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MACRODYNAMICAL APPROACH TO THE ANALYSIS OF NEURAL NETWORKS

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"Макродинамический подход к анализу нейронных сетей"

Описывается математический метод для анализа сложных систем типа нейронных сетей. Метод представляет собой обобщение известного подхода "среднего поля", и позволяет анализировать не только стационарные состояния, но и динамические свойства сети. Метод может быть также интерпретирован как Галеркинская процедура по отношению к уравнению эволюции для функции распределения. Обсуждаются типы нейронных сетей и задачи, для которых может быть применен данный метод. Показано, что в его рамках могут быть описаны процессы синхронизации, сети возбудимых нейронов, сети из неодинаковых нейронов и с неодинаковыми связями.

Abstract. General features of a mathematical method of analyzing complex systems like neural networks are presented. The method is a generalization of the mean-field approach and allows analyzing not only "steady-states" but also dynamical properties of the network. The method can be also interpreted as a Galerkin procedure for the master equation. The types of neural networks and related problems to which the method can be applied are discussed. It is shown that method can treat synchronization processes, networks of excitable neurons and ones of non-identical neurons and synapses.

Key words: complex systems, neural networks, coupled oscillators, macrodynamics, Galerkin procedure, master equation, mean-field equation, pair interaction, summatory functions.

Contents

1. Introduction

The analysis of complex systems is a very important problem in various fields of science. There are many systems for which the direct simulations is not possible and, perhaps, never will be, because of huge number of elements. On the other side, we usually do not need to know the detailed behavior of such a system, but only interested in studying some general (macroscopical) properties.

Therefore, it is very useful to get examples of systems to which some exact or asymptotical statements in the limit $N \longrightarrow \infty$ can be made, where N is the number simple subsystems (elements, for instance) constituting the whole system.

The neural networks are significant examples of systems for which the specified problem is of principal. Many of mathematical models of neural networks can be formulated in the form

$$\dot{\varphi}_i = a_i(\varphi_i) + \sum_{j=1}^{N} b^{N}_{i,j}(\varphi_i, \varphi_j), \quad i=1..N \qquad (1)$$

where dynamical variables φ_i denote states of neurons (or neuron clusters) and are vectors of a few dimensions; the functions a_i describe their "own" dynamics and $b^N_{i,j}$ describe functioning of synapses (or intercluster interaction).

Many types of complex systems of another physical origin can also be presented in the form (1).

We want to study the "macroscopical" properties of the system (1) in the limit $N \to \infty$. In the most general formulation, even the definition what are the "macroscopical properties" is a problem and should be specially stated; we shall call this statement as an *observation postulate*. Usually these properties should not vary considerably if a relatively small number of neurons is removed or essentially change their properties (see, e.g. [1]).

Another related problem is what is the rule that we should choose and change the functions a_i, $b_{i,j}$ as $N \to \infty$. Such a rule we call a *structure postulate*.

These two postulates determine the specifics of any approach to the analysis of a complex system. For instance, the so called autowave media may be interpreted as neural lattices (see e.g. [2]) with nearest neighbors, symmetric and linear connections (this is the structure postulate), are macroscopically described by partial differential equations, and therefore the observation postulate implies that we neglect the fine structure of the lattice dynamics.

Another example is the network where all the neurons are identical and all connected to each other in an identical way. For this case the mean-field approach seems natural enough [3].

We suggest a generalization of the mean-field approach which leads to the description of the network in terms of the neurons states distribution density in "microscopical" phase space $\{\varphi\}$, since these spaces are identical for all neurons.

In this description the *observation postulate* claims: macroscopical properties are defined by the (weak) limit of the distribution density as $N \to \infty$. This is really a generalization of the mean-field approach, because the mean-field variable defined in [3] is simply one of the Fourier-components of this function. Regarding all or sufficiently large number of these components removes the applicability restriction inherent to the mean-field approach that the systems should be in the vicinity of a stationary point.

Moreover, this generalization allows, in principle, treatment of a much more wide class of systems than the networks of identical and identically connected neurons. This becomes possible after a special choice of the *structure postulate.*

The two basic ideas of this approach, namely 1) the interaction functions $b_{i,j}$ should be represented in factorized form; 2) the summatory functions can be used as macrovariables - have been already implemented to some special systems (including the Hamiltonian ones) in previous papers of A.M.M. [4-6], and descend from the ideas of E.Schmidt (see e.g. in [7]) and A.Ya.Khinchin [8,9].

