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# MACRODYNAMICAL APPROACH TO THE ANALYSIS OF NEURAL NETWORKS

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"Макродинамический подход к анализу нейронных сетей"

Описывается математический метод для анализа сложных систем типа нейронных сетей. Метод представляет собой обобщение известного подхода "среднего поля", и позволяет анализировать не только стационарные состояния, но и динамические свойства сети. Метод может быть также интерпретирован как Галеркинская процедура по отношению к уравнению эволюции для функции распределения. Обсуждаются типы нейронных сетей и задачи, для которых может быть применен данный метод. Показано, что в его рамках могут быть описаны процессы синхронизации, сети возбудимых нейронов, сети из неодинаковых нейронов и с неодинаковыми связями.

Abstract. General features of a mathematical method of analyzing complex systems like neural networks are presented. The method is a generalization of the mean-field approach and allows analyzing not only "steady-states" but also dynamical properties of the network. The method can be also interpreted as a Galerkin procedure for the master equation. The types of neural networks and related problems to which the method can be applied are discussed. It is shown that method can treat synchronization processes, networks of excitable neurons and ones of non-identical neurons and synapses.

Key words: complex systems, neural networks, coupled oscillators, macrodynamics, Galerkin procedure, master equation, mean-field equation, pair interaction, summatory functions.

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## 1. Introduction

The analysis of complex systems is a very important problem in various fields of science. There are many systems for which the direct simulations is not possible and, perhaps, never will be, because of huge number of elements. On the other side, we usually do not need to know the detailed behavior of such a system, but only interested in studying some general (macroscopical) properties.

Therefore, it is very useful to get examples of systems to which some exact or asymptotical statements in the limit $N \to \infty$ can be made, where $N$ is the number simple subsystems (elements, for

instance) constituting the whole system.

The neural networks are significant examples of systems for which the specified problem is of principal. Many of mathematical models of neural networks can be formulated in the form

$$\dot{\varphi}_i = a_i(\varphi_i) + \sum_{j=1}^{N} b^N_{i,j}(\varphi_i, \varphi_j), \quad i=1..N \qquad (1)$$

where dynamical variables $\varphi_i$ denote states of neurons (or neuron clusters) and are vectors of a few dimensions; the functions $a_i$ describe their "own" dynamics and $b^N_{i,j}$ describe functioning of synapses (or intercluster interaction).

Many types of complex systems of another physical origin can also be presented in the form (1).

We want to study the "macroscopical" properties of the system (1) in the limit $N\to\infty$. In the most general formulation, even the definition what are the "macroscopical properties" is a problem and should be specially stated; we shall call this statement as an *observation postulate*. Usually these properties should not vary considerably if a relatively small number of neurons is removed or essentially change their properties (see, e.g. [1]).

Another related problem is what is the rule that we should choose and change the functions $a_i$, $b_{i,j}$ as $N\to\infty$. Such a rule we call a *structure postulate*.

These two postulates determine the specifics of any approach to the analysis of a complex system. For instance, the so called autowave media may be interpreted as neural lattices (see e.g. [2]) with nearest neighbors, symmetric and linear connections (this is the structure postulate), are macroscopically described by partial differential equations, and therefore the observation postulate implies that we neglect the fine structure of the lattice dynamics.

Another example is the network where all the neurons are identical and all connected to each other in an identical way. For this case the mean-field approach seems natural enough [3].

We suggest a generalization of the mean-field approach which leads to the description of the network in terms of the neurons states distribution density in "microscopical" phase space $\{\varphi\}$, since these spaces are identical for all neurons.

In this description the *observation postulate* claims: macroscopical properties are defined by the (weak) limit of the distribution density as $N\to\infty$. This is really a generalization of the mean-field approach, because the mean-field variable defined

in [3] is simply one of the Fourier-components of this function. Regarding all or sufficiently large number of these components removes the applicability restriction inherent to the mean-field approach that the systems should be in the vicinity of a stationary point.

Moreover, this generalization allows, in principle, treatment of a much more wide class of systems than the networks of identical and identically connected neurons. This becomes possible after a special choice of the *structure postulate*.

The two basic ideas of this approach, namely 1) the interaction functions $b_{i,j}$ should be represented in factorized form; 2) the summatory functions can be used as macrovariables - have been already implemented to some special systems (including the Hamiltonian ones) in previous papers of A.M.M. [4-6], and descend from the ideas of E.Schmidt (see e.g. in [7]) and A.Ya.Khinchin [8,9].

