# Spiral wave meander and symmetry of the plane

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## Abstract

We present a general group-theoretic approach that explains the main qualitative features of the meander of spiral wave solutions on the plane. The approach is based on the well known space reduction method and is to separate the motions in the system into superposition of those 'along' orbits of the Euclidean symmetry group, the group of all isometric transformations of the plane, and 'across' the group orbits. It has the visual interpretation as passing to a reference frame attached to the spiral wave's tip. The system of ODEs governing the tip movement is obtained. It is the system that describes the movements along the group orbits. The motions across the group orbits are described by a PDE which lacks the Euclidean symmetry. Consequences of the Euclidean symmetry on the spiral wave dynamics are discussed. In particular, we explicitly derive the model system for bifurcation from rigid to biperiodic rotation, suggested earlier by Barkley [1994] from *a priori* symmetry considerations.

## 1 Brief history of the question

Spiral waves provide one of the most striking examples of pattern formation in nonlinear active media and have attracted attention since their description in the Belousov-Zhabotinsky (BZ) chemical reaction medium [Zaikin, 1970]. They have been described in many other systems and models [Swinney & Krinsky, 1990; Holden *et al.* 1991], and in cardiac muscle they are of potentially vital interest, as they underly some lethal arhythmmias [Gray & Jalife 1996].

Spiral waves appear as rotating waves of chemical or other activity, that propagate through a stationary medium as a wave of synchronization of phase of self-oscillations or excitation, without any bulk movement of the medium. Typically they can be approximated by an Archimedean spiral, rotating with roughly constant speed. Soon after the discovery of spiral waves, Winfree [1973] examined in detail the dynamics of the tips of the spirals in the BZ reaction and found that these dynamics may not necessarily be steady, but that the spiral tip may describe complicated trajectories, which he called 'meander'. An extensive computational, phenomenological study of meander patterns [Zykov 1986; Karma 1990; Barkley *et al.* 1990; Winfree 1991; Plesser& Müller 1995] has shown that they exhibit common features. Early computational work used simple, two-component FitzHugh-Nagumo-like models; however, essentially the same meander patterns have been observed in high order, detailed models of excitation propagation in heart muscle [Efimov *et al.* 1995; Biktashev & Holden 1996], where the particular type of meander relates to physiological properties of the tissue (Fig. 1).

Barkley *et al.* [1990] have studied numerically the change in type of meander in a variant of the FitzHugh-Nagumo model, and found that the transition from steady periodic rotation to the meander can exhibit features typical for a supercritical Andronov-Hopf bifurcation, namely, a linear decrease of the ab-

