only via the combination $\epsilon t$. Here and later we omit explicit dependence of functions on $\epsilon$, for the sake of brevity. The problem is to determine asymptotically the functions $a(t)$ and $v(t)$.

The solution of this problem depends upon required precision and observation interval. If we are interested in the time scales of the order of 1 or less, then we may put $a = \text{const}$, or any other $\epsilon$-slowly varying function, and $v = 0$, or any other function $\mathcal{O}(1)$. The discrepancy between left- and right-hand sides *(residual)* in (46) will be of the order of $\epsilon$, and we can be sure that in some $\epsilon$-vicinity of our "approximate" solution there exists an actual solution of (46). Moreover, we can obtain more precise solutions at these time scales, if we put $a(t) = \text{const}$ and find $v(t)$ by the simplest perturbation procedure.

If we are interested in larger time scales, for instance, $\mathcal{O}(\epsilon^{-1})$, it is not sufficient to find appropriate $v(t)$, since it will increase to the values of the order of $\epsilon^{-1}$ unless we consider the dynamics of $a(t)$. In what follows we show that for the problem of finding appropriate $v$ to be solvable, the vector $a$ must satisfy an evolution equation of the form

$$a_t = \epsilon \mathcal{G}(a) \tag{49}$$

Below we suggest an asymptotic procedure for deriving the equations, which enables approximation of exact solutions in arbitrary large time scales. As an example, we derive such an equation valid for time scale of the order $o(\epsilon^{-3})$.

## 4.2 The requirement of orthogonality

The central idea of the evolution equation derivation is related to the following observation. Note that the representation (48) is ambiguous: we may introduce new variables $\tilde{a}$, $\tilde{v}$:

$$\left\{ \begin{array}{rcl} \tilde{a}(t) & = & a(t) + \epsilon a^{(1)}(t), \\ \\ \tilde{v}(t) & = & v(t) - \partial U(a)/\partial a \, a^{(1)}(t) + \mathcal{O}(\epsilon), \end{array} \right. \tag{50}$$

that will correspond to the same function $u(t)$. In other words, any point in the $\epsilon$-vicinity of the manifold $U$ can be represented by the sum of a vector on $U$ and a vector of the order of $\epsilon$, in different ways, the only necessary condition is that the point on $U$ should lie in some region of size of the order of $\epsilon$. If we want to obtain the right-hand side in evolution (49) with precision better than $\mathcal{O}(\epsilon)$, we should remove this ambiguity. In our example, the precision $\mathcal{O}(\epsilon^3)$ is necessary to have the error $o(1)$ at the time scale $o(\epsilon^{-3})$.

A possible, although by no means the only, choice is that the small correction $\epsilon v$ should be orthogonal to the manifold at the point $U(a)$. Before we put this requirement more formally, let us define the basis of eigenvectors $V_j$, $j = 1..n$ of the Jacobi matrix $F$ of $f$ on the manifold $U(a)$:

$$F(a) = \left. \frac{\partial f(u)}{\partial u} \right|_{u=U(a)}; \qquad F(a)V_j(a) = \lambda_j(a)V_j(a). \tag{51}$$

The equilibria on the manifold $U$ are indifferent with respect to shifts along this manifold, and the tangent space to $U$ is the null-space of the Jacobian $F$,

$$V_j(a) = \partial U(a)/\partial a_j, \qquad \lambda_j(a) = 0, j = 1 \dots m. \tag{52}$$

We assume that the manifold (47) is stable, so

$$\mathrm{Re}\,(\lambda_j) < 0, \quad j = m+1 \dots n. \tag{53}$$

And at last, let us define the basis biorthogonal to $\{V_j\}$. This consists of the eigenvectors $W_j$ of the transposed Jacobian matrix:

$$F^T(a)W_j(a) = \overline{\lambda_j}(a)W_j(a), \tag{54}$$

$$\big\langle W_j(a), V_{j'}(a)\big\rangle = \delta_{j,j'}, \tag{55}$$

where $\langle , \rangle$ is the inner product in $\mathbb{R}^n$, and $\overline{\lambda}$ denotes the complex conjugate of $\lambda$.

Now the requirement of orthogonality, making the representation (48) unambiguous, is written as

$$\langle W_j(a), v\rangle = 0, \quad j = 1..m. \tag{56}$$

**Remark** 4.2.1 To get rid of the ambiguity, the biorthogonality condition (55) is not necessary. Any other set of vectors $(W_j)$ would do as long as they are essentially different and not orthogonal to the null-space of $F$. However, it is usually convenient if $W_j$ all belong to the null-space of $F^T$.

## 4.3 Derivation of the evolution equation

Substitution of the Ansatz (48) into (46) gives

$$\dot{u} = \sum_j^m V_j \dot{a}_j + \epsilon\dot{v} = \epsilon F(a)v + \epsilon^2 f^{(2)}(a)\cdot v\cdot v + \epsilon h(U(a)) + \epsilon^2 h^{(1)}(a)v + \mathcal{O}(\epsilon^3) \tag{57}$$

Here we have used (47) and the Taylor expansions of $f$ and $h$. Note that by using more terms in the Taylor expansion, we could could obtain a higher precision. Since $\mathrm{d}a/\mathrm{d}t = \mathcal{O}(\epsilon)$, we can put

$$\dot{a}_j = \epsilon\mathcal{G}_j, \quad \mathcal{G} = \mathcal{O}(1). \tag{58}$$

Vector equation (57) is a system of $n$ equations for $n+m$ real variables $v$ and $a$, i.e. is under-determined. Let us rewrite it as an equation for $v$, depending on $a$ as a parameter:

$$\dot{v} = F(a)v + (h(U(a)) - \sum_j^m V_j\mathcal{G}_j) + \epsilon\left(f^{(2)}\cdot v\cdot v + h^{(1)}v\right) + \mathcal{O}(\epsilon^2) \tag{59}$$

and expand $v(t)$ in the basis $(V_j(a(t)))$:

$$v(t) = \sum_j^n V_j(a(t)) v_j(t), v_j \in \mathbb{R}. \tag{60}$$

In this section, we shall call the components $v_j$ modes, distinguishing between the *unstable modes* $v_j$, $j = 1 \dots m$ and the *stable modes* $v_j$, $j = m+1 \dots n$. Then the orthogonality condition (56) means that unstable modes are all zero. All $v_j$ satisfy

$$\dot{v}_j = \lambda_j(a) v_j + (h_j(a) - \mathcal{G}_j(t)) + \epsilon \left\{ \sum_k^n h_{jk} v_k + \sum_{k,l}^n (K_{jkl} \mathcal{G}_k v_l + f_{jkl} v_k v_l) \right\} + \mathcal{O}(\epsilon^2). \tag{61}$$

