On the Movement of Excitation Wave Breaks ⋆

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Movement of excitation waves in active media in some cases can be described by a kinematic approach in terms of movement of curves, the wave crests, by neglecting other details such as wave profile and refractoriness. Of special interest are broken waves, *e.g.* spiral waves. In this case, additional equations for the wave tip movement are required. We derive such equations by singular perturbation techniques. These equations differ from those proposed earlier from semi-phenomenological arguments [10,11], are more complicated and diverse and admit a broader variety of solutions. As an illustration, we apply these equations to the problem of a stationary rotating spiral wave. In this particular example, the 'traditional' equations have happened to be a special case.

1 Introduction

Waves of propagating excitation are observed in active media of physical, chemical and biological origin [1–5]. The simplest solution to the underlying system of equations, usually of the reaction-diffusion type, is a solitary propagating pulse. Autowave media are characterised by all-or-none nature of these waves or at least a discrete spectrum of their amplitude and profile, as these parameters are determined by the balance between energy income and dissipation.

The first nontrivial two-dimensional generalisation of the solitary one-dimensional pulse, after the solitary plane wave, is a smoothly curved wave, which in every small domain looks like a solitary plane wave propagating in some direction.

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The local propagation velocity of such a wave differs from that of the corresponding plane wave for various reasons, the most important of which is the wave curvature. The dependence of wave velocity on its curvature (usually, in a linear approximation) delivers closed equations of motion of its front or crest line, which constitutes the essence of the kinematic approach [7,10,11]. The term 'kinematic description' was introduced to stress the fact that all the underlying physics has been concentrated in a few phenomenological constants, defining the dependence of the velocity on the curvature, whereafter the prediction of the movement of the wave becomes a purely geometric problem in space-time. The curvature-velocity dependence can be obtained by singular perturbation techniques, starting from the solution of steadily propagating pulse [6,7]. In some circumstances this may not be enough, as it is the case for the Kuramoto-Sivashinsky equation [7,8], where inverse dependence of velocity on curvature requires account of higher derivatives of curvature along the wave.

A more complicated 2D pattern is a broken wave. The most important case of such structures are rotating spiral waves of excitation, observed in many autowave media [1–5], and are of significant practical interest in cardiac muscle [9], where they underlie dangerous pathologies like tachyarrhythmias and fibrillation. If the waves are rare, so that next wave does not feel the traces of the previous one that propagated through the same point, and slightly curved, then the crest line of the wave can be defined, which is now not a closed line or line ending on medium boundaries, but has an end or 'tip' inside the medium. [2] This line crosses a region with the boundary, drawn by the tip. In this region, motion of the crest line can be described by the kinematic equations. These equations now require boundary conditions at the tip trajectory, and some more conditions are required to determine the trajectory itself, *i.e.* the movement of the tip. Both types of equations relate geometry and motion of the crest line near the tip, and we shall call both wave tip motion equations.

The 'classical' kinematic approach, developed by Davydov, Mikhailov, Zykov *et al.* [10,11], is based on the following main equations:

$$V(s,t) = V_* - DK(s,t) \tag{1}$$
$$G_0 = \gamma(K_0 - K_c) \tag{2}$$
$$\partial_t K_0 = -G_0 \partial_s K_0 \tag{3}$$

where the wave form is described in terms of time t and arclength s measured from the wave tip, $V(s,t)$ and $G(s,t)$ are normal and tangential components of

[2] We use the term crest line, not front line, to avoid confusion with the close, but distinct 'eikonal-curvature' approach or Fife limit [12–14], which describes the broken excitation wave by two curves, its front and back, with their junction being the wave tip.

wave velocity, $K(s,t)$ is crest line curvature, and subscript 0 refers to the value at the tip, *i.e.* at $s = 0$. Constants D, V_*, γ and K_c are medium parameters (more formal definitions will be given below).

Equation (1) is well known in many areas of physics, *e.g.* in flame propagation and crystal growth (see references in [13]), and for excitation pulses in reaction-diffusion systems it has been derived *e.g.* by Kuramoto [7]. Equation (2) has been first proposed from phenomenological considerations; in [10,11] it has been substantiated by perturbation technique similar to that used by Kuramoto [7], starting from the solution in the form of a broken plane wave propagating steadily and in the direction orthogonal to itself, and neglecting, at some stage, the curvature variations along the wave. The last equation in this system, (3), is the weakest basis of the existing theory; as in [11], it is in fact an arbitrary suggestion introduced to close the system of equations. In the stationary case, equation (3) can be satisfied in two different ways; usually it was assumed that

$$G_0 = 0. \tag{4}$$

These equations, together with definition of geometrical quantities involved, constitute a well posed 'kinematic' problem, where all underlying physics is concentrated in a few coefficients. This theory and its generalization to curved surfaces, to inhomogeneous, refractory and nonstationary media and to three dimensions has been used to analyse the dynamics of spiral and scroll waves. A recent review of the results can be found in [11].

In this paper, we discard the simplification and arbitrary suggestion mentioned above, and derive the motion equations for the wave tip consistently by singular perturbation techniques using generic assumptions (Section 2). The resulting equations of the wave tip have proved more complicated and diverse than the traditional ones (2, 3). As an example, we apply the new equations to the problem of a stationary spiral wave (Section 3). In this simplest nontrivial problem, the traditional equations have proved to be a special case, giving a unique solution while, in general, there may be many, and different asymptotical magnitudes of certain characteristic quantities. The last section is devoted to the discussion of most interesting physical consequences and further directions.

2 Derivation of the wave tip motion equations

We consider a reaction-diffusion system in the plane,

$$\partial_t u = \hat{\mathbf{D}}\Delta u + f(u) + \delta f(u), \tag{5}$$

where $u \in \mathbf{R}^l$, $l \geq 2$ is a column-vector of state variables, $f + \delta f \in \mathbf{R}^l$ describes local kinetics, $\hat{\mathbf{D}}$ is an $l \times l$ matrix of diffusion coefficients and Δ is the two-dimensional Laplacian operator. For a chemical excitable medium, u are concentrations of reagents and $f + \delta f$ reaction rates; for cardiac muscle u are transmembrane voltage, ionic concentrations and gating variables.