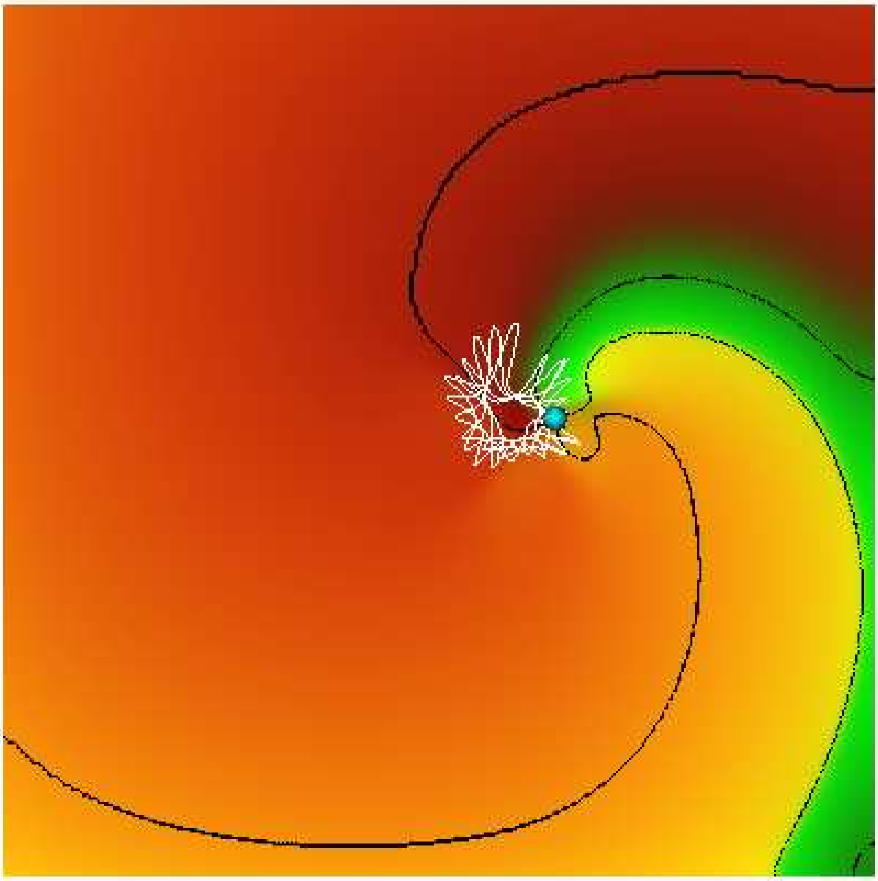

Figure 1: Spiral wave, isolines, tip and its trajectory, in a model of ventricular tissue. Medium is colour-coded according to values of two field components, an activation variable (transmembrane voltage, red colour component intensity) and a recovery variable ($[Ca^{++}]$ inactivation gate, green colour component intensity), details of the model can be found in (Biktashev & Holden 1996). The wave irregularly rotates counterclockwise. Black lines are the chosen isolines of the two variables, the blue ball at their intersection is the tip of the spiral wave. The white line shows the trajectory of the tip over last few rotations. The rotation is approximately biperiodic, modulated by some slow processes.

solute value of the real part of the leading eigenvalue as the bifurcation is approached and an $\epsilon^{1/2}$-growth of deviation of the meander pattern from steady rotation in the supercritical region. This led to the hypothesis that the onset of meander is due to a secondary Hopf bifurcation from a periodic motion (the steady rotation) to biperiodic one (the meander). A detailed quantitative experimental study of tip trajectories in a mathematical model of BZ reaction [Plesser & Müller 1995] has shown that, with a good precision, tip meander can often be decomposed into two periodic motions.

The question of possible resonances between the two periods arises naturally. Consideration of the 1 : 1 resonance led Barkley [1994] to propose a model form ODE system for this bifurcation. This model system was chosen to mimick both the bifurcation and the Euclidean symmetry by means of the simplest possible ODE system. Subsequent exploration of this system has led to the observation that no locking or entrainment between the two frequencies occur, which has proved to result from the symmetry built into the foundation of this system. Thus, it has become evident that the explanation of the phenomenon of spiral wave meander lies in the Euclidean symmetry of the plane on which they rotate. This is in agreement with general rigorous results by Rand [1982] within the context of rotating fluids in hydrodynamics and Renardy [1982] on bifurcations from rotating waves in abstract evolution equations possessing continuous symmetry groups.

In this paper, we apply ideas of the theory of dynamical systems with symmetry to the explanation of spiral wave meander. The abstract theoretical construct we use has a very clear "physical interpretation", which is to reduce the system with symmetry to a generic one, by moving to the frame of reference attached to the tip of the spiral. If the dynamics of the reduced system is "coherent", *i.e.* has a low dimensional attractor, then the motion of the tip can be described by an appropriate closed ODE system.

After this paper was submitted, we were informed of the related work of Fiedler [1995], Wulff [1996a,b] and Golubitsky *et al.* [1996].

## 2 Problem formulation

The simplest and commonest class of mathematical models generating spiral waves is the reaction diffusion systems on the plane,

$$\partial_t u = \mathbf{D}\nabla^2 u + f(u), \tag{1}$$

with $u(\mathbf{r},t) = (u_1, u_2, ...) \in \mathrm{R}^l$, $l \geq 2$, $\mathbf{r} = (x, y) \in \mathrm{R}^2$. Our arguments can be extended to a much wider class of systems, as the essential point lies in the symmetry. Apart from the invariance against shifts in time, (1) is invariant under the group of the isometric transformations of the plane $\mathrm{R}^2 \to \mathrm{R}^2$, the Euclidean group denoted $E(2)$ (we shall neglect reflections and consider only the orientation-preserving transformations). That is, if $u(\mathbf{r},t)$ is a solution to (1), then $\tilde{u}(\mathbf{r},t) = T(g)u(\mathbf{r},t)$ is another solution, for any $g \in E(2)$, where action $T(g)$ of $g \in E(2)$ on the function $u$ is defined as