Symbols $\mathcal{G}_j$ for $j \leq m$ are defined by (49), and for convenience we defined $\mathcal{G}_j \equiv 0$ for $j > m$. Symbols $h_{jk}$, $f_{jkl}$ etc. are Taylor coefficients of the functions $h$ and $f$, and $K_{jkl}$ are defined as

$$K_{jkl} = \langle \partial W_j(a)/\partial a_k, V_l(a) \rangle = - \langle W_j, \partial V_l(a)/\partial a_k \rangle . \tag{62}$$

Following the reasoning of Section 4.2, we need to find the functions $a_j(t)$ so as (61) would have a uniformly bounded solution $v(t)$ for all $t$ in the prescribed time interval. Let us find the conditions when

$$v_j(t) \equiv 0, \quad j = 1 \dots m; \quad v_j(t) = \mathcal{O}(1) \quad j = m+1 \dots n; \qquad \forall t \in (-\infty, +\infty). \tag{63}$$

To do this, we determine iteratively $a_j(t)$, $v_j(t)$ with higher and higher precision. We do it by considering alternately the stable modes equations, as equations for $v_j$, and unstable mode equations, as equations for $\mathcal{G}_j$.

Stable-mode equations in leading terms give

$$\mathcal{G}_j(t) = h_j(a(t)) + \mathcal{O}(\epsilon), j = 1 \dots m. \tag{64}$$

According to (58), this is the first-order approximation for the desired evolution equation:

$$\dot{a}_j = \epsilon h_j(a) + \mathcal{O}(\epsilon^2). \tag{65}$$

Solution of this equation will be close to exact solutions of (46) at time scales of the order of $o(\epsilon^{-2})$.

Using higher-order terms in unstable-mode equations, we could obtain more precise approximations for $\mathcal{G}_j$, if only we knew the stable modes $v_j$, $j = m+1 \dots n$ with sufficient precision. If we know $v$ with the precision $\mathcal{O}(\epsilon^\ell)$ for some $\ell$, it is sufficient to get the precision $\mathcal{O}(\epsilon^{\ell+1})$ for $\mathcal{G}$.

Next, let us consider equation (61) for stable modes. If we know $\mathcal{G}_j$ as functions of $a_j$, as in (64), with a precision $\mathcal{O}(\epsilon^\ell)$ for some $\ell$, then we can obtain uniformly bounded solution for $v_j$ with the precision $\mathcal{O}(\epsilon^\ell)$, using $a_j$ as known functions.

To prove this, let us show that if $\mathcal{G}_j$ are known, we can build arbitrary precision approximations of solutions $v$ to equation (61) uniformly at all $t \in (-\infty, +\infty)$ (assuming, of course, that such solutions exist in the vicinity of the manifold (47)). We rewrite equation (61), for brevity, in the form

$$\dot{v}_j = \lambda_j(t) v_j + h_j(t) + \epsilon H_j(t, v(t)), \tag{66}$$

where the term $H_j = \mathcal{O}(1)$ includes, besides dependence upon $\mathcal{G}_j$, any needed number of terms in Taylor expansions of $f$ and $h$. Considering $H_j$ as a known function of time, we formally solve resulting linear equation for $v_j$, and obtain the integral equation

$$v_j(t) = \int_{-\infty}^{t} e^{\Lambda_j(t) - \Lambda_j(\tau)} (h_j(\tau) + \epsilon H_j(\tau, v(\tau))) \mathrm{d}\tau, \tag{67}$$

where $\Lambda$ are primitives of $\lambda$,

$$\mathrm{d}\Lambda_j(t)/\mathrm{d}t \equiv \lambda_j(t). \tag{68}$$

We have selected here the particular solution which is bounded for all $t$, using the fact that all $\mathrm{Re}\,(\lambda_j) < 0$.

This integral equation can be solved iteratively, which yields uniform approximations for exact solutions with successively growing precision in $\epsilon$.

Moreover, due to the stability condition (53) and slowness of $\lambda_j$, $h_j$ and $H_j$ we can present the resulting expressions for $v_j$ in non-integral form. Let us show this for the lowest-order approximation of $v_j$. Since the exponents in (67) decay at time scales longer than $\mathcal{O}(1)$, the major contribution to the integrals is made by time moments $\tau$ close enough to $t$ so that $\tau - t = \mathcal{O}(1)$. Expanding $\Lambda_j$ and $h_j$ in Taylor series in $t$, $\tau$ and, therefore, in $\epsilon$, we get the asymptotic estimates

$$\Lambda_j(\tau) = \Lambda_j(t) + \lambda_j(t)(\tau - t) + \mathcal{O}(\epsilon), \tag{69}$$

$$h_j(\tau) = h_j(t) + \mathcal{O}(\epsilon), \tag{70}$$

valid for these $\tau$. Substituting these expressions into (67) and calculating the integral, we obtain

$$v_j(t) = -h_j(a(t))/\lambda_j(a(t)) + \mathcal{O}(\epsilon). \tag{71}$$

Note that with $a_j(t)$ known, this is a uniform approximation for exact solution of (46) valid for all $t$.