2. Uniform system of coupled oscillators

2.1. The equations of macrodynamics

Firstly, we consider the following special case. Let the "neurons" to be oscillators in the so called "phase description", as it is done, e.g., in [3] with φ_j being their phases, with identical frequencies, so we can put a_j=0, and all the connections to be identical and depend only upon the phase differences:

$$\dot{\varphi}_i = \frac{1}{N} \sum_{j=1}^{N} b\ (\varphi_i - \varphi_j),\ \ i=1..N \tag{2}$$

Let us expand the function $b(\varphi)$ into the Fourier series:

$$b\ (\varphi) = \sum_{\alpha} B_\alpha\ e^{i\alpha\varphi} \tag{3}$$

And now we introduce the macro-variables S_α instead of microvariables φ_j:

$$S_\alpha = \frac{1}{N} \sum_j e^{i\alpha\varphi_j},\ \alpha=0,1,2... \tag{4}$$

Automatically, we get that $S_0 \equiv 1$ is a first integral, and $S_{-j} \equiv (S_j)^*$. Differentiating the system (4) by time and substituting (2) and (4), we get the closed system of equations for macrovariables:

$$\dot{S}_\alpha = i\ \alpha \sum_\beta\ B_\beta\, S_{\alpha+\beta}\, S_{-\beta} \tag{5}$$

(the macrodynamic system). Note, that the formal derivation of (5) from (2) is exact and is valid for any N, because it has used only the rearrangement of summation through indices j and α, while j runs over a finite number of values $1..N$.

When the macrodynamic system is solved, then the individual dynamics of j-th neuron is governed by the equation

$$\dot{\varphi}_j = \sum_\alpha B_\alpha\, e^{i\alpha\varphi_j}\, S_j^* \tag{2'}$$

Note that the system (2') together with equation (4), when reduced to a finite number of S_α, is a special case of a system of oscillators coupled only due to interaction with common environment that was studied in [1].

It is easy to see that S_α are the Fourier components of the phase distribution density defined as

$$s(\varphi,t) = \sum_j \delta\ (\varphi-\varphi_j(t)) \tag{6}$$

So the system (5) is simply the Fourier transform of the master equation for $s(\varphi,t)$. If the weak limit of $s(\varphi,t)$ as $N\to\infty$ is a smooth function, then we can analyze only a few number of S_α, i.e. we have reduced the system of N equations for individual neurons to the system of a few number of equations for macrovariables. This is the goal of the theory.

For the special case of

$$b(\varphi) = 2B\cos\ (\varphi+\theta) \tag{7}$$

we get from (5) that

$$\dot{S}_\alpha = i\ \alpha\ B\ \left(\ e^{i\theta}\, S_{\alpha+1}\, S_{-1}\ -\ e^{-i\theta}\, S_{\alpha-1}\, S_1\ \right) \tag{8}$$

i.e. the evolution of a macrovariable is coupled only with "neighboring" macrovariables and with the "principal" macrovariable S_1. Note, that S_1 is that very macrovariable considered as mean-field in ref. [3].

2.2. The simplest example of using the theory

The system (2) has two obvious types of "stationary" solutions. One of them is

$$\varphi_j \equiv \varphi_* \ \forall j \tag{9}$$

where

$$\dot{\varphi}_* = b\ (0) \tag{10}$$

– "total synchronization" and is valid for any N, and the other can be written as

$$\varphi_j = 2\pi j/N + t \oint b(\varphi)d\varphi \tag{11}$$

– "total disorder" and is only asymptotically valid as $N \longrightarrow \infty$.

The stability analysis of the solution (9) for any N was made (for a more general model) in the paper by Shnol [1]. Concisely speaking, the result reads: the synchronous regime for the system of N oscillators is stable if and only if such a regime is stable for 2 oscillators. The stability analysis for (10) is a more difficult problem because it cannot be done for any finite N but must be performed in the limit $N \longrightarrow \infty$.