$$T(g)u(\mathbf{r},t) = u(g^{-1}\mathbf{r},t). \tag{2}$$

We are interested in spiral wave solutions of such systems, though it is not easy to specify this class of solution formally. Rigidly rotating waves are independent of time in an appropriately rotating frame of reference; however, as we are interesting in meandering not rigidly rotating waves, this circumstance is not much helpful. In fact, for our current purposes it is enough to mention that the isotropy subgroup of spiral wave solutions is trivial, *i.e.* they are not invariant under any nontrivial Euclidean transformation,

$$\forall t, \forall g \neq I \quad T(g)u(\mathbf{r},t) \neq u(\mathbf{r},t). \tag{3}$$

It is well known that the behaviour of dynamical systems with symmetries can be drastically different from those without symmetry, *i.e.* generic systems [Anosov & Arnold, 1985], and a standard way to study symmetrical systems is to reduce them to generic ones and then apply the results of the generic theory. For continuous groups, this can be made by separation of the system movement onto superposition of that 'along the group' and 'across the group', the second being described by a generic vector field that lacks the symmetry of the parent system (1).

## 3 Abstract scheme

Let us consider a differential equation

$$\frac{\mathrm{d}U}{\mathrm{d}t} = F(U), \quad U \in \mathcal{B}, \tag{4}$$

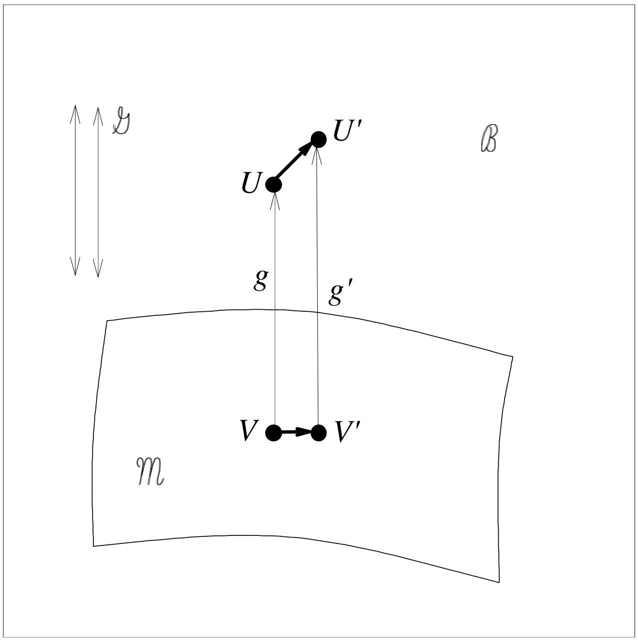


Figure 2: Decomposition of the movement in space $\mathcal{B}$ onto movement along a manifold $\mathcal{M}$ and along the group $\mathcal{G}$. Here $V, V' \in \mathcal{M}$, $U, U' \in \mathcal{B}$, and $g, g' \in \mathcal{G}$.

in a Banach space $\mathcal{B}$, which is invariant under the action $T$ of a Lie group $\mathcal{G}$, dim $\mathcal{G} = k < \infty$, *i.e.*

$$F(T(g)U) \equiv T(g)F(U), \tag{5}$$

for all $U \in \mathcal{B}$ and $g \in G$, and so any transformation $T(g)$ maps any solution to another solution.

The original phase space $\mathcal{B}$ is foliated by the orbits of $\mathcal{G}$, and the well known space reduction method can be used to reduce an original differential equation admitting symmetry $\mathcal{G}$ to one without symmetries [Anosov & Arnold 1985, p.31]. The basic idea is to identify whole orbits of the symmetry group in phase space with points of the orbit manifold or orbit space. The original vector field, corresponding to (4), projects onto a vector field in the orbit space. The differential equation defined by this field has no symmetry and is called the quotient system.