System (5) is obtained as a perturbation of a special system,

$$\partial_t u = \hat{\mathbf{D}}\Delta u + f(u), \quad \delta f = \epsilon f_1, \quad \epsilon \ll 1, \tag{6}$$

that belongs to Winfree's 'rotor boundary' ∂R [15] in the parametric space. Near this boundary, the rotation period and core diameter of the spiral wave become very large, and we assume that at the boundary, a stationary propagating broken wave solution u_* exists. Denoting normal (propagation) velocity by v_* and sideward ('growth') velocity by g_*, this hypothetical solution can be defined as

$$u(x, y, t) = u_*(x - v_* t, y - g_* t). \tag{7}$$

The coordinate $\eta = y - g_* t$ is chosen so that the broken wave is at $\eta > 0$. The form (7) is the generic form of a stationary propagating broken wave solution taking account of the translational symmetry of the reaction-diffusion system. That it is not an arbitrary two-dimensional solution breaking the translational symmetry, but specifically the broken wave, is given by the requirements that

$$u_*(\xi, \eta) \rightarrow \bar{u}_*(\xi), \quad \eta \rightarrow +\infty, \tag{8}$$
$$\bar{u}_*(\xi) \rightarrow u_0, \quad \xi \rightarrow \pm\infty, \tag{9}$$

i.e. far from the tip, the broken wave $u_*(\xi, \eta)$ becomes a plane wave, with the profile of a solitary pulse $\bar{u}_*(\xi)$ propagating through resting state u_0. Then the perturbed system (5) also has a plane wave solution,

$$u(x, y, t) = U_*(x - V_* t), \tag{10}$$

with a profile U_* and a velocity V_* close to $\bar{u}_*$ and v_*.

We are interested in solutions of (5) in the form of the broken wave, slightly perturbed and smoothly curved. Let us introduce the coordinate system (s, q) related to the crest line (see Fig. 1(a)):

$$\mathbf{r} = \mathbf{R}(s, t) + q\mathbf{N}(s, t), \tag{11}$$

where $\mathbf{R}(s,t)$ is the crest line equation, $\mathbf{N}(s,t)$ is the unit normal and s is the arclength measured from the tip.

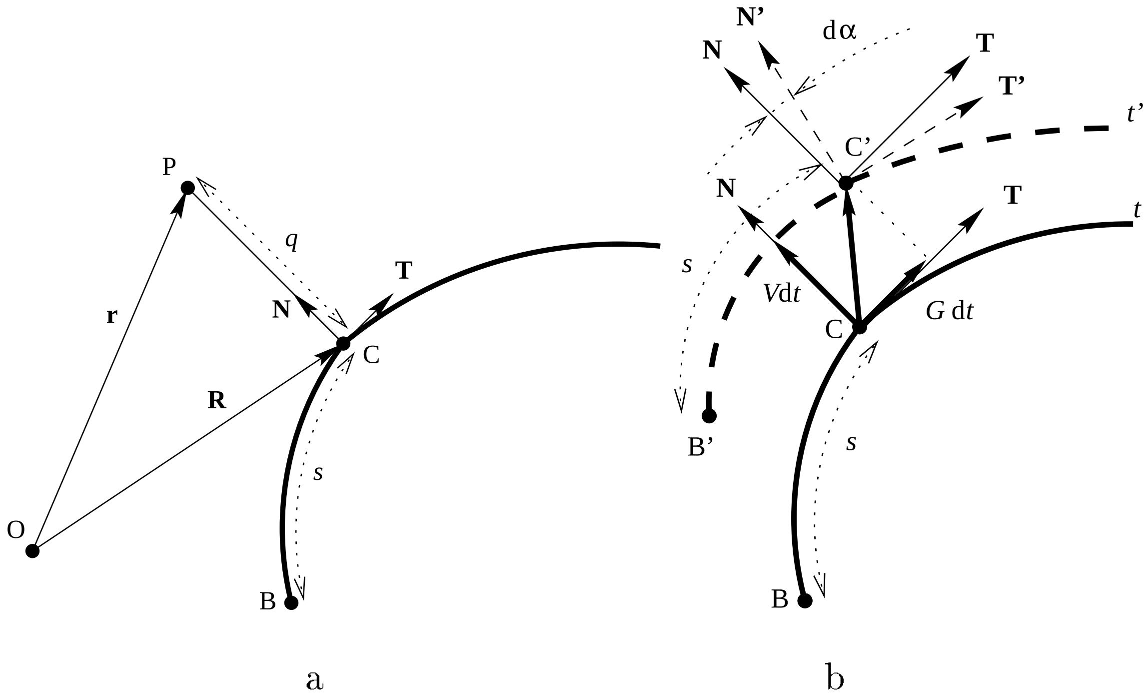

Fig. 1. Kinematic description of motion of the crest line. (a) The curvilinear coordinates related to the crest line shown by the bold line. O is the origin, B — the wave break point (tip), C — a point on the crest-line at the distance s from the tip, $\mathbf{R}$ — its radius-vector, $\mathbf{T}$ and $\mathbf{N}$ — tangent and normal unit vectors, P — a point in the plane at the distance q from C in the directin of $\mathbf{N}$, $\mathbf{r}$ its radius-vector. Thus, s and q are the curvilinear coordinates of the point P. (b) Velocities related to the moving crest-line. Solid bold line is the position of the crest at the previous moment, t, and the dashed bold line at the next moment, $t' = t + \mathrm{d}t$. The point C with chosen coordinate s has shifted to C' with the velocity V normal to the crest-line and G along the crest-line, and orientation of the crest-line at that point has turned by the angle $\mathrm{d}\alpha = (\omega + GK)\mathrm{d}t$.

Then the required solution is

$$u(s,q,t) = u_*(s,q) + \delta u(s,q,t), \quad \delta u = \epsilon u_1. \tag{12}$$

Now we transform (5) into the coordinate system (11), and perform the substitution (12). Linear approximation in δu yields

$$\partial_t u_1 = \hat{\mathbf{L}} u_1 + h(s,q,t) \tag{13}$$

where the time-independent linear operator $\hat{\mathbf{L}}$ is

$$\hat{\mathbf{L}} = \hat{\mathbf{B}}(s,q) + \hat{\mathbf{D}}(\partial_s^2 + \partial_q^2) + v_*\partial_q + g_*\partial_s, \tag{14}$$
$$(\hat{\mathbf{B}})_{ij} = (\partial f_i/\partial u_j)_{u=u_*(s,q)} \tag{15}$$

The translational symmetry of (5) means that $\hat{\mathbf{L}}$ has two null-eigenfunctions

$$\Psi_1 = \partial_q u_*(q,s), \quad \Psi_2 = \partial_s u_*(q,s), \quad \hat{\mathbf{L}}\Psi_1 = \hat{\mathbf{L}}\Psi_2 = 0, \tag{16}$$

and far from the tip, the perturbed solution approaches the plane wave, so

$$\begin{aligned}
\Psi_1(s,q) &\to \Phi_1(q) \equiv \partial_q \bar{u}_*(q), \quad s \to +\infty, \\
\Psi_1(s,q) &\to 0, \quad s \to -\infty, \\
\Phi_1(q) &\to 0, \quad q \to \pm\infty, \\
\Psi_2(s,q) &\to 0, \quad s \to \pm\infty, \\
\Psi_2(s,q) &\to 0, \quad q \to \pm\infty,
\end{aligned} \tag{17}$$

In what follows, we shall assume that these limits are approached rapidly enough.