Let us study the solution (11) in the case (7) in terms of macroequations (8). In terms of macrovariables the solution (11) has the form

$$S_\alpha = 0,\ \alpha \neq 0 \tag{12}$$

Linearized equations are

$$\dot{S}_1 = i\ B\ N\ e^{-i\theta}\ S_1;$$

$$\dot{S}_\alpha = 0,\ \alpha \neq 1 \tag{13}$$

Therefore, the necessary stability condition of disorder is

$$\theta \in (-\pi, 0). \tag{14}$$

Note, that if we consider stochastical system instead of dynamical, e.g. if a_j are not zero, but independent Gaussian white noises, even though arbitrarily small, then (14) becomes also the sufficient condition.

It is interesting, that the criterion obtained is analogous to that of stability of synchronized solution (9) cited above: the "total disorder" solution for many neurons is stable if and only if the counter-phase solution is stable for 2 oscillators.

For general $b(\varphi)$ the linearized system is

$$\dot{S}_{\alpha} = i\,\beta\,N\,(\,B_0 \;+\; B_{-\alpha}\,)\,S_{\alpha} \tag{15}$$

and necessary stability condition is

$$arg(B_{\alpha})\in(-\pi,0) \tag{16}$$

for all α.

2.3. The problem of synchronization

Generally speaking, the approach is reasonable if we can take into consideration only few number of macrovariables instead of very large number of microvariables. This is possible only if the distribution density is a smooth function, because the macrovariables are its Fourier components.

This implies, particularly, that the approach cannot be applied to studying the synchronization phenomena, because in that case the distribution density becomes singular. This pessimistic statement can be, however, disproved by the following two considerations.

Firstly, if we add to the system an additive noise, then the "singular" synchronization becomes impossible. In this case it is sufficient to consider the number of macrovariables inversely proportional to the noise amplitude. Perhaps, some analogous effect can yield the "internal" noises, i.e. statistical scattering of the neuron parameters (see in this respect [10]).

Secondly, let us recall that the "cut off" system (5) with finite number of S_{α} is the result of Galerkin projection to the trigonometrical basis. If we are to analyze the regimes that are hardly approximated in this basis, then we can chose another one, may be singular.

To illustrate the latter idea, let us consider the synchronization process in the system (2,7) with its macrodynamical version (8). It is already known that if $\theta\in(0,\pi)$ then the disorder state (10) is unstable and the synchronized state (9) is stable. So we might suggest that the typical trajectory of the system is from the vicinity of total disorder to the vicinity of totally synchronized state. To describe such a trajectory within the frames of the same approximation, we need to choose a (macroscopical) manifold that would comprise both these states. Let us choose this manifold in the form

$$s(\varphi,t) = \frac{1}{2\pi} s_0 + s_1 \, \delta(\varphi - \varphi_*) \tag{17}$$

It is 2-parametric, because the normalization requires

$$s_0 = 1 - s_1 \tag{18}$$

The total disorder corresponds to the limit $s_0=1$, and synchronization to $s_1=1$. On this manifold we have

$$S_0 = 1; \; S_\alpha = s_1 \, e^{i\alpha\varphi} \tag{19}$$

It is convenient to use S_1 as a complex coordinate on the manifold. Other macrovariables are expressed as

$$S_\alpha = S_1^{\alpha} \, / \, |S_1|^{\alpha-1} \tag{20}$$

By substituting S_2 obtained from eq. (20) into eq. (8), we get the closed equation for S_1:

$$\dot{S}_1 = i \, B \, S_1 (e^{i\theta} |S_1| - e^{-i\theta}) \tag{21}$$

which is the desired cut-off macrodynamical equation. It is easy to see that the equation has the expected properties: if (16) is violated, then the typical trajectory leaves the vicinity of the disorder state (11,12) and approaches to the synchronized state (9,10).

3. Generalizations

3.1. Excitable neurons

Now we proceed to generalizations of the "basic" model (2) in the direction of the "most general" model (1). Firstly, let us consider nontrivial "own" dynamics $a(\varphi)$, still identical for all neurons. Let it also be expanded into Fourier series:

$$a(\varphi) = \sum_\mu a_\mu e^{i\mu\varphi} \tag{22}$$

Then the macrovariables (4) satisfy the equations

$$\dot{S}_\mu = i\mu \sum_\nu S_{\mu+\nu} \, (\, a_\nu + b_\nu \, S_{-\nu} \,) \tag{23}$$

Since $a(\varphi)$ can have zeros, we can now take into consideration not only oscillators, but also excitable neurons. A simple model of this kind for nearly-stationary states was suggested in ref. [3].