To obtain the quotient system for (4) explicitly, let us "parameterize" the orbit space by a manifold $\mathcal{M} \in \mathcal{B}$ of codimension $k$ which is, for simplicity, everywhere transversal to the orbits. Then any point $U$ in the region $\mathcal{D} = \mathcal{G}\mathcal{M} \subset \mathcal{B}$ can be uniquely represented in the form (see Fig. 2)

$$U = T(g)V, \quad V \in \mathcal{M}, g \in \mathcal{G}, \tag{6}$$

*i.e.* $(g, V)$ are coordinates on $\mathcal{D}$ (here we neglect the possibility of the same group orbit crossing the manifold more than once; this will be addressed later in Sec. 5). Differentiating (6) by time and using commutativity (5), we immediately obtain

$$\frac{\mathrm{d}V}{\mathrm{d}t} + T^{-1}(g)\frac{\mathrm{d}T(g)}{\mathrm{d}t}V = F(V). \tag{7}$$

The vector field $F(V)$ can also be uniquely decomposed into the two components, $(F(V))_{\mathcal{M}}$ which is tangent to $\mathcal{M}$ at $V$, and $(F(V))_{\mathcal{G}}$ which is tangent to the group orbit crossing $\mathcal{M}$ at $V$,

$$F(V) = (F(V))_{\mathcal{M}} + (F(V))_{\mathcal{G}}\,. \tag{8}$$

Substituting this into (7) and equating components along $\mathcal{M}$ and along $\mathcal{G}$ separately, we obtain a differential equation on $\mathcal{M}$,

$$\frac{\mathrm{d}V}{\mathrm{d}t} = (F(V))_{\mathcal{M}}\,, \tag{9}$$

and another on $\mathcal{G}$,

$$\frac{\mathrm{d}T(g)}{\mathrm{d}t}V = T(g)\,(F(V))_{\mathcal{G}}\,. \tag{10}$$

Note, that Eq. (9) for $V$ does not depend on $g$, which is a consequence of commutativity (5). Equation (9) is the target quotient system, lacking the symmetry of the original system and determining the motion along the manifold $\mathcal{M}$ which is separated from the motion along the group which can be found afterwards by integrating (10).

If the transformations $T(g)$ are explicitly defined, the vector fields $(F(V))_{\mathcal{M}}$ and $(F(V))_{\mathcal{G}}$ can be found explicitly; the standard approach is to expand the vectors tangent to group orbits in the basis of the group representation generators.

To conclude, we have shown that Eq. (4) can be replaced by (9), which is a generic differential equation, *i.e.* it is not invariant under any nontrivial action of $\mathcal{G}$, defined on the manifold $\mathcal{M}$; and then the solution of the parent system (4) is restored through integration of (9), (10) and (6). This procedure is valid subject to two most important conditions,

- $\mathcal{M}$ is everywhere transversal to the orbits of $\mathcal{G}$,
- trajectories of (4) do not leave the region $\mathcal{D} = \mathcal{G}\mathcal{M}$.

# 4 Application for spiral waves

To apply the above construction to the spiral waves, we choose $\mathcal{G} = E(2)$ and its representation $T$ on $\mathcal{B}$ given by (2). The choice of the Banach space $\mathcal{B}$ is not quite obvious: *e.g.*, we cannot use $L_2(R^2, R^l)$ since spiral waves do not vanish at infinity and thus do not belong to this space, and we cannot use $C(R^2, R^l)$ since arbitrary small rotations can produce finite changes in functions of this space and so the representation $T$ is not differentiable. Hence, $\mathcal{B}$ consists of bounded continuous vector functions which are asymptotically "circular" at infinity, so that small rotations change them slightly; a formal construction of such a space can be found in [Wulff, 1996a,b].

Condition (3) means that the isotropy subgroup of spiral waves is trivial. So, all we need is to choose the reduction manifold $\mathcal{M}$ to satisfy the transversality condition, which for our problem means that we should define a class of functions $\{v(x, y)\}$ by conditions which would be violated by any motion of the plane. A simple and obvious choice of such conditions is

$$\begin{aligned} v_1(0,0) &= u_{10} \\ v_2(0,0) &= u_{20} \\ \partial_x v_1(0,0) &= 0 \end{aligned} \qquad (11)$$

with appropriately chosen constants $u_{10}$ and $u_{20}$; components 1 and 2 are chosen just for example. The idea is that first and second conditions make impossible translations — small or finite, if they originate, locally or globally, from a unique solution, while the third one makes impossible rotations — if gradient of $v_1$ at the origin is nonzero. For small transformations, this can be guaranteed *e.g.* by $(\nabla v_1, \nabla v_2) \neq 0$ (finite transformations will be discussed in the next section).