We introduce the following notations for local crest line-related quantities (see also Fig. 1(*b*)): $\mathbf{T} = \partial_s \mathbf{R}$ for unit tangent, $K = -\partial_s \mathbf{T} \cdot \mathbf{N}$ for local curvature, $V = \partial_t \mathbf{R} \cdot \mathbf{N}$ for normal velocity, $G = \partial_t \mathbf{R} \cdot \mathbf{T}$ for tangent velocity and $\omega = -\partial_t \mathbf{N} \cdot \mathbf{T}$ for angular velocity. It is easily seen that

$$\omega \equiv \partial_s V - GK. \tag{18}$$

We expect to find solutions depending on the small parameter ϵ not only through (12) but also *via* the shape of the crest line. This dependence may be different in different situations. To facilitate calculations, however, we stick to a certain dependence, which will later prove to be a self-consistent assumption in some cases, and still lead to correct consequences, if expressed in original variables, in other cases (including the special case corresponding to the 'traditional' equations). Namely, we assume that

$$\begin{aligned}
K &= \epsilon K_1 \\
\omega &= \epsilon \omega_1 \\
V - v_* &= \epsilon V_1 \\
G - g_* &= \epsilon G_1
\end{aligned} \tag{19}$$

where the quantities with subscript 1 are supposed to remain finite in the limit $\epsilon \to 0$. A more generic and accurate approach would consider each of these quantities as an independent small parameter; however, this would only enlarge the formulae and yet lead to the same results.

In a linear approximation in ϵ, the free term h in (13) is

$$h(s,q,t) = \left[G_1 - q\left(g_* K_1 + \partial_s K_1 \hat{\mathbf{D}} + \omega_1\right)\right] \Psi_2 + \\ + \left(V_1 + K_1 \hat{\mathbf{D}}\right) \Psi_1 - 2qK_1 \hat{\mathbf{D}} \partial_s \Psi_2 + f_1(u_*) + O(\epsilon) \qquad (20)$$

Let us define the two-dimensional inner product,

$$\langle u, v\rangle = \int\int u(s,q)v(s,q)\mathrm{d}q\mathrm{d}s. \qquad (21)$$

Then the conditions of solvability of (13) with respect to u_1 are

$$\left\langle h, \Psi^j \right\rangle = 0, \quad j = 1, 2, \qquad (22)$$

where Ψ^1, Ψ^2 are the null-eigenfunctions of the adjoint operator $\hat{\mathbf{L}}^+$,

$$\hat{\mathbf{L}}^+ \Psi^{1,2} = 0, \quad \left\langle \Psi_i, \Psi^j \right\rangle = 0,\ i \neq j, \quad \left\langle \Psi_2, \Psi^2 \right\rangle = 1. \qquad (23)$$

We assume without proof, that asymptotic behaviour of $\Psi^{1,2}(s,q)$ at large s is analogous to (17), *i.e.*

$$\begin{aligned} \Psi^1(s,q) &\to \Phi^1(q) \quad s \to +\infty, \\ \Psi^1(s,q) &\to 0, \quad s \to -\infty, \\ \Phi^1(q) &\to 0, \quad q \to \pm\infty, \\ \Psi^2(s,q) &\to 0, \quad s \to \pm\infty, \\ \Psi^2(s,q) &\to 0, \quad q \to \pm\infty, \end{aligned} \qquad (24)$$

Now, we come to the key point in the derivation. Let us consider equation (22) for $j = 1$:

$$\left\langle K_1 g_* q \Psi_2 + 2qK_1 \hat{\mathbf{D}} \partial_s \Psi_2 + \omega_1 q \Psi_2 + q \partial_s K_1 \hat{\mathbf{D}} \Psi_2, \Psi^1 \right\rangle - \\ - \left\langle \left(V_1 + K_1 \hat{\mathbf{D}}\right) \Psi_1 + f_1(u_*), \Psi^1 \right\rangle = O(\epsilon) \qquad (25)$$

It contains two singular integrals. Convergence of the first one is provided by the decay of Ψ_2 (17) and the boundedness of other factors. On the contrary, convergence of the second one is not guaranteed by any of the assumptions made so far. So, to satisfy Eq. (25), we should first provide the convergence of this integral, which requires that the integrand vanishes at large s. This requirement leads immediately to the classical wave motion equation (1), where

$$
\begin{aligned}
V_* &= v_* - \epsilon\left(F_1, \Phi^1\right)\left(\Phi_1, \Phi^1\right)^{-1}, \\
D &= \left(\hat{\mathbf{D}}\Phi_1, \Phi^1\right)\left(\Phi_1, \Phi^1\right)^{-1}, \\
F_1 &= \lim_{s\to\infty} f_1(u_*)
\end{aligned}
\tag{26}
$$

and the parentheses $(\cdot, \cdot)$ denote one-dimensional inner products,

$$
(u, v)(s) = \int u(s, q)v(s, q)\mathrm{d}q. \tag{27}
$$

With (1), (26) satisfied, both integrals in (25) converge and we may further require that their sum vanishes. This leads to another equation, now for the tip:

$$
\omega_1(0, t) = \lambda_0 - \lambda_1 K_1(0, t) - \lambda_2 \partial_s K_1(0, t) + O(\epsilon). \tag{28}
$$

where

$$
\lambda_0 = \left[\left\langle F_1, \Psi^1 - \Phi^1 \frac{(\Psi_1, \Psi^1)}{(\Phi_1, \Phi^1)}\right\rangle + \left\langle f_1(u_*) - F_1, \Psi^1\right\rangle\right]\left\langle q\Psi_2, \Psi^1\right\rangle^{-1} \tag{29}
$$

$$
\begin{aligned}
\lambda_1 = {}& g_* + 2\left\langle q\hat{\mathbf{D}}\partial_s\Psi_2, \Psi^1\right\rangle\left\langle q\Psi_2, \Psi^1\right\rangle^{-1} - \\
& - \left[\left\langle \hat{\mathbf{D}}\left(\Psi_1 - \Phi_1\right), \Psi^1\right\rangle + \left\langle \hat{\mathbf{D}}\Phi_1, \Psi^1 - \Phi^1\frac{(\Psi_1, \Psi^1)}{(\Phi_1, \Phi^1)}\right\rangle\right]\left\langle q\Psi_2, \Psi^1\right\rangle^{-1}
\end{aligned}
\tag{30}
$$

$$
\lambda_2 = \left\langle q\hat{\mathbf{D}}\Psi_2, \Psi^1\right\rangle\left\langle q\Psi_2, \Psi^1\right\rangle^{-1}. \tag{31}
$$

Note, that we now have obtained two motion equations, one for the crest line (1) and one for the tip (28), out of single equation (25).