Note, that the method can be easily generalized to the case when the phase space is not a circle, but any other manifold. The only condition is that we could approximate distributions on this manifold; that is, it should have not too large "total volume", otherwise the Galerkin procedure is hardly applicable. In particular, the phase space should have few dimensions.

So, further generalization for arbitrary functions $a(\varphi)$, $b(\varphi,\psi)$ in any phase space $\{\varphi\}$ is obvious, if only they are identical for all neurons and neuron pairs. To discard this restrictions, some essential assumptions must be made, i.e. we need a lucky *structure postulate*.

3.2. Non-identical neurons

As a next stage, let us assume that neurons are not identical, though the connections remain "uniform".

The main assumption, that will permit us to treat the caseof different neurons, is the following *structure postulate*:

$$a_j(\varphi) = a\ (\ \lambda_j,\ \varphi_j) \tag{24}$$

where λ_j are time-independent vectors of few dimensions. If $b_{i,j}(\varphi,\psi)\equiv b(\varphi-\psi)$, as before, then we can rewrite (1) in the form

$$\begin{cases} \dot\varphi_j = a\ (\lambda_j,\ \varphi_j)\ +\ b\ (\varphi_j,\ \varphi_k) \\ \dot\lambda_j\ =\ 0 \end{cases} \tag{25}$$

Now we consider "expanded" phase space

$$\{\Phi\}=\{\varphi\}\times\{\lambda\},\ \Phi_j = \begin{pmatrix} \varphi_j \\ \lambda_j \end{pmatrix} \tag{26}$$

In terms of variables Φ_j the system (24) is again the system with identical neurons, as in previous section. The variability of neuron properties is now transformed into the variability of initial conditions for "expanded" neurons, and the second part of the *structure postulate* requires that neuron distribution in $\{\lambda\}$ can be subject to Galerkin procedure.

Of course, the choice of the Galerkin basis in the parameter subspace $\{\lambda\}$ should be made depending upon the properties of this space.

Consider the simplest example. Let $\{\varphi\}$ be again the unit circle, and $a_j(\varphi)=a(\lambda_j,\varphi)\equiv\lambda_j$, i.e. the oscillators vary only in their own frequency. Generally, if the macrovariables are defined as

$$S_{\alpha,\mu} = \sum_j f_\alpha(\lambda_j)\, e^{i\mu\varphi_j} \tag{27}$$

where $\{f_\alpha\}$ is a basis in $\{\lambda\}$, then the macrosystem has the form

$$\dot{S}_{\alpha,\mu} = i\,\mu \sum_\beta \left\{ V_{\alpha,\beta}\, S_{\beta,\mu} + U_\beta \sum_\nu B_\nu\, S_{\alpha,\mu+\nu}\, S_{\beta,-\nu} \right\} \tag{28}$$

where the structure constants $V_{\alpha,\beta}$ and U_β are defined by

$$\sum_\alpha U_\alpha\, f_\alpha(\omega) = 1 \tag{29}$$

and

$$\sum_\beta V_{\alpha\beta}\, f_\beta(\omega) = \omega\, f_\alpha(\omega) \tag{30}$$

If f_α is the basis of power functions,

$$f_\alpha(\omega) = \omega^\alpha,\ \alpha=0,1,2\ldots \tag{31}$$

(which is appropriate if ω vary in a finite interval), we get structure constants

$$U_\alpha = \delta_{\alpha,0} \tag{32}$$

$$V_{\alpha,\beta} = \delta_{\alpha+1,\beta} \tag{33}$$

and the macroequations

$$\dot{S}_{\alpha,\mu} = i\,\mu \left\{ S_{\alpha+1,\mu} + \sum_\nu B_\nu\, S_{\alpha,\mu+\nu}\, S_{0,-\nu} \right\} \tag{34}$$

3.3. Non-identical synapses

The formal approach used in the previous subsection can also be applied to construct "macrodynamical" networks with non-identical connections between neurons. Again, we assume that the connection of i-th neuron to j-th one does not depend immediately upon i and j, but is a function of a few number of parameters, which, in turn, depend upon i and j (*structure postulate*):

$$\begin{cases} \dot{\varphi}_i = \sum_j b\,(\lambda_i,\ \lambda_j, \varphi_i,\ \varphi_j); \\ \dot{\lambda}_i = 0 \end{cases} \tag{35}$$

Here $\{\lambda_j\}$ are some parameters determining interneuronal connections.