Generators of the representation $T$ defined by (2) are $\partial_x$, $\partial_y$ and $\partial_\theta = y\partial_x - x\partial_y$, for translations along $x$, $y$ and rotation around the origin respectively. Expanding $(F(V))_{\mathcal{G}}$ in this basis, to

$$(F(V))_{\mathcal{G}} = (\mathbf{c}, \nabla)v + \omega\partial_\theta v$$

brings (9) to the form

$$\partial_t v = \mathbf{D}\nabla^2 v + f(v) - (\mathbf{c}, \nabla)v - \omega\partial_\theta v, \qquad (12)$$

where $\mathbf{c}(t) = (c_x(t), c_y(t))$ can be interpreted as a translation velocity and $\omega(t)$ as a rotation velocity. The system of PDE (12) and finite equations (11) can be viewed as a dynamical system in the phase space $\mathcal{M} = \{\mathbf{c}, \omega, v\}$ where $v$ is a vector-function of $\mathbf{r}$ and $c_x$, $c_y$ and $\omega$ are scalar variables. This is the target quotient system, corresponding to the abstract quotient system (9).

Equation (10) for $g(t)$ is easy to treat by using the isomorphism between the plane $\mathrm{R}^2$ on which the wave rotates, and the complex plane C. A natural representation $T_{\mathrm{C}}$ of $E(2)$ on C is the group of similar movements of the complex plane, *i.e.* if $g = \{\mathbf{R}, \Theta\}$ is rotation through an angle $\Theta$ around the origin, followed by translation by vector $\mathbf{R} = (X, Y)$, then $g$ is represented by

$$T_{\mathrm{C}}(\{\mathbf{R}, \Theta\}) : z \longmapsto R + z\mathrm{e}^{i\Theta}, \qquad (13)$$

where $R = X + iY \in \mathrm{C}$. Infinitesimal transformation $\mathrm{d}g = \{\mathrm{d}\mathbf{R}, \mathrm{d}\Theta\}$ is represented in C by $T_{\mathrm{C}}(\mathrm{d}g) : z \longmapsto z + \mathrm{d}R + iz\mathrm{d}\Theta$, and in the functional space by $T(\mathrm{d}g) = I + (\mathrm{d}\mathbf{R}, \nabla) + \mathrm{d}\Theta \cdot \partial_\theta$. Thus Eq. (9) is represented in C by

$$T_{\mathrm{C}}(\{\mathbf{R} + \mathrm{d}\mathbf{R}, \Theta + \mathrm{d}\Theta\}) = T_{\mathrm{C}}(\{\mathbf{R}, \Theta\})\ T_{\mathrm{C}}(\{\mathbf{c}\mathrm{d}t, \omega\mathrm{d}t\})$$

Substition of the definition of $T_{\mathrm{C}}$ (13) gives

$$R + \mathrm{d}R + \exp(i\Theta + i\mathrm{d}\Theta)\cdot z = R + \exp(i\Theta)\cdot(c\mathrm{d}t + \exp(i\omega\mathrm{d}t)\cdot z)$$

where obviously $c = c_x + ic_y$, which leads to

$$\begin{aligned} \partial_t\Theta &= \omega(t), \\ \partial_t R &= c(t)\mathrm{e}^{i\Theta}. \end{aligned} \qquad (14)$$

This is the equation on the group in the coordinates $(R, \Theta)$.

# 5 Visual interpretation

The first two condition of (11) say that the origin is an intersection point of two isolines, that of $v_1$ and $v_2$, and the third one says that the isoline of $v_1$ is tangent to the $x$-axis. Function $v(\mathbf{r})$ is just function $u(\mathbf{r})$ moved somehow (by $T(g^{-1})$) along the plane. Intersection of two isolines is often used as a definition of the tip of the spiral wave. So, in other words, conditions (11) say that function $v$ is function $u$ considered in a frame of reference with its origin at the tip of the spiral wave and with $y$ axis along the gradient of $u_1$ at the tip (see Fig. 3).