Equation (22) at $j = 2$ also contains singular integrals, but their convergence is already guaranteed by (17) and (24). So, this equation leads to just one more condition,

$$
G_1(0, t) = \mu_0 + \mu_1 K_1(0, t) + \mu_2 \partial_s K_1(0, t) + O(\epsilon), \tag{32}
$$

where

$$\mu_0 = \frac{\langle q\Psi_2, \Psi^2\rangle}{\langle q\Psi_2, \Psi^1\rangle} \left[\left\langle F_1, \Psi^1 - \Phi^1 \frac{(\Psi_1, \Psi^1)}{(\Phi_1, \Phi^1)} \right\rangle + \left\langle f_1(u_*) - F_1, \Psi^1 \right\rangle \right] - \\ - \left\langle f_1(u_*), \Psi^2 \right\rangle \tag{33}$$

$$\mu_1 = -\left\langle \hat{\mathbf{D}}\Psi_1, \Psi^2 \right\rangle + 2\left\langle q\hat{\mathbf{D}}\partial_s\Psi_2, \Psi^2 - \Psi^1 \frac{\langle q\Psi_2, \Psi^2\rangle}{\langle q\Psi_2, \Psi^1\rangle} \right\rangle + \\ + \frac{\langle q\Psi_2, \Psi^2\rangle}{\langle q\Psi_2, \Psi^1\rangle} \left[\left\langle \hat{\mathbf{D}}(\Psi_1 - \Phi_1), \Psi^1 \right\rangle + \left\langle \hat{\mathbf{D}}\Phi_1, \Psi^1 - \Phi^1 \frac{(\Psi_1, \Psi^1)}{(\Phi_1, \Phi^1)} \right\rangle \right] \tag{34}$$

$$\mu_2 = \left\langle q\hat{\mathbf{D}}\Psi_2, \Psi^2 - \frac{\langle q\Psi_2, \Psi^2\rangle}{\langle q\Psi_2, \Psi^1\rangle}\Psi^1 \right\rangle \tag{35}$$

Equations (28) and (32) give the required system of equations at the wave tip and supplement the wave motion equation (1). In this system, boundary conditions for $K(s,t)$ and equations of the tip motion are mixed together. With help of (18), it can be rewritten in an equivalent form[3],

$$\begin{aligned} 0 &= \lambda_0 + (D - \lambda_2)K_1' + (g_* + \epsilon\mu_0 - \lambda_1)K_1 \\ &\quad + \epsilon(\mu_1 - \lambda_3)K_1^2 + \epsilon(\mu_2 - \lambda_4)K_1K_1' - \epsilon\lambda_5 K_1'^2 + O\left(\epsilon^2\right), \\ \omega_1 &= \lambda_0 - \lambda_1 K_1 - \lambda_2 K_1' - \epsilon\lambda_3 K_1^2 - \epsilon\lambda_4 K_1 K_1' - \epsilon\lambda_5 K_1'^2 + O\left(\epsilon^2\right), \\ G_1 &= \mu_0 + \mu_1 K_1 + \mu_2 \partial_s K_1 + O(\epsilon), \end{aligned} \tag{36}$$

In original variables this is

$$\begin{aligned} 0 &= \epsilon\lambda_0 + (D - \lambda_2)K' + (g_* + \epsilon\mu_0 - \lambda_1)K \\ &\quad + (\mu_1 - \lambda_3)K^2 + (\mu_2 - \lambda_4)KK' - \lambda_5 K'^2 + O\left(\epsilon^3\right) \\ \omega &= \epsilon\lambda_0 - \lambda_1 K - \lambda_2 K' - \lambda_3 K^2 - \lambda_4 KK' - \lambda_5 K'^2 + O\left(\epsilon^3\right) \\ G &= g_* + \epsilon\mu_0 + \mu_1 K + \mu_2 K' + O\left(\epsilon^2\right) \end{aligned} \tag{37}$$

where functions K, K', G and ω are assumed to have arguments $(0,t)$. Here the first equation is the boundary condition for the evolution of $K(s,t)$, and the two others determine the motion of the tip given the evolution of $K(s,t)$.

In these equations, we have retained terms of different asymptotical orders in ϵ, to keep within the scope the traditional equations (2), (3). Equation (2) can be considered as a special case of (32) at $\mu_2 = 0$. Equation (28) is new, and is to replace the traditional equation (3), which does not fit to the new system at all, or requires too many assumptions to have sense. However, the stationary versions of the motion equations are comparable, and will be compared the next section and in Discussion.

[3] Here we include higher-order terms with coefficients $\lambda_{4,5}$ which can be obtained in the next order of the same perturbation technique

3 The stationary spiral wave

3.1 Problem formulation and general solution.

A stationary spiral wave rotates rigidly around a fixed point, and its tip describes a circle, called the core of the spiral, centered at that point. The rigid rotation means that the shape of the wave remains constant and only its position on the plane changes, so K and G do not depend on t. In this case, the wave evolution equation (1) can be transformed to an integro-differential one [10,11],

$$K(s)\left[-G(0)+\int_0^s K(s_1)\left(V_*-DK(s_1)\right)\mathrm{d}s_1\right]-DK'(s)=\omega \tag{38}$$

where ω is now the constant angular velocity of the whole spiral; for definiteness we consider only spirals rotating counter-clockwise, so that increasing s means moving from left to right with respect to propagation direction, and $\omega > 0$.