For instance, consider the equations like (2) but with varying connection weights:

$$b\ (\lambda_i,\ \lambda_j,\ \varphi_i,\ \varphi_j) =$$

$$\circeq \sum_{\alpha,\beta,\nu} B_{\alpha,\beta}\, f_\alpha(\lambda_i)\, f^*_\beta(\lambda_j)\, e^{i\nu(\varphi_i-\varphi_j)} + \text{c.c.}; \qquad (36)$$

(here * denotes complex conjugation and "c.c." is the term complex conjugate to that written down). Here $\{f_\alpha\}$ is again a basis on the parameter space $\{\lambda\}$. Macrovariables are defined as

$$S_{\alpha,\mu} = \sum_j f_\alpha(\lambda_j)\, e^{i\mu\varphi_j} \qquad (37)$$

Let

$$f_\alpha(\lambda)\ f_\beta(\lambda) = \sum_\gamma F_{\alpha\beta\gamma}\, f_\gamma(\lambda) \qquad (38)$$

Then the macrosystem is

$$\dot{S}_{\alpha\mu} = i\ \mu \sum_{\nu,\gamma,\delta} B_{\alpha\gamma\delta}\, S_{\delta,\mu+\nu}\, S^*_{\gamma,\nu} \qquad (39)$$

where $\hat{B}_{\alpha\beta\gamma}$ are some combinations of $F_{\alpha\beta\gamma}$ and $B_{\alpha\beta\nu}$.

4. Discussion

We have proposed an asymptotical approach to the problem of analysis of large neural networks and another complex systems. Informally speaking, the main postulates of the approach claim: the variability of elements and their connections remains "finite" when $N \longrightarrow \infty$; and therefore the macroscopical properties of the system can be described in terms of the weak limit of the neuron distribution density in the expanded space "phase+parameters".

If this weak limit is sufficiently smooth, or, more generally, is close in a sense to a finite-dimensional manifold, then the master equation can be approximated by a finite-dimensional system through the Galerkin procedure. This finite-dimensional "macrodynamical" system may help in qualitative analysis of simplest cases and may serve as a basis for numerical procedures in general.

The physical (biological) sense of the "structure parameters" λ in (25), (35) depends upon the problem being solved. An example of the parameter of the neurons variety was shown in Section 3.2 - it is the own frequency. If the mathematical model of a neuron contains parameters then any or all of them can be used.

But biological sense of parameterization of interconnections is much less obvious, and probably is very specific to the problem, and in some problems may be absent. One of possible physiological interpretations may be related with the Dale's principle declaring that the type of a synapse is determined by the types of the pre- and post-synaptic membranes.

In this respect, it is interesting to study whether the networks of the Hopfield type can be described with this formalism or not. It appears probable, because the *structure postulate* (36) seems analogous to the Hebbian rule. Here the structure parameters λ have no relations with physiology but are determined by network hystory (learning). The popular question about the information capacitance of such a network has no sense in our formalism, because we imply $N\rightarrow\infty$. But all the questions about convergence dynamics, non-symmetric connections etc. remain sensible.

Another possible meaning of the structure parameters λ of the connections variety is the space location of connected neurons. In this terms, it becomes possible to describe the networks with quasilocal, space-dependent connections.

A natural generalization of the formalism can be made for stochastical systems. Say, if the system (1) is supplemented with additive independent white Gaussian noise, then the master equation becomes Focker-Plank equation. As we have already mentioned, in this case the synchronization stops being a singularity, and the Galerkin procedure becomes simpler. From this point of view, the proposed formalism if even more adequate to stochastic systems than to deterministic ones.

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