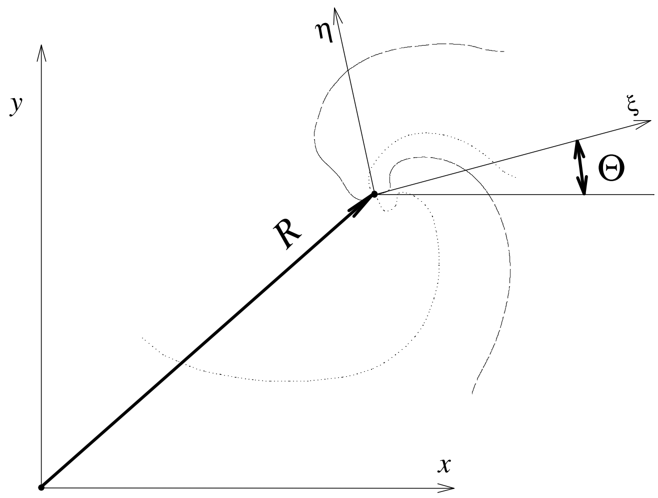


Figure 3: Frame of reference $(\xi, \eta)$ related to the tip of the spiral. The tip is the intersection of isoline $u = u_0$ (dashed) with isoline $v = v_0$ (dotted). The origin of $(\xi, \eta)$ is at the tip, shifted by vector $R$ from $(x, y)$-origin, $\xi$-axis is tangent to the $u$-isoline, rotated by angle $\Theta$ from $x$-axis.

Coordinates $\xi$, $\eta$ in the tip frame are related to those $x$, $y$ of the laboratory frame by

$$
\begin{aligned}
x &= X + \xi\cos\Theta - \eta\sin\Theta \\
y &= Y + \xi\sin\Theta + \eta\cos\Theta
\end{aligned}
$$

Performing this change of variables in the original system (1), with $X$, $Y$ and $\Theta$ varying with time, after elementary though tedious calculations we can directly obtain Eqs. (14, 12).

To conclude, the physical interpretation of the newly obtained equations is: (11) is a definition of the spiral tip, (14) is its motion equation, and (12) is an equation for the field in the tip's frame of reference.

Now we can easily interpret the assumption made in Sec. 3 that any group orbit crosses the manifold only once. In terms of this application, this simply means that we consider only solutions with one wave tip.

# 6 Some simple consequences

The simplest attractor in the quotient system is an equilibrium $(\omega, \mathbf{c}, v) = (\omega_0, \mathbf{c}_0, v_0)$. In this case, all the evolution is motion along a closed group orbit, *i.e.* the spiral moves as a rigid body, and this movement is rotation:

$$
\begin{aligned}
\Theta &= \Theta_0 + \omega_0 t, \\
R &= R_0 + c_0 \exp(i\Theta).
\end{aligned} \tag{15}
$$

Let us suppose that the equlibrium $(\omega_0, \mathbf{c}_0, v_0)$ undergoes a Hopf bufurcation. Then the dynamics of the quotient system can be described by

$$
\begin{aligned}
\dot{z} &= Z(z) &&= \epsilon z + i\mu z - az|z|^2 + O(|z|^4) \\
\omega &= \Omega(z) &&= \omega_0 + \omega_1 z + \omega_{-1}\bar{z} + O(|z|^2) \\
c &= C(z) &&= c_0 + c_1 z + c_{-1}\bar{z} + O(|z|^2) \\
v &= V(z) &&= v_0 + v_1 z + v_{-1}\bar{z} + O(|z|^2),
\end{aligned} \tag{16}
$$

where $z$ is a normal-form complex coordinate on the central manifold, $Z(z)$ is a vector field on this manifold, and functions $\Omega(z)$, $C(z)$ and $V(z)$ determine the shape of the central manifold in the space $\mathcal{M} = \{\omega, \mathbf{c}, v\}$.

System (14, 16) with the last equation left out, is formally equivalent to Barkley's [1984] model system obtained from symmetry considerations. To see this, let us introduce new variables $s \in \mathrm{R}$ and $\Phi$ by

$$c\exp(i\Theta) = s\exp(i\Phi),$$

resolve the system with respect to $z$ to determine the function $z(s, w)$, where $w = \dot{\Phi}$ (which is a well defined operation if $\omega_1$ and $c_1$ are nonzero), and then exclude $z$. We then obtain

$$
\begin{aligned}
\dot{R} &= s\exp(i\Phi) \\
\dot{\Phi} &= w\ H(s, w) \\
\dot{s} &= F(s, w) \\
\dot{w} &= G(s, w)
\end{aligned} \tag{17}
$$

where $H(s, w) = 1$. Meanwhile, system (3) from [Barkley 1994] reads

$$
\begin{aligned}
\dot{R} &= s\exp(i\Phi) \\
\dot{\Phi} &= w\ h(s^2, w^2) \\
\dot{s} &= s\ f(s^2, w^2) \\
\dot{w} &= w\ g(s^2, w^2)
\end{aligned} \tag{18}
$$

(the particular choice of the form of $F()$ and $G()$ was for parity purposes, to represent the symmetry due to reflections).