The boundary conditions of (38) at $s = 0$ are (37), and at infinity

$$K(+\infty) = +0. \tag{39}$$

Make a change of variables,

$$\begin{aligned} s &= DV_*^{-1}\sigma, \\ K(s) &= V_* D^{-1} y(\sigma), \\ \omega &= V_*^2 D^{-1}\Omega. \end{aligned} \tag{40}$$

We are interested in solutions with K, and hence y, small. Transformation of equation (38) to an ODE and substitution (40) lead to the following equation:

$$yy'' - y'^2 - \Omega y' - y^3 = O\left(y^4\right) \tag{41}$$

where prime $'$ denotes differentiation by the new independent variable σ, with boundary conditions

$$\begin{aligned} 0 = {} & \epsilon\nu_0 + (1-\nu_2)y' + (\nu_3-\nu_1)y \\ & +(\nu_4-\nu_6)y^2 + (\nu_5-\nu_7)yy' - \nu_8 y'^2 + O\left(\epsilon^3\right), \qquad \sigma = 0, \end{aligned}$$

$$\Omega = \epsilon\nu_0 - \nu_1 y - \nu_2 y' - \nu_6 y^2 - \nu_7 y y' - \nu_8 {y'}^2 + O\left(\epsilon^3\right), \qquad \sigma = 0, \tag{42}$$

$$y \to +0, \qquad \sigma \to +\infty, \tag{43}$$

This poses a nonlinear eigenvalue problem for eigenvalue Ω and function $y(\sigma)$. Here the dimensionless medium parameters are defined as

$$\begin{aligned} &\nu_0 = \lambda_0 D V_*^{-2}, && \nu_1 = \lambda_1 V_*^{-1}, && \nu_2 = \lambda_2 D^{-1}, \\ &\nu_3 = (g_* + \epsilon\mu_0) V_*^{-1}, && \nu_4 = \mu_1 D^{-1}, && \nu_5 = \mu_2 V_* D^{-2}, \\ &\nu_6 = \lambda_3 D^{-1}, && \nu_7 = \lambda_4 V_* D^{-2}, && \nu_8 = \lambda_5 V_*^2 D^{-3}. \end{aligned} \tag{44}$$

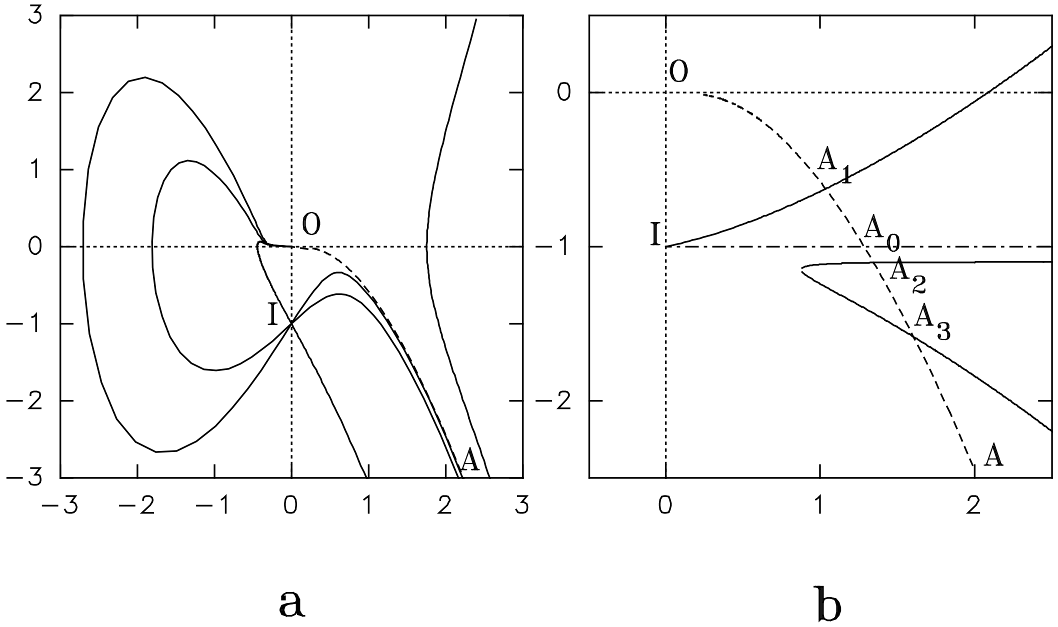

Fig. 2. (a) Phase portrait of equation (41) in coordinates $\left(y/\Omega^{2/3}, y'/\Omega\right)$. O is a complex equilibrium, I is a non-equilibrium singular point. Dashed line OA is the separatrix of the origin. (b) The boundary-value problem in the same coordinates. The separatrix OA (dashed) is the only integral curve, obeying (43); conditions (42) select points on it, corresponding to the tip. Dash-dotted line IA_0 corresponds to (2,3) and gives the only intersection point A_0. Solid lines correspond to $\nu_0 = 0$, $\nu_1 = -0.1$, $\nu_2 = 1$, $\nu_3 = 0.2$, $\nu_4 = -1.1$, $\nu_5 = 1$. In this case, there are three intersections, $A_{1,2,3}$, and thus three solutions, $\Omega_{1,2,3} \approx -0.95094, -0.64050, -0.49185$, and $y_{1,2,3}(0) \approx 0.078918, 2.2233, 0.12060$.

A phase portrait of the differential equation (41) without the term $O(y^4)$ is shown on Fig. 2(a). The only integral curve obeying the condition at infinity (43) is the separatrix OA. This curve can be described analytically; it is convenient to do that in a piecewise manner,

$$y(\sigma) = (9/2)^{1/3} \Omega^{2/3} \zeta^{2/3} \left(1 + Y^2(\zeta)\right),$$

$$y'(\sigma) = 3\Omega\zeta Y(\zeta)\left(1 + Y^2(\zeta)\right) - \Omega,$$
$$\zeta = \frac{\sqrt{2}}{3}\Omega^{1/2}(\sigma_0 - \sigma)^{3/2},$$
$$Y(\zeta) = \left(J_{-2/3}(\zeta) - J_{+2/3}(\zeta)\right) / \left(J_{-1/3}(\zeta) + J_{+1/3}(\zeta)\right), \tag{45}$$

for $\sigma \leq \sigma_0$, and

$$y(\sigma) = -(9/2)^{1/3}\Omega^{2/3}\zeta^{2/3}\left(1 - Y^2(\zeta)\right),$$
$$y'(\sigma) = -3\Omega\zeta Y(\zeta)\left(1 - Y^2(\zeta)\right) - \Omega,$$
$$\zeta = \frac{\sqrt{2}}{3}\Omega^{1/2}(\sigma - \sigma_0)^{3/2},$$
$$Y(\zeta) = \left(I_{-2/3}(\zeta) - I_{+2/3}(\zeta)\right) / \left(I_{-1/3}(\zeta) - I_{+1/3}(\zeta)\right)$$
$$\equiv K_{2/3}(\zeta)/K_{1/3}(\zeta), \tag{46}$$

for $\sigma \geq \sigma_0$, where σ_0 is an integration constant related to the position of the tip point on the separatrix OA. We will call these pieces the J-branch and the I-branch respectively (it is easy to see that one is the analytical continuation of the other).