After we get the asymptotic expressions (71), we can return to consideration of unstable-mode equations (61) at $j = 1 \dots m$. Substituting the (71) into (61), we conclude that

$$\begin{aligned} \dot{a}_j &= \epsilon \mathcal{G}_j \\ &= \epsilon h_j(a) + \epsilon^2 \left\{ -\sum_k h_{jk} h_k / \lambda_k + \sum_{k,l} \left( K_{jkl} h_k h_l / \lambda_l + f_{jkl} h_k h_l / (\lambda_k \lambda_l) \right) \right\} \\ &\quad + \mathcal{O}(\epsilon^3) \end{aligned} \tag{72}$$

Discarding the terms $\mathcal{O}(\epsilon^3)$ in this evolution equation implies that its solutions, when substituted into (48) together with $v$ from (71) will produce the residual in the exact equation (46) of the order of $\epsilon^3$, therefore, they approximate exact solutions at time scales $o(\epsilon^{-3})$.

So, it is possible to get arbitrarily precise solutions for stable modes, given sufficiently precise expressions for $\mathcal{G}_j$. The expression (72), therefore, can be used to built the stable modes with the precision $\mathcal{O}(\epsilon^2)$ instead of $\mathcal{O}(\epsilon)$. This solution can be substituted back into unstable-mode equation, which yields the expression for $\mathcal{G}_j$ with the precision of $\mathcal{O}(\epsilon^3)$, and consequently for $da_j/dt$ with the precision $\mathcal{O}(\epsilon^4)$ etc.

We see, that this iterative procedure yields successively more and more precise evolution equations. The significant feature of the procedure is that their solutions approximate exact solutions not only with successively decreasing (in asymptotic sense) error, but also become valid at successively growing time scales. This becomes possible only by *keeping terms of different order in the same equation.*

## 4.4 Associations with other popular methods

### 4.4.1 Systems with slowly varying coefficients

The problem of asymptotic description of slowly varying systems of the form

$$u_t = f(u, \epsilon t, \epsilon), \quad u \in \mathbb{R}^n \tag{73}$$

is trivially reduced to the case already considered: we introduce new dynamic variable $\tau(t)$ governed by the equation

$$\tau_t = 0 + \epsilon \tag{74}$$

and immediately face the problem (46,47) for the vector $(u, \tau)$, with the manifold of stationary points in $\mathbb{R}^{n+1}$ given by the equation

$$f(u, \tau, 0) = 0. \tag{75}$$

### 4.4.2 Tikhonov's fast-slow systems

This is also true for the Tikhonov's standard form of singularly perturbed systems [10, chapter 39]:

$$\begin{aligned} \epsilon u_t &= f(u,v), \\ v_t &= g(u,v) \end{aligned}$$

if we perform the time scaling

$$\begin{aligned} u_{t'} &= f(u,v) \\ v_{t'} &= \epsilon g(u,v) \end{aligned}$$

and note that this system at $\epsilon = 0$ has the manifold of stationary points given by the equation

$$f(u,v) = 0. \tag{76}$$

### 4.4.3 Enslaving principle

The power expansion of an autonomous ODE system near bifurcation of a stable equilibrium may be written in the form

$$\begin{aligned} \dot{v}^u &= \epsilon v^u + N^u(v^u, v^s), \\ \dot{v}^s &= -Lv^s + N^s(v^u, v^s). \end{aligned}$$

Here $v^s$ is a vector, $L$ is a matrix, $\mathrm{Re}\,(\lambda(L)) > 0$, $\epsilon \to 0$ is the supercriticity and $N^{s,u}$ have zero linear parts in their Taylor expansions. This system is transformed by scaling into that of the form (46), with the stationary manifold $v_s = 0$. The iteration procedure described above will involve expression for $v^s$ via $v^u$, which is just the "slaving principle of synergetics" [11]. The exact form of the dependence $v^s(v^u)$ is closely related to the famous concept of the central manifold [9]; see also Section 4.5 below.

### 4.4.4 Krylov-Bogoliubov averaging

If the unperturbed system has a manifold of periodical solutions instead of stationary points, then literal translation of the above formalism seems impossible, because cycles are more complex mathematical objects than equilibria. Nonetheless, the basic ideas of the above derivation: variation of arbitrary constants, i.e. the coordinates on the manifold of unperturbed solutions, and the requirement of orthogonality — remain applicable. The difference is that the coefficients of the evolution equations include integration of scalar product $\langle , \rangle$ over a period. This method was first used by Krylov and Bogoliubov [7].

Finally, it should be noted that the case of stationary manifold in (46) is structurally unstable, and there should be some special reasons for such a problem to occur in applications, while the case of isolated periodic solution (limit cycle) is generic, and we always have at least one arbitrary constant, the initial phase, associated with the fact that $f()$ does not depend explicitly on time.

### 4.5 Subcenter manifold

Especially important is the association of the described method is with the method of the (sub)center manifold, which is both a fundamentally important theoretical concept and an efficient practical tool.

The procedure of alternating increase of asymptotic precision of the $\mathcal{G}(a)$ and $v(a)$ leads to building asymptotic series in $\epsilon$ for these relationships. If these series converge for some $\epsilon$, this means that each of those $\epsilon$ we will have an invariant manifold

$$\mathcal{U}(a,\epsilon) = U(a) + \epsilon v(a,\epsilon) \tag{77}$$

and the motion on that manifold given by

$$\dot{a} = \mathcal{G}(a,\epsilon). \tag{78}$$

The unique dependence of $v$ on $a$ arises when we resolve the integral equation (67), since this procedure chooses a unique solution to the differential equation.

In the extended phase space $\mathbb{R}^n \times \mathbb{R} = \{u, \epsilon\}$, manifold $U$ is an intersection of the manifold $\mathcal{U}$ by the hyperplane $\epsilon = 0$. It be easily seen that the difference between $\mathcal{U}$ and $U$ at small $\epsilon$ is along $V_0$, which means that $\mathcal{U}$ is tangent to the center subspace of $U$. Thus it is a center manifold or a sub-center manifold; the technical difference is not important for us here, as all we use is the formalism.

This motivates an alternative approach to building the asymptotic evolution equation: from the very beginning, to look for representations of (77) and (78) in the form of power series in $\epsilon$ straightaway, instead of coming to (77) via the complicated procedure described above. This certainly is a very efficient method from a practical viewpoint, which will be demonstrated later for the SVW equation. The slight disadvantage of this method for theory and for study purposes is that since for every $a$ and $\epsilon$ it focuses on the single solution out of an infinite variety, it completely ignores the dynamics around the (sub)-center manifold, and masks its origin. Another drawback of the subcenter manifold approach is that it is hardly applicable to problems explicitly depending on time. And as far as actual calculations rather than their motivation are concerned, both methods are strikingly similar, as we shall see on the same example.