Further bifurcations will normally lead to the increase in the dimensionality of the embedding space of the attractor. For instance, a secondary Hopf bifurcation can give birth to an invariant torus which can subsequently break up leading to dynamical chaos. This scenario would naturally be described in model systems of higher dimensionality. Note that this viewpoint differs from that of Barkley [1994] who tried to reproduce the whole of Winfree's [1991] parametric portrait of the FitzHugh-Nagumo system in full in terms of the same model system, and in particular, to describe the bifurcation of meander into hypermeander.

This "meander-hypermeander" bifurcation may be explained as the birth of a chaotic attractor in the reduced system. To the extent that a chaotic signal $c(t) \exp \int \omega(t) \mathrm{d}t$ has properties analogous to that of truely stochastic noise, it would be natural to expect that its time integral $R(t)$ would have properties analogous to that of Wiener processes, — *i.e.* grow at large times in average as $\mathcal{O}(t^{1/2})$. Hence, the hypermeander patterns described by Winfree [1991] and Nagy-Ungvarai *et al.* [1993] could be explained as a Brownian walk along the symmetry group. The fact that a torus breakup into a chaotic attractor may occur soon after a secondary Hopf bifurcation [Afrajmovich & Shilnikov, 1983] is consistent with Winfree's observation of no other boundaries between the bifurcation lines "rigid rotation-meander" and "meander-hypermeander".

# 7 Discussion

In this paper, we have proposed a way to study systematically spiral wave meander. The idea starts from a group-theoretic construct using abstraction of the manifold of group orbits, and leads to a system of equations (11, 12, 14) which has a clear interpretation and is suitable both for a theoretical analysis and for simulation. As a partial case, this technique results in a model system for the bifurcation from rigid rotating to biperiodic meander, which is formally equivalent to the previously proposed model system [Barkley 1994]. However, here we have explicitly derived, this system, and the construct described is a general approach to treat a broader variety of related problems, *e.g.* the problem of hypermeander.

There are still someopen questions. The reduced dynamical system defined by (11,12) lacks the symmetry of the original one, so we have assumed that it can be treated as a 'generic' system, and all results of the dynamical system theory without symmetry can be used. However, this system is defined in an unusual way, and an accurate mathematical consideration is required in particular cases. For the case of the Hopf-Barkley bifurcation considered above, such a rigorous consideration within the approach based on the Lyapunov-Schmidt reduction has been made by Wulff [1996a,b]. It is interesting to mention in this respect that, as Barkley [1994] has pointed out, in the cases considered so far, the bifurcation to meander happens supercritically, while in generic systems both supercritic and subcritic cases are 'equally possible'.

Another open question is related to the alternative theory reducing description of perturbed spiral wave dynamics to ODEs (see *e.g.* [Biktashev & Holden 1995]), based on another basic mathematical idea, that of central (inertial) manifold. As it was discussed in [Biktashev & Holden 1995], the applicability of that approach depends upon the solvability of eigenvalue problems for the adjoint linearized operator in spaces of functions rapidly decaying far from the spiral core; the physical interpretation of this condition is that the spiral wave is sensitive only to perturbations located near the core. In the context of the theory of [Biktashev & Holden 1995], this property can be considered as definitive for 'proper' spiral waves. However, in the present paper, the conditions of transversality (11) or, more generally, (3) seem fairly generic, and the coherent low-dimensional behaviour considered in Sec. 6 is completely due to the vicinity of bifurcation. A natural interpretation is that meander patterns like those observed in spiral waves may be much more widespread than observed so far.

Combination of the two theories can be performed as either as generalization of [Biktashev & Holden 1995] for meandering spirals, or development of the present theory to account symmetry breaking perturbations. Such a combination can be helpful *e.g.* in studying meandering spirals under external periodic forcing, which has been studied recently from a phenomenological viewpoint [Mantel & Barkley 1996; Grill *et al.* 1995].

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