The tip point ($\sigma = 0$) may be at the J-branch if $\sigma_0 > 0$ or at the I-branch if $\sigma_0 < 0$, in the latter case only the I-branch plays a rôle. Boundary conditions at the tip from (42) in these two cases are

$$\begin{aligned} 0 = {} & \nu_3 + 6^{1/3}Y_0\zeta_0^{1/3}\Omega^{1/3} \pm (9/2)^{1/3}\nu_4\left(1 \pm Y_0^2\right)\zeta_0^{2/3}\Omega^{2/3} \\ & \pm 3\nu_5\zeta_0 Y_0\left(1 \pm Y_0^2\right)\Omega - \nu_5\Omega + O\left(\left(1 \pm Y_0^2\right)^{-1}(\zeta_0\Omega)^{-2/3}\epsilon^3\right), \\ 0 = {} & \epsilon\nu_0 \mp (9/2)^{1/3}\nu_1\zeta_0^{2/3}\left(1 \pm Y_0^2\right)\Omega^{2/3} \mp 3\nu_2\zeta_0 Y_0\left(1 \pm Y_0^2\right)\Omega + \\ & +(\nu_2 - 1)\Omega - \nu_6(9/2)^{2/3}\Omega^{4/3}\zeta_0^{4/3}\left(1 \pm Y_0^2\right)^2 \mp \\ & \mp\nu_7(9/2)^{1/3}\Omega^{2/3}\zeta_0^{2/3}\left(1 \pm Y_0^2\right)\left[\pm 3\Omega\zeta_0 Y_0\left(1 \pm Y_0^2\right) - \Omega\right] - \\ & -\nu_8\left[\pm 3\Omega\zeta_0 Y_0\left(1 \pm Y_0^2\right) - \Omega\right]^2 + O\left(\epsilon^3\right), \end{aligned} \tag{47}$$

where the upper sign is for J-branch and lower sign is for I-branch,

$$Y_0 = Y(\zeta_0), \qquad \zeta_0 = \frac{\sqrt{2}}{3}\Omega^{1/2}|\sigma_0|^{3/2}, \tag{48}$$

and $Y(\zeta)$ is the corresponding function from (45) or from (46).

So (47) is a system of finite (non-differential) equations for unknown variables

ζ_0 and Ω, which, in principle, solves the problem. All kinematical parameters of the spiral wave are expressed *via* ζ_0 and Ω:

$$
\begin{aligned}
\omega &= V_*^2 D^{-1} \Omega, \\
\tan \varphi_0 &= -\left(\nu_3 + \nu_4 y(0) + \nu_5 y'(0)\right) / \left(1 - y(0)\right), \\
r_0 &= D(V_* \Omega)^{-1} \left[\left(\nu_3 + \nu_4 y(0) + \nu_5 y'(0)\right)^2 + \left(1 - y(0)\right)^2 \right]^{1/2}, \\
K(0) &= V_* D^{-1} y(0).
\end{aligned} \tag{49}
$$

Here r_0 is the core radius and φ_0 is the orientation angle of the tip with respect to its radius-vector.

System (47) in its general form is, however, rather complicated:

- The first of these equations is a cubic equation with respect to $\Omega^{1/3}$, and there may be up to three different spiral wave solutions in the same medium.
- Functions $Y(\zeta)$ are transcendental, and explicit expression for $\zeta_0(\nu_i)$ cannot be written for the general case.
- It depends on small parameter ϵ and should have solution with uniformly small $y(\sigma)$, and the problem is still more complicated if there are small parameters besides ϵ.

Fig. 2(b) illustrates these difficulties: it shows an example with three solutions, while two of them have $y(0)$ small.

In following subsections, we consider two most important special cases, where explicit results can be obtained.

3.2 The generic case.

Suppose that all dimensionless parameters of the medium given by (44) are of the order of 1, and so ϵ is the only small parameter of the problem. Note that, in particular, $\nu_3 \sim 1$ requires that $g_* \sim V_*$.

One can see that due to smallness of ϵ when $g_* \neq 0$, boundary conditions (47) can be satisfied only with small Ω and large ζ_0 and with the tip $\sigma = 0$ lying at the I-branch. Then

$$
y(\sigma) = (6\zeta)^{-1/3} \Omega^{2/3} + O\left(\Omega^{2/3} \zeta^{-4/3}\right), \tag{50}
$$

$$
y'(\sigma) = -(6\zeta)^{-1} \Omega + O\left(\Omega \zeta^{-2}\right), \tag{51}
$$

$$\zeta = \frac{\sqrt{2}}{3}(\sigma - \sigma_0)^{3/2}\Omega^{1/2}, \tag{52}$$

$$Y(\zeta) = 1 + (6\zeta)^{-1} + O\left(\zeta^{-2}\right), \tag{53}$$

With this precision, curvature distribution $Y(\zeta)$ is the same as that of the involute of a circle. If substituted into equation (41), these asympotics make terms yy'' and y'^2 much less than others, and can be obtained as a solution of this equation with these terms left out. Note that neglecting these very terms corresponds to independence of the normal front velocity on front curvature, which is natural for very small curvature, and consistent with the spiral being an involute of a circle.

If we look for the solution in the form

$$\begin{aligned} \zeta_0 &= A\epsilon^{-\alpha} + O(1), \qquad \alpha > 0, \\ \Omega &= B\epsilon^{\beta} + O\left(\epsilon^{2\beta}\right), \qquad \beta > 0, \end{aligned} \tag{54}$$

boundary conditions become

$$\begin{aligned} 0 &= \nu_3 + (6AB)^{1/3}\epsilon^{(\beta-\alpha)/3}, \\ &\quad + O\left(\epsilon^{3-(2\beta+\alpha)/3}, \epsilon^{(2\beta+\alpha)/3}, \epsilon^{(\beta+2\alpha)/3}, \epsilon^{(4\beta-\alpha)/3}\right), \\ 0 &= \epsilon\nu_0 - \nu_1 B^{2/3}(6A)^{-1/3}\epsilon^{(2\beta+\alpha)/3} - B\epsilon^{\beta}, \\ &\quad + O\left(\epsilon^3, \epsilon^{\alpha+\beta}, \epsilon^{2(2\alpha+\beta)/3}, \epsilon^{2(\alpha+2\beta)/3}, \epsilon^{2\beta}, \epsilon^{(5\beta+\alpha)/3}\right), \end{aligned} \tag{55}$$

and so

$$\begin{aligned} \alpha &= \beta = 1, \\ A &= \nu_3^2(6\nu_0)^{-1}(\nu_1 - \nu_3), \\ B &= -\nu_0\nu_3(\nu_1 - \nu_3)^{-1}. \end{aligned} \tag{56}$$

Our solution has a physical sense only at positive ζ_0 and ω. Hence, the following inequalities should be fulfilled:

$$\nu_3 < 0, \tag{57}$$

$$\epsilon\nu_0(\nu_1 - \nu_3) > 0. \tag{58}$$

Inequality (57) means

$$g_* < 0, \tag{59}$$

i.e. spiral wave solution can be found in this way only if the original half-wave solution of the unperturbed system is growing but not shrinking. And (58) shows that the spiral wave solutions are found only to one side of the manifold ∂R of the parametric space, which in our notations is determined by equation $\epsilon = 0$.