## 5 Method of the detecting operator

### 5.1 Preliminary comments

In this section, we will restrict our consideration to one spatial dimension, $n = 1$; this simplifies the formulae but addresses the main difficulties. Correspondingly, we will parametrise the periodic solutions $U$ and its frequency $\omega$ with $k$ which is now the wavenumber rather than wavevector, instead of $\eta$. Thus the key formulae slightly change their shape. The two-dimensional manifold of periodic wave solutions is

$$u(x,t) = U(kx - \omega(k)t + \phi_0, k) = U(\xi, k), \tag{79}$$

with the coordinates of the initial phase $\phi_0$ and the wavenumber $k$, lying in an interval.

Function $U(\xi, k)$ satisfies the equation

$$k^2 D U_{\xi\xi}(\xi, k) + \omega(k) U_\xi(\xi, k) + f(U(\xi, k)) = 0. \tag{80}$$

and the periodicity condition

$$U(\xi + 2\pi, k) \equiv U(\xi, k). \tag{81}$$

The linearised operator $\mathcal{L}$ now has the form

$$\mathcal{L}(k) = k^2 D \partial_\xi^2 + \omega(k) \partial_\xi F(U(\xi, k)), \tag{82}$$

and its shift (Goldstone) eigenfunction is

$$\mathcal{L} V_0 = 0, \qquad V_0(\xi, k) \equiv U_\xi(\xi, k). \tag{83}$$

The gauge transformation now looks

$$U^K(\xi, k) = U(\xi + K(k), k). \tag{84}$$

The SVW Ansatz (8) becomes

$$u = U(\phi, \phi_x) + v \tag{85}$$

where

$$\phi = \epsilon^{-1} \Phi(\epsilon x, \epsilon t, \epsilon), \quad \epsilon \ll 1, v \ll 1. \tag{86}$$

An arbitrary function of the form (85), (86) when substituted into original equation yields a residual of the order of 1, and, therefore, is inapplicable already on the time scales $\mathcal{O}(1)$.

For the base periodic waves (79), their phase $\phi$ obeys

$$\phi_t = \omega(\phi_x) \tag{87}$$

If we consider (87) formally as an evolution equation for $\phi$, and substitute a solution of it to (85,1), it will lead to a residual in (1) of the order of $\epsilon$, so it is applicable at the time scales $\mathcal{O}(1)$ but not $\mathcal{O}(\epsilon^{-1})$. Finding a proper function $v(x,t)$ can improve the precision at the times $\mathcal{O}(1)$, as it was done in papers [3, 5], but cannot enlarge the time scale in which it is applicable.

This situation is similar to that of Section 4.1, and to enlarge the time scale, we should get and solve a more precise evolution equation for $\phi$, that would give a smaller residual in (1), say, of the order of $\epsilon^2$. To do that we shall use the technique developed in Section 4, properly modifying it to this spatially distributed problem.

## 5.2 The orthogonality requirement and the detecting operator

As it was made for finite-dimension examples, let us note, that the representation (85) is ambiguous, when considered with the precision $\mathcal{O}(\epsilon)$. That is, we can add some arbitrary sufficiently small function of the order of $\epsilon$ to $\phi$ and then compensate this change by changing the correction $v$ so that the function $u$ remains the same:

$$\begin{aligned} u &= U(\phi, \phi_x) + v \\ &= U(\phi + \epsilon\chi, (\phi_x + \epsilon\chi_x)) + \left[v - \epsilon U_\xi(\phi, \phi_x)\chi - 2\epsilon U_k(\phi, \phi_x)\phi_x\chi_x + \mathcal{O}(\epsilon^2)\right] \end{aligned} \tag{88}$$

The assumptions (86) are not violated by this transformation, if the function $\chi$ is also $\epsilon$-slow.

If we want to obtain results with the required accuracy, this ambiguity should be eliminated. Note that the transformation (88) leads, in particular, to change in the "amount" of shift mode (83) in the term $v$, locally in every place. So, the representation becomes unambiguous if we require that this "amount" is zero with the required precision, as it was made in Section 4.

Since this requirement should be defined locally, we cannot use simply the scalar product with the adjoint modes. Therefore we define a linear operator with the sense of "detector" of the shift mode, which "measures" the "amount" of the shift mode separately in every place. Since we deal with oscillating basic solution, we should use time integration (see Section 4.4.4). However, this can be avoided, if we use independent variables $\xi = \phi(x, t, \epsilon)$, $\tau = t$ instead of $x$, $t$. In coordinates $\xi, \tau$, the perturbed solution is not oscillating, but only slowly varying in time, and time averaging is no longer required.

In the coordinates $(\xi, \tau)$ the correction $v$ is governed by an equation with coefficients approximately periodic in space and slowly varying in time. Therefore, after sufficient span of time the solution $v$ also becomes approximately periodic and can be expanded in the sum

$$v(\xi, t, \epsilon) = \sum_j A_j(\epsilon\xi, \epsilon t, \epsilon) V_j(\xi, k(\epsilon\xi, \epsilon t, \epsilon)), \tag{89}$$

where $V_j$ are the $2\pi$-periodic eigenfunctions of linear operator $\mathcal{L}$ given by (82), and

$$k \equiv (\partial x/\partial\phi)^{-1} = k(\epsilon\xi, \epsilon\tau, \epsilon). \tag{90}$$