## 2. Uniform system of coupled oscillators

### 2.1. The equations of macrodynamics

Firstly, we consider the following special case. Let the "neurons" to be oscillators in the so called "phase description", as it is done, e.g., in [3] with $\varphi_j$ being their phases, with identical frequencies, so we can put $\alpha_j \equiv 0$, and all the connections to be identical and depend only upon the phase differences:

$$\dot{\varphi}_i = \frac{1}{N} \sum_{j=1}^{N} b\ (\varphi_i - \varphi_j),\ i=1..N \qquad (2)$$

Let us expand the function $b(\varphi)$ into the Fourier series:

$$b\ (\varphi) = \sum_{\alpha} B_\alpha\ e^{i\alpha\varphi} \qquad (3)$$

And now we introduce the macro-variables $S_\alpha$ instead of microvariables $\varphi_j$:

$$S_\alpha = \frac{1}{N} \sum_{j} e^{i\alpha\varphi_j},\ \alpha=0,1,2... \qquad (4)$$

Automatically, we get that $S_0 \equiv 1$ is a first integral, and $S_{-j} \equiv (S_j)^*$. Differentiating the system (4) by time and substituting (2) and (4), we get the closed system of equations

for macrovariables:

$$\dot{S}_\alpha = i\,\alpha \sum_\beta B_\beta\, S_{\alpha+\beta}\, S_{-\beta} \qquad (5)$$

(the macrodynamic system). Note, that the formal derivation of (5) from (2) is exact and is valid for any $N$, because it has used only the rearrangement of summation through indices $j$ and $\alpha$, while $j$ runs over a finite number of values $1..N$.

When the macrodynamic system is solved, then the individual dynamics of $j$-th neuron is governed by the equation

$$\dot{\varphi}_j = \sum_\alpha B_\alpha\, e^{i\alpha\varphi_j}\, S_j^* \qquad (2')$$

Note that the system (2') together with equation (4), when reduced to a finite number of $S_\alpha$, is a special case of a system of oscillators coupled only due to interaction with common environment that was studied in [1].

It is easy to see that $S_\alpha$ are the Fourier components of the phase distribution density defined as

$$s(\varphi,t) = \sum_j \delta\,(\varphi-\varphi_j(t)) \qquad (6)$$

So the system (5) is simply the Fourier transform of the master equation for $s(\varphi,t)$. If the weak limit of $s(\varphi,t)$ as $N\to\infty$ is a smooth function, then we can analyze only a few number of $S_\alpha$, i.e. we have reduced the system of $N$ equations for individual neurons to the system of a few number of equations for macrovariables. This is the goal of the theory.

For the special case of

$$b(\varphi) = 2B\cos(\varphi+\theta) \qquad (7)$$

we get from (5) that

$$\dot{S}_\alpha = i\,\alpha\,B\left( e^{i\theta}\, S_{\alpha+1}\, S_{-1} - e^{-i\theta}\, S_{\alpha-1}\, S_1 \right) \qquad (8)$$

i.e. the evolution of a macrovariable is coupled only with "neighboring" macrovariables and with the "principal" macrovariable $S_1$. Note, that $S_1$ is that very macrovariable considered as mean-field in ref. [3].

## 2.2. The simplest example of using the theory

The system (2) has two obvious types of "stationary" solutions. One of them is

$$\varphi_j \equiv \varphi_* \quad \forall j \tag{9}$$

where

$$\dot{\varphi}_* = b\ (0) \tag{10}$$

– "total synchronization" and is valid for any $N$, and the other can be written as

$$\varphi_j = 2\pi j/N + t \oint b(\varphi) d\varphi \tag{11}$$

– "total disorder" and is only asymptotically valid as $N \to \infty$.

The stability analysis of the solution (9) for any $N$ was made (for a more general model) in the paper by Shnol [1]. Concisely speaking, the result reads: the synchronous regime for the system of $N$ oscillators is stable if and only if such a regime is stable for 2 oscillators. The stability analysis for (10) is a more difficult problem because it cannot be done for any finite $N$ but must be performed in the limit $N \to \infty$.

Let us study the solution (11) in the case (7) in terms of macroequations (8). In terms of macrovariables the solution (11) has the form

$$S_\alpha = 0, \ \alpha \neq 0 \tag{12}$$

Linearized equations are

$$\dot{S}_1 = i\ B\ N\ e^{-i\theta}\ S_1;$$

$$\dot{S}_\alpha = 0, \ \alpha \neq 1 \tag{13}$$

Therefore, the necessary stability condition of disorder is

$$\theta \in (-\pi, 0). \tag{14}$$

Note, that if we consider stochastical system instead of dynamical, e.g. if $a_j$ are not zero, but independent Gaussian white noises, even though arbitrarily small, then (14) becomes also the sufficient condition.

It is interesting, that the criterion obtained is analogous to that of stability of synchronized solution (9) cited above: the "total disorder" solution for many neurons is stable if and only if the counter-phase solution is stable for 2 oscillators.

For general $b(\varphi)$ the linearized system is

$$\dot{S}_\alpha = i\,\beta\,N\,(\,B_0 \;+\; B_{-\alpha}\,)\,S_\alpha \qquad (15)$$

and necessary stability condition is

$$arg(B_\alpha)\in(-\pi,0) \qquad (16)$$

for all $\alpha$.