Finally, the dimensional parameters of the spiral wave are

$$\begin{aligned}
\omega &= V_*^2 D^{-1} \nu_0 \nu_3 (\nu_3 - \nu_1)^{-1} \epsilon + O\left(\epsilon^2\right), \\
\tan\varphi_0 &= -\nu_3 - \nu_0(\nu_3 - \nu_4)(\nu_3 - \nu_1)^{-1}\epsilon + O\left(\epsilon^2\right), \\
r_0 &= D V_*^{-1} (1+\nu_3)^{1/2} \nu_3^{-1} \left[(\nu_3 - \nu_1)\nu_0^{-1}\epsilon^{-1} - (1 - \nu_3\nu_4)(1+\nu_3)^{-1}\right] + O(\epsilon) \\
K(0) &= -V_* D^{-1} \nu_0 (\nu_3 - \nu_1)^{-1}\epsilon + O\left(\epsilon^2\right). \qquad (60)
\end{aligned}$$

3.3 The case of zero tip growth rate.

One of necessary conditions for the results of previous subsection is that parameter g_* is nonzero — and, moreover, is negative. Let us consider the special case

$$g_* = 0. \qquad (61)$$

Now we look for solutions to the finite problem (47) in the form

$$\begin{aligned}
\zeta_0 &= \zeta_* + A\epsilon^\alpha + O\left(\epsilon^{2\alpha}\right), \qquad \alpha > 0, \\
\Omega &= B\epsilon^\beta + O\left(\epsilon^{2\beta}\right), \qquad \beta > 0, \qquad (62)
\end{aligned}$$

at the J-branch, where $\zeta_* \approx 0.68555$ is the least positive root of equation

$$J_{2/3}(\zeta_*) = J_{-2/3}(\zeta_*). \qquad (63)$$

Then $Y_0 = -A\epsilon^\alpha + O\left(\epsilon^{2\alpha}\right)$ and hence boundary conditions (47) become

$$\begin{aligned}
&\bar\nu_3\epsilon - A(6B\zeta_*)^{1/3}\epsilon^{\alpha+\beta/3} + 2^{-1/3}\nu_4(3B\zeta_*)^{2/3}\epsilon^{2\beta/3} + O\left(\epsilon^{2\alpha+\beta/3}, \epsilon^{\alpha+2\beta/3}, \epsilon^\beta, \epsilon^{3-2\beta/3}\right) = 0, \\
&\nu_0\epsilon - 2^{-1/3}\nu_1(3B\zeta_*)^{2/3}\epsilon^{2\beta/3} + O\left(\epsilon^3, \epsilon^{\alpha+2\beta/3}, \epsilon^\beta\right) = 0. \qquad (64)
\end{aligned}$$

where $\bar\nu_3 = \mu_0/V_*$. This gives

$$
\begin{aligned}
\alpha &= 1/2, \\
\beta &= 3/2, \\
A &= (\nu_1 \bar{\nu}_3 + \nu_0 \nu_4)(2\nu_0\nu_1)^{-1/2}, \\
B &= 2^{1/2}(3\zeta_*)^{-1}(\nu_0/\nu_1)^{3/2},
\end{aligned} \tag{65}
$$

and the dimensional parameters of the spiral wave solution are

$$
\begin{aligned}
\omega &= 2^{1/2}(3\zeta_*)^{-1} V_*^2 D^{-1} (\nu_0/\nu_1)^{3/2} \epsilon^{3/2} + O\left(\epsilon^3\right), \\
\tan\varphi_0 &= -\left[\bar{\nu}_3 + \nu_4 \nu_0/\nu_1\right] \epsilon + O\left(\epsilon^{3/2}\right), \\
r_0 &= 2^{-1/2} 3\zeta_* D V_*^{-1} \left[(\nu_1/\nu_0)^{3/2} \epsilon^{-3/2} - (\nu_1/\nu_0)^{1/2} \epsilon^{-1/2}\right] + O(1), \\
K(0) &= V_* D^{-1} (\nu_0/\nu_1) \epsilon + O\left(\epsilon^{3/2}\right).
\end{aligned} \tag{66}
$$

This solution formally coincides, in the main orders, with that presented in [10,11]:

$$
\begin{aligned}
\omega &= V_*^2 D^{-1} \xi p^{3/2}, \\
\tan\varphi_0 &= 0, \\
r_0 &= D V_*^{-1} \xi^{-1} p^{-3/2}, \\
K(0) &= V_* D^{-1} p,
\end{aligned}
$$

where p is a small parameter and $\xi \approx 0.685$ was found by numerical integration of equation (38). This coincidence is achieved by identifying p with $\nu_0 \nu_1^{-1} \epsilon$ and ξ with $2^{1/2}(3\zeta_*)^{-1} \approx 0.68763$. In this 'weak' sense we can say that the 'traditional' case is of codimension 1 relative to the generic case, as the 'traditional' spiral wave *solution* can be achieved if one additional conditions (61) is fulfilled. Note, however, that this is only a formal correspondence, as the small parameter p of [10,11] has different physical sense from ϵ: while ϵ shows the instantaneous turning rate of a tip of an uncurved half-wave, with its growth rate being zero in this case, p is proportional to the growth rate while the turning rate is assumed identically zero. This is because coincidence of the *solutions* was achived not by coincidence of *equations*.

4 Discussion.

In this paper we have derived motion equations both for the crest line of the excitation wave and for its tip, within a single perturbation procedure and using only assumptions of smallness of typical wave curvature and proximity to the manifold of stationary propagating half-wave solutions in parametric space, supposedly corresponding to Winfree's boundary ∂R. These motion equations are obtained here for the first time. The motion equations depend upon coefficients determined by the properties of the linearised operator at these basic solutions.

It is interesting to compare our results with the traditional 'kinematic' theory of [10,11], built partly from asymptotical and partly from phenomenological consideration. The traditional tip motion equations (2), (3) do not coincide, nor are they a partial case of our new equations (36), as (3) has different functional form. However, if we restrict consideration to stationary solutions, they can be considered as a special case. How special is it? We answer this questions in terms of relative codimension, *i.e.* number of additional equalities for parameters required to obtain this case. If we look for conditions when the spiral wave solutions are identical to those obtained from the traditional approach, then the only additional condition (61) is required, and in this sense the traditional approach has relative codimension one. However, as it was noted in Section 3.3, this provides only formal correspondence of the solutions, and even the small parameters in these solutions have different physical sense.