Now the requirement of orthogonality may be formulated as follows: this expansion of $v$ should never contain the shift mode, i.e. $A_0 \equiv 0$. So we need a tool to determine the function $A_0(\epsilon\xi, \epsilon)$, from a given function $v(\xi, \epsilon)$, at a fixed time moment. It would be a linear operator with the property

$$\mathcal{D}v = A_0 \tag{91}$$

for every $v$ of the form (89). We shall call $\mathcal{D}$ *detecting operator*. It seems to be impossible to ensure (91) exactly, but it would be sufficient for us, if it is valid to the precision $\epsilon^2$:

$$\mathcal{D}_2 v = A_0 + \mathcal{O}(\epsilon^2). \tag{92}$$

To construct this operator, we define the biorthogonal basis to the eigenfunctions of $\mathcal{L}$, which consists of eigenfunctions $W_j$ of the adjoint operator $\mathcal{L}^+$,

$$\mathcal{L}^+(k)W_j(\xi,k) = \overline{\lambda_j}(k)W_j(\xi,k) \tag{93}$$

so that

$$\int_0^{2\pi} \big\langle W_j(\xi,k), V_{j'}(\xi,k)\big\rangle \,\mathrm{d}\xi = \delta_{j,j'}. \tag{94}$$

Then we make the first attempt to define the detecting operator, as the local projector onto the shift (Goldstone) mode:

$$\mathcal{D}_1 v(\xi) = \int\limits_{\xi-\pi}^{\xi+\pi} \langle W_0(\zeta, k(\zeta)), v(\zeta)\rangle \,\mathrm{d}\zeta. \tag{95}$$

Substituting (89) into (95) and using Taylor expansions of the slowly varying functions around the middle of the integration periods, we get

$$\begin{aligned}
\mathcal{D}_1 v \;&=\; A_0(\epsilon\xi) && (96)\\
&\quad + \epsilon \sum_j (A_j'(\epsilon\xi) + A_j(\epsilon\xi)k'(\epsilon\xi)\partial_k) \int\limits_{\xi-\pi}^{\xi+\pi} \langle W_0(\eta,k), V_j(\eta,k)\rangle\,(\eta-\xi)\,\mathrm{d}\eta\Big|_{k=k(\epsilon\xi)} && (97)\\
&\quad + \mathcal{O}(\epsilon^2) && (98)
\end{aligned}$$

where for brevity the prime $'$ denotes differentiation by the shown argument.

We see that $\mathcal{D}_1$, indeed, is a detecting operator, but the detection error is of the order of $\epsilon$ and not of $\epsilon^2$ as we wanted. The analysis of (98) also shows the way how to improve the situation. Note that the error is an oscillating function of $\xi$ in the order $\mathcal{O}(\epsilon)$. So an operator of sliding averaging, e.g.

$$\mathcal{A}g(\xi) = \frac{1}{2\pi} \int\limits_{\xi-\pi}^{\xi+\pi} g(\zeta)\,\mathrm{d}\zeta, \tag{99}$$

would cut this error off, without affecting the main signal $A_0$. This is proved, again, by using Taylor expansions of the slowly varying integrands.

Thus we see that the operator

$$\mathcal{D}_2 = \mathcal{A} \circ \mathcal{D}_1 \tag{100}$$

obeys the property (92), i.e. it is the detecting operator needed. Recall, this operator works in the space of functions of one spatial variable $\xi$; it depends on time $\tau$ and on $\epsilon$ as parameters, via $k()$.

Now we can specify the requirement of unambiguity of the representation (85) with the help of detecting operator in the following way:

$$\mathcal{D}_2 v = \mathcal{O}(\epsilon^2) \tag{101}$$

It is possible to make every SVW (85) to satisfy (101), by using the transformation (88). To do that, we would measure the amplitude of the Goldstone mode in $v$ by the detecting operator $\mathcal{D}_2$, and then use this very amplitude as the correction $\chi$ to the phase $\phi$. To show this, it is convenient to make $\partial U/\partial k$ orthogonal to $W_0$ via a gauge transformation (84) with appropriately chosen $K$ (see below, (105)).

**Remark** 5.2.1 To eliminate the ambiguity of $u$ with the precision $\epsilon^2$ it might be sufficient to use the simpler detecting operator of the order of $\epsilon$, since the correction $\epsilon v$ itself is small of the order of $\epsilon$. The second order of the precision will be required later, to detect the free term $h$ in the linear equation for $v$, which contains terms $\mathcal{O}(1)$.

**Remark** 5.2.2 The above construction shows, how the process of increasing the precision of detection can be continued iteratively. Such an increase will be required if we need more and more precise evolution equations.

**Remark** 5.2.3 Detecting operators for modes other than $V_0$ can be constructed similarly, by using other adjoint eigenfunctions $W_j$ instead of $W_0$.

## 5.3 The evolution equation

By the assumptions made, the function $v$ obeys the following asymptotic equation, in coordinates $\xi = \phi$, $\tau = t$:

$$v_\tau = k^2(\epsilon\xi, \epsilon\tau, \epsilon) D v_{\xi\xi} + \Omega(\epsilon\xi, \epsilon\tau, \epsilon) v_\xi + F(\epsilon\xi, \epsilon\tau, \epsilon) v + h(\xi, k(), \partial k()/\partial\xi, \Omega()) + \mathcal{O}(\epsilon^2) \tag{102}$$

Here $k()$, defined by (90) and $\Omega()$, defined as

$$\Omega(\epsilon\xi, \epsilon\tau, \epsilon) = -(\partial t/\partial\phi)^{-1} \tag{103}$$

are known slowly varying functions, and $h$ has the form

$$h = U_\xi(\Omega - \omega(k)) + U_k(\Omega k_\xi - k_\tau) + D(U_\xi + U_{\xi k} k)\, k\, k_\xi \tag{104}$$

It is now the time to choose the initial phases of $U$ for different $k$ for the gauge transformation (84). Namely, we require that

$$\oint \langle W_0(\xi, k), U_k(\xi, k)\rangle \, \mathrm{d}\xi = 0, \qquad \forall k. \tag{105}$$

When this is fulfilled, we apply the operator $\mathcal{D}_2$ to both sides of (102) and conclude that

$$[\Omega(\epsilon\xi, \epsilon\tau, \epsilon) - \omega(k(\epsilon\xi, \epsilon\tau, \epsilon))]\, \mathcal{D}_2 v_\xi + \mathcal{D}_2 h = \mathcal{O}(\epsilon^2) \tag{106}$$