## 2.3. The problem of synchronization

Generally speaking, the approach is reasonable if we can take into consideration only few number of macrovariables instead of very large number of microvariables. This is possible only if the distribution density is a smooth function, because the macrovariables are its Fourier components.

This implies, particularly, that the approach cannot be applied to studying the synchronization phenomena, because in that case the distribution density becomes singular. This pessimistic statement can be, however, disproved by the following two considerations.

Firstly, if we add to the system an additive noise, then the "singular" synchronization becomes impossible. In this case it is sufficient to consider the number of macrovariables inversely proportional to the noise amplitude. Perhaps, some analogous effect can yield the "internal" noises, i.e. statistical scattering of the neuron parameters (see in this respect [10]).

Secondly, let us recall that the "cut off" system (5) with finite number of $S_\alpha$ is the result of Galerkin projection to the trigonometrical basis. If we are to analyze the regimes that are hardly approximated in this basis, then we can chose another one, may be singular.

To illustrate the latter idea, let us consider the synchronization process in the system (2,7) with its macrodynamical version (8). It is already known that if $\theta\in(0,\pi)$ then the disorder state (10) is unstable and the synchronized state (9) is stable. So we might suggest that the typical trajectory of the system is from the vicinity of total disorder to the vicinity of totally synchronized state. To describe such a trajectory within the frames of the same approximation, we need to choose a (macroscopical) manifold that would comprise both these states. Let us choose this manifold in the form

$$s(\varphi,t) = \frac{1}{2\pi} s_0 + s_1 \delta(\varphi - \varphi_*) \qquad (17)$$

It is 2-parametric, because the normalization requires

$$s_0 = 1 - s_1 \qquad (18)$$

The total disorder corresponds to the limit $s_0=1$, and synchronization to $s_1=1$. On this manifold we have

$$S_0 = 1;\ S_\alpha = s_1\, e^{i\alpha\varphi} \qquad (19)$$

It is convenient to use $S_1$ as a complex coordinate on the manifold. Other macrovariables are expressed as

$$S_\alpha = S_1^{\alpha} / |S_1|^{\alpha-1} \qquad (20)$$

By substituting $S_2$ obtained from eq. (20) into eq. (8), we get the closed equation for $S_1$:

$$\dot{S}_1 = i\, B\, S_1 (e^{i\theta} |S_1| - e^{-i\theta}) \qquad (21)$$

which is the desired cut-off macrodynamical equation. It is easy to see that the equation has the expected properties: if (16) is violated, then the typical trajectory leaves the vicinity of the disorder state (11,12) and approaches to the synchronized state (9,10).

## 3. Generalizations

### 3.1. Excitable neurons

Now we proceed to generalizations of the "basic" model (2) in the direction of the "most general" model (1). Firstly, let us consider nontrivial "own" dynamics $a(\varphi)$, still identical for all neurons. Let it also be expanded into Fourier series:

$$a(\varphi) = \sum_\mu a_\mu e^{i\mu\varphi} \qquad (22)$$

Then the macrovariables (4) satisfy the equations

$$\dot{S}_\mu = i\mu \sum_\nu S_{\mu+\nu} ( a_\nu + b_\nu S_{-\nu} ) \qquad (23)$$

Since $a(\varphi)$ can have zeros, we can now take into consideration not only oscillators, but also excitable neurons. A simple model of this kind for nearly-stationary states was suggested in ref. [3].

Note, that the method can be
when the phase space is not a cir
only condition is that we could a
manifold; that is, it should hav
otherwise the Galerkin proced
particular, the phase space shou

So, further generalization
$b(\varphi,\psi)$ in any phase space $\{\varphi\}$
identical for all neurons and
restrictions, some essential as
need a lucky *structure postulate*

## 3.2. Non-identical neurons

As a next stage, let us assum
though the connections remain "u

The main assumption, that wi
different neurons, is the follow

$$a_j(\varphi) = c$$

where $\lambda_j$ are time-independent
$b_{i,j}(\varphi,\psi)\equiv b(\varphi-\psi)$, as before, the

$$\begin{cases} \dot{\varphi}_j = a(\lambda_j, \varphi_j) + b(\varphi_j \\ \dot{\lambda}_j = 0 \end{cases}$$

Now we consider "expanded" phas

$$\{\Phi\}=\{\varphi\}\times\{\lambda$$

In terms of variables $\Phi_j$ the sy
identical neurons, as in prev
neuron properties is now tra
initial conditions for "expande
the *structure postulate* requir
can be subject to Galerkin proc

Of course, the choice of th
subspace $\{\lambda\}$ should be made de
space.

Consider the simplest exa
circle, and $a_j(\varphi)=a(\lambda_j,\varphi)\equiv\lambda_j$,
their own frequency. Generally