Another possible interpretation of this question is, when the *equations* rather than *solutions* become identical to the traditional ones. To see it, during derivation of (36) we had retained more terms than were really used; and some terms did not play any role because of their additional smallness caused by slow variation of curvature along the crestline, not accounted for by ansatz (19). Let us consider now the minimal cut-off system, obeying the following requirements: (i) it has all the terms necessary to achieve the generic solution of Section 3.2, (ii) it has all the terms necessary to achieve the 'non-growing' solution of Section 3.3, formally equivalent to the 'traditional' spiral wave solutions, and (iii) it has all the terms necessary to have the 'traditional' tip motion equations (2), (4) as a special case. This minimal system is (we drop symbols $O\,()$)

$$\begin{aligned} 0 &= \epsilon\lambda_0 + (D-\lambda_2)K' + (g_* + \epsilon\mu_0 - \lambda_1)K + (\mu_1 - \lambda_3)K^2, \\ \omega &= \epsilon\lambda_0 - \lambda_1 K - \lambda_2 K' - \lambda_3 K^2, \\ G &= g_* + \epsilon\mu_0 + \mu_1 K, \end{aligned} \tag{67}$$

Now, consider a special case defined by the following five conditions

$$\begin{aligned} g_* &= 0 \\ \lambda_0 &= 0 \\ \lambda_1 &= 0 \\ \lambda_2 &= D \\ \lambda_3 &= 0 \end{aligned} \tag{68}$$

and make a change of parameters

$$
\begin{aligned}
K_c &= -\epsilon\mu_0/\mu_1 \\
\gamma &= -\mu_1
\end{aligned} \tag{69}
$$

Then the minimal system gets the form

$$
\begin{aligned}
0 &= K_c - K(0) \\
\omega &= -DK'(0) \\
G &= \gamma(K_c - K(0))
\end{aligned} \tag{70}
$$

coinciding with that of (4), (18), (2). Thus, in terms of motion equations, the traditional case has the much higher codimension five. This means that, while there is a certain probability that in a particular system condition (61) may be fulfilled with reasonable accuracy, and stationary spiral wave solution would have the properties predicted by the traditional equations, there is much less hope that the five conditions (68) would be fulfilled simultaneously, even approximately, and so the usefullness of the traditional equations for the study of parametric dependencies is much more doubtful. The applicability of those equations for nonstationary regimes is still more restricted, as in that case the 'traditional' equations do not match the new ones.

Having found spiral wave solutions in the vicinity of the manifold $\epsilon = 0$ of stationary propagating half-wave solutions, we now can rethink its relationship with ∂R. The following properties of the line $\epsilon = 0$ are similar to that of ∂R:

- Spiral wave solution exist only to one side of this manifold, which is expressed by the condition (58).
- Approaching this manifold is accompanied by growth of spiral wave period and core radius as ϵ^{-1}.

Some new properties of this parametric region are predicted by the new theory, for instance

- The tip angle φ_0 should be varying along this boundary, being always positive. In other words, growing half-waves can give birth to spiral waves *via* perturbation of parameters.
- There may be a codimension 2 submanifold $\partial^2 R \subset \partial R$ on this boundary (*e.g.* a point in a two-dimensional parametric space) where $\varphi_0 \to +0$, corresponding to the half-wave neither growing nor shrinking. Near this submanifold, the asymptotics of spiral wave solutions are different, *e.g.* period and radius grow as $\epsilon^{-3/2}$ rather than ϵ^{-1} (this is the 'traditional' solution).
- Analytical continuation of ∂R through $\partial^2 R$, if any, is no longer a 'rotor boundary'. In other words, perturbation of shrinking half-waves does not produce spiral waves. So, ∂R, generically, is a manifold with a border.

These new properties can be tested by numerical experiment; however, due to their asymptotical nature, it may be of considerable computational cost. A qualitative prediction of existence of $\partial^2 R$ should be easier to test. It is not observed in Winfree's [15] parametric map of the FitzHugh-Nagumo system, nor in Barkley's [16] map of his 'vertical isoclines' system. However, it is worth noticing that while in [16] ∂R is well separated from the meander boundary ∂M, *i.e.* the boundary between rigidly rotating spirals and biperiodic spirals, in [15] the rotor boundary ∂R after some point, goes very close to ∂M, and to distinguish them reliably, very careful computations are required. So, we can put forward a hypothetical alternative interpretation of that diagram, that at $\partial^2 R$, manifolds ∂R and ∂M join, and what goes next and shown as ∂R and ∂M going close to each other, is actually single boundary between meandering (biperiodic) spiral waves and the absence of any spiral waves solutions. Joining ∂R and ∂M would mean that in the vicinity of $\partial^2 R$, there is transition from simple (rigid) to compound (meandering) rotation *not* due to interaction of the wave tip with the refractory tail (which is commonly considered as physical mechanism of meander). The possibility of such a transition has been recently hypothesized by Starobin & Starmer [17]. Note, that in this case, the transition would be described within the new kinematic theory, and seen as Hopf bifurcation of a stationary solution in the evolution equations for the 'natural equation' $K(s,t)$ of the crest line. We believe that all these question deserve further study.

From the viewpoint of practical importance of the kinematic theory, the assumption of the proximity to the ∂R boundary may seem rather exotic. However, in terms of properties of cardiac tissue, it corresponds to reduced excitability and/or shortened action potential and refractoriness, which are features of certain pathological conditions, and this makes this limit interesting from the practical viewpoint. For instance, numerical experiments of Efimov *et al.* [18] with a model of ventricular tissue show that transition through ∂M (which is the way leading to the ∂R boundary in Winfree's [15] diagram) can be achieved by reducing the number of functioning Na channels, and such a reduction is known to correlate with certain cardiac pathologies, such as ischaemia and influence of some pharmaceutical agents.

In this respect, it is interesting to compare the "kinematic" theory considered here with the ideologically close approach of [12–14]. Despite the fact that these two approaches are close relatives and even have equation (1) in common, the "Fife limit" theory is based on the consideration that excitation wave has a sharp front and a sharp back, and description is made in terms of motion of these two lines. Though the notion of sharp front is quite relevant to cardiac excitation wave, the notion of the waveback is rather questionable, the tip is not a junction of the front with the back and the "Fife" theory is hardly applicable to heart tissue at all. On the contrary, the kinematic approach needs neither a sharp back nor even a sharp front, and only assumes that the crest line remains smooth and pulse profile across this line perturbed slightly, and these conditions may be relevant to certain conditions in heart.

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