Since $v = \mathcal{O}(\epsilon)$, equation (106) means that

$$\mathcal{D}_2 h = \mathcal{O}(\epsilon). \tag{107}$$

Selecting here terms $\mathcal{O}(1)$, we get

$$\Omega(\epsilon\xi, \epsilon\tau, \epsilon) - \omega(k(\epsilon\xi, \epsilon\tau, \epsilon)) = \mathcal{O}(\epsilon) \tag{108}$$

After substituting this result back into (106), we conclude that

$$\mathcal{D}_2 h = \mathcal{O}(\epsilon^2). \tag{109}$$

In principle, this is already the required result, just slightly disguised. After some algebra, involving the detecting property (101) of $\mathcal{D}_2$, slowness of the functions $k$ and $\Omega$, and then changing to the original independent variables $x$, $t$, equation (109) leads to

$$\phi_t = -\omega(\phi_x) + R(\phi_x)\phi_{xx} + \mathcal{O}(\epsilon^2), \tag{110}$$

where

$$R(k) = \oint \langle W_0(\xi, k), D(U_\xi(\xi, k) + 2kU_{\xi k}(\xi, k))\rangle \; \mathrm{d}\xi. \tag{111}$$

Equation (110) is the desired evolution equation of the precision $\epsilon^2$. It coincides with the equation obtained by heuristic considerations in Section 2.

**Remark** 5.3.1 The technique of the detecting operator can help in deriving more precise evolution equations, in the way it was done for the finite-dimension example in Section 4. Some new features occur on this way:

- More precise detecting operators are needed,
- Nonlinear terms in the Taylor expansion of the function $f(u)$ should be taken into account,
- The amplitudes of the stable modes should be determined, therefore the detectors for these modes are needed.

### 5.4 Degenerated (conservative) waves: an example

As already noted, autowaves are an example of "generic" waves, not depending on special properties of the PDE system, like conservation laws. Thus the family of the basic, plane periodic wave solutions is the poorest possible. In conservative systems, the conservation laws add new parameters to the families, corresponding to the conserved quantities. For instance, in systems where total energy is conserved, waves of different amplitudes are possible, that have different energy content.

In this section, we will show that the developed technique of the detecting operator can be used to treat slowly varying waves in such systems, by taking into account the available multi-parametric family of basic waves, but without directly exploiting the conservation laws.

As an example, we choose the well known nonlinear Klein-Gordon equation

$$u_{tt} - u_{xx} + f(u) = 0 \tag{112}$$

for the scalar field $u$; here $f(u)$ is a nonlinear function. The 2-parameter family of periodical wave solutions, depending on parameters $k$ and $a$, is determined by equations

$$\begin{aligned} &u = U(\xi, a), \qquad \xi = kx - \omega t, \qquad \omega^2 = E(a) + k^2, \\ &E(a)U_{\xi\xi} + f(U) = 0, \qquad U(\xi + 2\pi, a) \equiv U(\xi, a). \end{aligned} \tag{113}$$

where $k$ is the wave number, $\omega(k, a)$ is the frequency and $E$ is an arbitrary constant, that is related to the conserved quantity in (112) and could be used to distinguish between the waves of different amplitudes $a$.

**Remark** 5.4.1 There is no unique way to define the amplitude $a$, as the shape of different basic waves in the same equation may be different. In different situation different choices of parameter $a$ may be preferable. For instance, it might seem natural to choose quantity $E$ as the amplitude parameter, thus allowing $E(a) \equiv a$. But this does not work for linear waves, where $E$ is the same for all amplitudes and thus cannot be used to distinguish between them.

We present the derivation of the evolution equations briefly, as the technique was already described in detail, and now we only present the main milestones and points specific to this particular problem.

The SVW Ansatz is

$$\begin{aligned} u(x,t,\epsilon) &= U(\phi(x,t,\epsilon), a(x,t,\epsilon)) + v(x,t,\epsilon), \\ \phi(x,t,\epsilon) &= \frac{1}{\epsilon}\Phi(\epsilon x, \epsilon t, \epsilon), \\ a(x,t,\epsilon) &= A(\epsilon x, \epsilon t, \epsilon), \\ v = \mathcal{O}(\epsilon), &\quad \Phi, A = \mathcal{O}(1). \end{aligned} \tag{114}$$

Linear approximation for $v$ is

$$\mathcal{L}v + h = 0, \tag{115}$$

where

$$\mathcal{L}v \equiv v_{tt} - v_{xx} + f'(U(\phi, a))v \tag{116}$$

and

$$h = (\phi_t^2 - \omega^2(a, \phi_x))U_{\xi\xi} + (\phi_{tt} - \phi_{xx})U_\xi + 2(\phi_t a_t - \phi_x a_x). \tag{117}$$

The representation (114) is ambiguous with respect to the transformation

$$\begin{aligned} \phi &\to \phi + \epsilon\delta\phi \\ a &\to a + \epsilon\delta a \\ v &\to v - \epsilon U_a \delta a - \epsilon U_\xi \delta\xi + \mathcal{O}(\epsilon^2) \end{aligned}$$

So we should impose the conditions implying $\delta a = 0$, $\delta\xi = 0$.

The calculations will be simplified if we use the following identities specific to the evolution equation (112):

$$\oint U_\xi U_a \,\mathrm{d}\xi = \oint U_\xi U_{\xi\xi} \,\mathrm{d}\xi = \oint U_a U_{\xi a} \,\mathrm{d}\xi = 0, \tag{118}$$

$$\oint U_{\xi\xi} U_a \,\mathrm{d}\xi = -\oint U_\xi U_{\xi a} \,\mathrm{d}\xi = \frac{\mathrm{d}G}{\mathrm{d}a} \tag{119}$$

where

$$G(a) = \oint U_\xi^2 \,\mathrm{d}\xi. \tag{120}$$

Note that these identities are not crucial for the method as such, only simplify the calculations.

**Remark** 5.4.2 Since choice of the amplitude parameter $a$ is arbitrary (recall Remark 5.4.1), equation (120) does *not* really define function $G(a)$; rather, it defines the quantity $G$ for a given shape of $U$, and what value of $a$ designates the shape is another issue.

From these identities we can see, in particular, that to construct the detectors we may use the kernels $\partial U/\partial\xi$ and $\partial^2 U/\partial\xi^2$ for detecting $\delta\phi$ and $\delta a$, respectively. Here we recall Remark 4.2.1 that the projectors do not have to be build from the adjoint eigenfunctions.

So we impose the following conditions of unambiguity:

$$\mathcal{D}\{U_\xi\}v = \mathcal{D}\{U_{\xi\xi}\}v = 0. \tag{121}$$

Here $\mathcal{D}\{\varkappa\}$ denotes the second-order detecting operator constructed from function $\varkappa$ as a kernel, in the same was as it was done in (95), (99) and (100). From the equation

$$\mathcal{D}\{U_\xi\}(\mathcal{L}v + h) = \mathcal{D}\{U_\xi\}h = 0 \tag{122}$$

we get the following evolution equation:

$$G(a)(\phi_{tt} - \phi_{xx}) + \frac{dG}{da}(\phi_t a_t - \phi_x a_x) = 0, \tag{123}$$

and from

$$\mathcal{D}\{U_a\}(\mathcal{L}v + h) = \mathcal{D}\{U_{\xi\xi}\}v + \mathcal{D}\{U_a\}h = \mathcal{D}\{U_a\}h = 0 \tag{124}$$

get the second evolution equation:

$$\phi_t^2 = \omega^2(\phi_x, a) = E(a) + \phi_x^2. \tag{125}$$

Suppose the wave propagates to the right and the function $\omega(k, a)$ is positive, and introduce the local wave number $k \equiv \partial\phi/\partial x$. Then the second evolution equation (125) can be rewritten as

$$\phi_t = -\omega(\phi_x, a) \tag{126}$$

or

$$k_t + \omega_k k_x + \omega_a a_x = 0. \tag{127}$$

Using (126), we express $\partial\phi/\partial t$ and $\partial^2\phi/\partial t^2$ in (123) via spatial derivatives, and get the equation

$$(\omega G_a + G\omega_a)a_t + (kG_a - \omega_k\omega_a G)a_x + G(1 - \omega_k^2)k_x = 0. \tag{128}$$

Equations (127) and (128) are the required SVW equations for the nonlinear Klein-Gordon equation (112). It is easy to see that a linear case they correspond to the well known eikonal equation for the phase and transport equation for the amplitude:

$$(a^2)_t + (\omega_k a^2)_x = 0. \tag{129}$$

Here we made the most usual choice of amplitude parameter, $G(a) = a^2$ (recall Remark 5.4.2), and used the explicit form of the dispersion relationship, $\omega^2 = E + k^2$.

For a nonlinear case, it is convenient to choose the amplitude parameter $a \equiv G(a)$. Then equation (128), with the help of (126), can be rewritten in the form

$$(\omega(k, a)G(a))_t + (kG(a))_x = 0 \tag{130}$$

As it should be, equations (127), (130) coincide, up to the notations difference, with evolution equations obtained by the Whitham method, see e.g. monograph [2, page 516].

## 6 The subcenter manifold approach

In the previous section, we extended the asymptotic procedure used in the model finite dimensional problem of Section 4 to the PDEs, by applying it in every point of space. Another way to do such extension was suggested by Roberts [12]. This approach considers the whole phase distribution $\phi(x)$, in its entirety, as a coordinate on the manifold $U$. Thus, instead of a finite dimensional manifold, we consider a functional manifold.

The formalism is based on consideration of the starting PDE system (1) as an ordinary differential equation in a Banach space $\mathcal{Y}$:

$$\frac{\mathrm{d}\hat{u}}{\mathrm{d}t} = \hat{f}(\hat{u}) \tag{131}$$

where $\hat{u}(t)$ is a function of time with values in a Banach space $\mathcal{Y}$, representing the spatial distribution of the field $u(x,t)$:

$$\hat{u} : \mathbb{R} \to \mathcal{Y}; \quad t \mapsto u(x,t) \tag{132}$$

The procedure is inspired the (sub)center version of the asymptotic procedure, as discussed above in Section 4.5. It is based on the Ansatz

$$u(x,t) = \mathcal{U}[\phi(x,t)], \tag{133}$$

where square brackets [] denote functional dependence. We will need only local dependence, so we will assume that $[\phi]$ denotes dependence on $\phi$ and *all its spatial derivatives*, $\mathcal{U}(\phi, \partial\phi/\partial x_i, \partial^2\phi/\partial x_i \partial x_j, \dots)$. Ansatz (133) can be written in Banach-vector form as

$$\hat{u}(t) = \hat{\mathcal{U}}(\hat{\phi}(t)), \tag{134}$$

where $\hat{\phi}$ is the vector of a smaller functional space $\mathcal{X}$, representing the spatial distribution of the phase $\phi(x,t)$ at a particular time instant $t$:

$$\hat{\phi} : \mathbb{R} \to \mathcal{X}; \quad t \mapsto \phi(x,t) \tag{135}$$

and $\hat{\mathcal{U}} : \mathcal{X} \to \mathcal{Y}$.

Space $\Phi$ is "smaller" than $\mathcal{Y}$, e.g. in the sense that it consists of scalar functions rather than $\mathbb{R}^\ell$-valued functions as $\mathcal{Y}$. Another non-formalised difference is that $\hat{\phi}$ will represent functions $\phi(x,t)$ as in (9), i.e. with slowly varying spatial derivatives.

The evolution of the phase $\phi(x,t)$ is sought in the form

$$\frac{\partial\phi}{\partial t} = \mathcal{G}[\phi(x,t)] \tag{136}$$

or

$$\frac{\mathrm{d}\hat{\phi}}{\mathrm{d}t} = \hat{\mathcal{G}}(\hat{\phi}(t)) \tag{137}$$

where $\hat{\mathcal{G}} : \mathcal{X} \to \mathcal{X}$.

Then, the operators $\hat{\mathcal{U}}$ and $\hat{\mathcal{G}}$ are sought in the form of formal (asymptotic) power series in the small parameter $\epsilon$,

$$\hat{\mathcal{U}} = \sum_{n=0}^{\infty} \hat{\mathcal{U}}^{(n)} \epsilon^n, \qquad \hat{\mathcal{G}} = \sum_{n=0}^{\infty} \hat{\mathcal{G}}^{(n)} \epsilon^n. \tag{138}$$

or

$$\mathcal{U} = \sum_{n=0}^{\infty} \mathcal{U}^{(n)} \epsilon^n, \qquad \mathcal{G} = \sum_{n=0}^{\infty} \mathcal{G}^{(n)} \epsilon^n. \tag{139}$$

Then we use the procedure described in Section 4, expressed in terms of $\hat{\mathcal{U}}$ and $\hat{\mathcal{G}}$ as a guidance, but immediately translate obtained expressions to the straightforward form, without hats.

The orbit derivative of (133) by the system (136) yields

$$\partial u / \partial t = \mathcal{U}_\phi \mathcal{G} + \mathcal{U}_{\nabla\phi} \nabla \mathcal{G} + \mathcal{O}(\epsilon^2) \tag{140}$$

since, as we mentioned, $\mathcal{U}$ is function of $\phi$ and all its derivatives, and derivatives of higher orders are, according to (9), of higher asymptotic orders in $\epsilon$.

In turn, differentiation of $\mathcal{G}$ by spatial variables, using the chain rule, yields

$$\partial \mathcal{G} / \partial x_i = \mathcal{G}_\phi \phi_i + \mathcal{G}_{\phi_j} \phi_{ij} + \mathcal{O}(\epsilon^2). \tag{141}$$

Here and below, we denote spatial derivatives of $\phi$ by subscripts corresponding to the spatial coordinate, so that $\phi_i$ means $\partial\phi/\partial x_i$ etc.; and also assume summation by repeated indices.

Substituting now (141) and (140) into (1), with account of expansions (139) yields

$$\begin{aligned} u_t \quad &= \quad \mathcal{U}_\phi^{(0)} \mathcal{G}^{(0)} + \mathcal{U}_\phi^{(1)} \mathcal{G}^{(0)} + \mathcal{U}_\phi^{(0)} \mathcal{G}^{(1)} + \mathcal{U}_{\phi_i}^{(0)} \mathcal{G}_{\phi_j}^{(0)} \phi_{ij} + O(\epsilon^2) \\ = f(u) + D u_{x_i, x_i} \quad &= \quad f(\mathcal{U}^{(0)}) + F(\mathcal{U}^{(0)}) \mathcal{U}^{(1)} \\ & \quad + D \left( \mathcal{U}_{\phi\phi}^{(0)} \phi_i \phi_i + 2 \mathcal{U}_{\phi\phi_j}^{(0)} \phi_i \phi_{ij} + \mathcal{U}_\phi^{(0)} \phi_{ii} + \mathcal{U}_{\phi\phi}^{(1)} \phi_i \phi_i \right) \\ & \quad + \mathcal{O}(\epsilon^2) \end{aligned} \tag{142}$$

Then we consider sequentially different orders in $\epsilon$ of this equation.

**Order $\mathcal{O}(1)$.** Equating terms of (142) of the order $\mathcal{O}(1)$ we have

$$-\mathcal{U}_{\phi^{(0)}} \mathcal{G}^{(0)} = f(\mathcal{U}^{(0)}) + D \mathcal{U}_{\phi\phi}^{(0)} \phi_i \phi_i. \tag{143}$$

This coincides with the equation (5) for the basic waves if

$$\mathcal{U}^{(0)} = U; \qquad \mathcal{G}^{(0)} = -\omega; \tag{144}$$

and $\phi_i\phi_i$ is identified with $\eta$. So, together with the requirement of periodicity in $\phi$, this provides a nonlinear eigenvalue problem determining the basic solution $\mathcal{U}^{(0)} = U$ and the main frequency $-\mathcal{G}^{(0)} = \omega$ as functions of the local value of the slowly varying phase gradient $\eta = \phi_i\phi_i = (\nabla\phi)^2$.

**Order $\mathcal{O}(\epsilon)$.** In the order $\mathcal{O}(\epsilon)$, equation (142) gives an equation which is convenient to interpret as a linear equation for $\mathcal{U}^{(1)}$:

$$\mathcal{L}\mathcal{U}^{(1)} = h \tag{145}$$

where the linear operator $\mathcal{L}$ is

$$\mathcal{L} = \eta D\partial_{\phi\phi} + \omega(\eta)\partial_\phi + F \tag{146}$$

and the free term $h$ depends on $\mathcal{G}^{(1)}$,

$$h = \mathcal{G}^{(1)}\mathcal{U}^{(0)}_\phi + \mathcal{U}^{(0)}_{\phi_i}\mathcal{G}^{(0)}_{\phi_j} - D\mathcal{U}^{(0)}_{\phi\phi}\phi_{ii} - 2D\mathcal{U}^{(0)}_{\phi\phi_j}\phi_i\phi_{ij}. \tag{147}$$

Linear operator $\mathcal{L}$ defined by (146) is singular, and its zero eigenvalue is provided by the Goldstone mode $V_0 = \partial U/\partial\phi$. By the Fredholm alternative, (145) is solvable iff the free term is orthogonal to the eigenfunction $W_0$,

$$\mathcal{L}^+ W_0 = 0 \tag{148}$$

of the adjoint operator $\mathcal{L}^+$,

$$\mathcal{L}^+ = \eta D\partial_\phi^2 - \omega(\eta)\partial_\phi + F^T \tag{149}$$

As before, we fix the arbitrariness in $U$ and $W_0$ by requiring (21) and (22), and then the answer for $\mathcal{G}^{(1)}$ looks

$$\mathcal{G}^{(1)} = \left(\oint \left\langle W_0, D\mathcal{U}^{(0)}_\phi \right\rangle \mathrm{d}\phi\right)\phi_{ii} + 2\left(\oint \left\langle W_0, D\mathcal{U}^{(0)}_{\phi\phi_j} \right\rangle \mathrm{d}\phi\right)\phi_i\phi_{ij} \tag{150}$$

Noting that $\mathcal{U}^{(0)}$ depends on $\phi$ only via $\eta = \phi_i^2$, we arrive ultimately to the evolution equation precisely coinciding, up to the notation difference, with (23), with identical definitions of $P$ and $Q$.

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